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Symmetry and monotonicity of positive solutions to Schrödinger systems with fractional p-Laplacians

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Abstract. In this paper, we first establish narrow region principle and decay at infinity theorems to extend the direct method of moving planes for general fractional p-Laplacian systems. By virtue of this method, we investigate the qualitative properties of positive solutions for the following Schrödinger system with fractional p-Laplacian

$$\begin{cases} (-\Delta)_p^s u + au^{p-1} &= f(u,v), \\ (-\Delta)_p^t v + bv^{p-1} &= g(u,v), \end{cases}$$

where $0 < s, t < 1$ and $2 < p < \infty$. We obtain the radial symmetry in the unit ball or the whole space $\mathbb{R}^N (N \geq 2)$, the monotonicity in the parabolic domain and the nonexistence on the half space for positive solutions to the above system under some suitable conditions on f and g, respectively.

§1 Introduction

In this paper, we are concerned with the Schrödinger system as follows

$$\begin{cases} (-\Delta)_p^s u + au^{p-1} &= f(u,v), \quad \text{in } \Omega\,, \\ (-\Delta)_p^t v + bv^{p-1} &= g(u,v), \quad \text{in } \Omega\,, \\ u > 0, & v > 0, \quad \text{on } \Omega\,, \end{cases} \tag{1.1}$$

where the fractional p-Laplacian $(-\Delta)_p^s$ and $(-\Delta)_p^t$ are the nonlinear nonlocal pseudo differential operators of the types

$$(-\Delta)_p^s u(x) := C_{N,sp} PV \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]}{|x-y|^{N+sp}} dy \tag{1.2}$$

and

$$(-\Delta)_p^t u(x) := C_{N,tp} PV \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]}{|x-y|^{N+tp}} dy. \tag{1.3}$$

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Here, PV stands for the Cauchy principal value, $C_{N,sp}$ and $C_{N,tp}$ are normalization positive constants, $0 < s,\, t < 1$ and $2 < p < \infty$. The coefficients a and b are positive constants when Ω is a unit ball or the whole space. While Ω is the half space or an unbounded parabolic domain defined by

$$\Omega := \left\{ x = (x', x_N) \in \mathbb{R}^N \mid x_N > |x'|^2,\, x' = (x_1, x_2, ..., x_{N-1}) \right\}, \tag{1.4}$$

$a = a(x')$ and $b = b(x')$ are the functions that do not depend on x_N and have lower bounds in Ω. Let

$$\mathcal{L}_{sp} := \{u \in L^{p-1}_{loc} \mid \int_{\mathbb{R}^N} \frac{|1 + u(x)|^{p-1}}{1 + |x|^{N+sp}} dx < \infty\}$$

and

$$\mathcal{L}_{tp} := \{v \in L^{p-1}_{loc} \mid \int_{\mathbb{R}^N} \frac{|1 + v(x)|^{p-1}}{1 + |x|^{N+tp}} dx < \infty\},$$

we assume that

$$u \in C^{1,1}_{loc} \cap \mathcal{L}_{sp} \quad \text{and} \quad v \in C^{1,1}_{loc} \cap \mathcal{L}_{tp},$$

which are necessary to guarantee the integrability of (1.2) and (1.3). Obviously, for $p = 2$ the fractional p-Laplacian coincides with the fractional Laplace operator, which is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes. With respect to $p \neq 2$, the nonlinear and nonlocal fractional p-Laplacian also arises in some important applications such as the non-local "Tug-of-War" game (cf. [1, 2]). In particular, Laskin [14, 15] originally proposed the fractional Schrödinger equation that provides us with a general point of view on the relationship between the statistical properties of the quantum mechanical path and the structure of the fundamental equations of quantum mechanics.

During the last decade the elliptic equations and systems with fractional Laplacian $(-\Delta)^s$ have enjoyed a growing attention. To overcome the difficulty caused by the non-locality of the fractional Laplacian, Caffarelli and Silvestre [4] introduced an extension method to reduce the nonlocal problem into a local one in higher dimensions. This method has been applied successfully to investigate the equations with $(-\Delta)^s$, a great number of related problems have been studied extensively from then on (cf. [3, 11] and the references therein). Another effective method to handle the higher order fractional Laplacian is the method of moving planes in integral forms, which turns a given pseudo differential equations into their equivalent integral equations, we refer [5, 9, 10, 19] for details. However, in some cases, one needs to assume $\frac{1}{2} \le s < 1$ or impose additional integrability conditions on the solutions by using the extension method or the integral equations method. Meanwhile, the aforementioned methods are not applicable to other nonlinear nonlocal operators, such as the fully nonlinear nonlocal operator and fractional p-Laplacian ($p \neq 2$). Recently, Chen et al. [7] developed a direct method of moving planes which can conquer these difficulties. Later a lot of articles have been devoted to the investigation of various equations and systems with fractional Laplacian by virtue of this direct method. Among them, it is worth mentioning some works on generalizing the direct method of moving planes to the fractional Laplacian system (cf. [18]) and the Schrödinger system with fractional Laplacian (cf. [16], [21]).

Afterwards, Chen et al. [8] extended this direct method to consider the following fully nonlinear nonlocal equation

$$F_\alpha\left(u(x)\right) := C_{N,\alpha}\, PV \int_{\mathbb{R}^N} \frac{G\left(u(x)-u(y)\right)}{|x-y|^{N+\alpha}} dy = f\left(x,u\right),$$

where G is a local Lipschitz continuous function, and the operator F_α is non-degenerate in the sense that

$$G'(w) \geq c > 0. \tag{1.5}$$

Note that F_α becomes the fractional Laplacian when $G(\cdot)$ is an identity map.

Indeed, the fractional p-Laplacian we considered in this paper is a particular case of the nonlinear nonlocal operator $F_\alpha(\cdot)$ for

$$\alpha = sp \ \text{ and } G(w) = |w|^{p-2}w,$$

which is degenerate if $p > 2$ or singular if $p < 2$. For simplicity, we will adopt this notation $G(\cdot)$ to denote the fractional p-Laplacian in what follows. In this case, $G'(w) = (p-1)|w|^{p-2} \geq 0$, we have

$$G'(w) \to \begin{cases} 0, & p > 2, \\ \infty, & 1 < p < 2, \end{cases}$$

as $w \to 0$. It indicates that (1.5) is not satisfied for the fractional p-Laplacian. Unfortunately, the methods introduced in either [7] or [8] relies heavily on the non-degeneracy of $G(\cdot)$, hence they cannot be applied directly to the fractional p-Laplacian. That is why there have been only few papers concerning the qualitative properties of the solutions for the fractional p-Laplacian. In this respect, Chen and Li [12] established some new arguments to prove the symmetry and monotonicity of positive solutions for the nonlinear equations with fractional p-Laplacian. After Chen and Liu [12] extended their results to the fractional p-Laplacian system (1.1) with $s = t$ and $a = b = 0$. Very recently, Wu and Niu [22] established a narrow region principle to the equation involving fractional p-Laplacian. In the spirit of [22], Ma and Zhang [20] proved the symmetry of positive solutions for the Choquard equations involving the fractional p-Laplacian.

However, the research on the narrow region principle for the fractional p-Laplacian systems and the qualitative properties of positive solutions for the Schrödinger system (1.1) have not been carried out to our knowledge. The main purpose of this paper is to extend the direct method of moving planes for general fractional p-Laplacian systems by establishing a narrow region principle and a decay at infinity theorem. Then we can apply this method to derive the symmetry, monotonicity and nonexistence of positive solutions to the Schrödinger system involving the fractional p-Laplacian in various domains.

Now we are in position to state our main results of this paper as follows.

Theorem 1.1. *Let $u \in C^{1,1}_{loc}(\mathbb{R}^N) \cap \mathcal{L}_{sp} \cap C(\mathbb{R}^N)$ and $v \in C^{1,1}_{loc}(\mathbb{R}^N) \cap \mathcal{L}_{tp} \cap C(\mathbb{R}^N)$ be a positive solution pair of*

$$\begin{cases} (-\Delta)^s_p u + au^{p-1} &= f(u,v), \quad in \ \ \mathbb{R}^N, \\ (-\Delta)^t_p v + bv^{p-1} &= g(u,v), \quad in \ \ \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $0 < s,\ t < 1$, $2 < p < +\infty$, $a,\ b > 0$ and $f,\ g \in C^1\left((0,+\infty)\times(0,+\infty),\mathbb{R}\right)$. Suppose that

(i) $\frac{\partial f}{\partial v} > 0$ and $\frac{\partial g}{\partial u} > 0$ for $\forall\, u,\ v > 0$;

(ii) $\frac{\partial f}{\partial u} \le u^{m-1}v^n$ *and* $\frac{\partial f}{\partial v} \le u^m v^{n-1}$ *as* $(u,v) \to (0^+,0^+)$;

(iii) $\frac{\partial g}{\partial u} \le u^{q-1}v^r$ *and* $\frac{\partial g}{\partial v} \le u^q v^{r-1}$ *as* $(u,v) \to (0^+,0^+)$;

(iv) $\frac{\partial f}{\partial u} - a(p-1)u^{p-2}$ *is increasing with respect to* u *as* $u \to 0^+$ *and* $\frac{\partial g}{\partial v} - b(p-1)v^{p-2}$ *is increasing with respect to* v *as* $v \to 0^+$;

(v) $u(x) \sim \frac{1}{|x|^\gamma}$ *and* $v(x) \sim \frac{1}{|x|^\tau}$ *as* $|x| \to \infty$,

where m, r, n, $q \ge 1$ *and* γ, $\tau > 0$ *satisfy*

$$\min\{\gamma(m-1)+\tau n,\, \gamma m+\tau(n-1)\} > \gamma(p-2)+sp \tag{1.7}$$

and

$$\min\{\tau(r-1)+\gamma q,\, \tau r+\gamma(q-1)\} > \tau(p-2)+tp. \tag{1.8}$$

Then u *and* v *are radially symmetric and monotone decreasing about some point in* $\mathbb{R}^n$.

Remark 1.2. Due to the presence of the fractional p-Laplacian and a, $b \neq 0$, the Kelvin transform is no longer valid, so we need to impose the additional assumptions on the behavior of u and v at infinity.

Theorem 1.3. *Let* $u \in C^{1,1}_{loc}(B_1(0)) \cap \mathcal{L}_{sp} \cap C(B_1(0))$ *and* $v \in C^{1,1}_{loc}(B_1(0)) \cap \mathcal{L}_{tp} \cap C(B_1(0))$ *be a positive solution pair of*

$$\begin{cases} (-\Delta)^s_p u + au^{p-1} = f(u,v), & x \in B_1(0)\,, \\ (-\Delta)^t_p v + bv^{p-1} = g(u,v), & x \in B_1(0)\,, \\ u = v = 0, & x \notin B_1(0)\,, \end{cases} \tag{1.9}$$

where $0 < s$, $t < 1$, $2 < p < +\infty$ *and* a, $b > 0$. *Suppose that* f, $g \in C^{0,1}([0,+\infty)\times[0,+\infty),\mathbb{R})$ *satisfy*

$$f(u,v_1) < f(u,v_2) \quad \text{for } \forall\, u \ge 0,\ 0 \le v_1 < v_2 \tag{1.10}$$

and

$$g(u_1,v) < g(u_2,v) \quad \text{for } \forall\, v \ge 0,\ 0 \le u_1 < u_2, \tag{1.11}$$

respectively. Then u *and* v *are radially symmetric and monotone decreasing about the origin.*

Theorem 1.4. *Let* $u \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{sp} \cap C(\Omega)$ *and* $v \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{tp} \cap C(\Omega)$ *be a positive solution pair of*

$$\begin{cases} (-\Delta)^s_p u + a(x')u^{p-1} = f(u,v), & x \in \Omega\,, \\ (-\Delta)^t_p v + b(x')v^{p-1} = g(u,v), & x \in \Omega\,, \\ u = v = 0, & x \notin \Omega\,, \end{cases} \tag{1.12}$$

where Ω *is an unbounded parabolic domain given in* (1.4), $0 < s$, $t < 1$, $2 < p < +\infty$ *and* $a(x')$, $b(x')$ *are bounded from below in* Ω. *Meanwhile,* f, $g \in C^{0,1}([0,+\infty)\times[0,+\infty),\mathbb{R})$ *satisfy* (1.10) *and* (1.11). *Then* u *and* v *are strictly increasing with respect to the* x_N*-axis.*

Theorem 1.5. *Let* $u \in C^{1,1}_{loc}(\mathbb{R}^N_+) \cap \mathcal{L}_{sp} \cap C(\mathbb{R}^N_+)$ *and* $v \in C^{1,1}_{loc}(\mathbb{R}^N_+) \cap \mathcal{L}_{tp} \cap C(\mathbb{R}^N_+)$ *be a nonnegative solution pair of*

$$\begin{cases} (-\Delta)^s_p u + a(x')u^{p-1} = f(u,v), & x \in \mathbb{R}^N_+\,, \\ (-\Delta)^t_p v + b(x')v^{p-1} = g(u,v), & x \in \mathbb{R}^N_+\,, \\ u = v = 0, & x \notin \mathbb{R}^N_+\,, \end{cases} \tag{1.13}$$

where $0 < s,\, t < 1,\ 2 < p < +\infty$ *and* $a(x'),\, b(x')$ *are bounded from below in* Ω*. Meanwhile,* $f,\, g \in C^{0,1}\left([0,+\infty)\times[0,+\infty),\mathbb{R}\right)$ *satisfy* (1.10), (1.11) *and*

$$f(0,0) = g(0,0) = 0. \tag{1.14}$$

Suppose that

$$\varliminf_{|x|\to\infty} u(x) = \varliminf_{|x|\to\infty} v(x) = 0, \tag{1.15}$$

then $u(x) = v(x) \equiv 0$ *in* $\mathbb{R}^N$*.*

The remainder of this paper is organized as follows. In section 2 , we establish the narrow region principle and decay at infinity theorem for the general fractional p-Laplacian systems. Section 3 contains the proof of Theorem 1.1 and 1.3 . Moreover, Theorem 1.4 and 1.5 are proved in the last section.

§2 Narrow Region Principle and Decay at Infinity

In this section, we construct the narrow region principle and the decay at infinity theorem for anti-symmetric functions, which play essential roles in carrying on the direct method of moving planes for the fractional p-Laplacian systems.

Before establishing two maximum principles, we first introduce the following notations to facilitate our description. Taking the whole space $\mathbb{R}^N$ as an example. Let

$$T_\lambda := \{x \in \mathbb{R}^N \mid x_1 = \lambda,\ \text{for}\lambda \in \mathbb{R}\}$$

be the moving planes,

$$\Sigma_\lambda := \{x \in \mathbb{R}^N \mid x_1 < \lambda\}$$

be the region to the left of T_λ and

$$x^\lambda := (2\lambda - x_1, x_2, ..., x_N)$$

be the reflection of x with respect to T_λ. Let (u, v) be a solution pair of Schrödinger system (1.6), we denote the reflected functions by $u_\lambda(x) := u(x^\lambda)$ and $v_\lambda(x) := v(x^\lambda)$. Moreover,

$$\begin{cases} U_\lambda(x) & := & u(x^\lambda) - u(x), \\ V_\lambda(x) & := & v(x^\lambda) - v(x), \end{cases}$$

represent the comparison between the values of $u(x)$, $u(x^\lambda)$ and $v(x)$, $v(x^\lambda)$, respectively. Evidently, U_λ and V_λ are anti-symmetric functions, i.e., $U_\lambda(x^\lambda) = -U_\lambda(x)$ and $V_\lambda(x^\lambda) = -V_\lambda(x)$. From now on, C denotes a constant whose value may be different from line to line, and only the relevant dependence is specified in what follows.

Now we start by establishing the following narrow region principle, which generalizes Theorem 1.1 in [22] to the fractional p-Laplacian systems.

Theorem 2.1. ***(Narrow region principle)*** *Let* Ω *be a bounded narrow region in* Σ_λ*, such that it is contained in* $\{x \mid \lambda - \delta < x_1 < \lambda\}$ *with a small* $\delta > 0$*. Assume that* $u \in \mathcal{L}_{sp} \cap C^{1,1}_{loc}(\Sigma_\lambda)$*,*

$v \in \mathcal{L}_{tp} \cap C^{1,1}_{loc}(\Sigma_\lambda)$ and U_λ, V_λ are lower semi-continuous on $\overline{\Omega}$, which satisfy

$$\begin{cases} (-\Delta)_p^s u_\lambda(x) - (-\Delta)_p^s u(x) + C_1(x)U_\lambda(x) + C_2(x)V_\lambda(x) \geq 0, & x \in \Omega\,, \\ (-\Delta)_p^t v_\lambda(x) - (-\Delta)_p^t v(x) + C_3(x)U_\lambda(x) + C_4(x)V_\lambda(x) \geq 0, & x \in \Omega\,, \\ U_\lambda(x) \geq 0,\ V_\lambda(x) \geq 0, & x \in \Sigma_\lambda \backslash \Omega\,, \\ U_\lambda(x^\lambda) = -U_\lambda(x),\ V_\lambda(x^\lambda) = -V_\lambda(x), & x \in \Sigma_\lambda\,, \end{cases} \tag{2.1}$$

where $C_1(x)$, $C_2(x)$, $C_3(x)$ and $C_4(x)$ have lower bounds as C_1, C_2, C_3, $C_4 \in \mathbb{R}$, respectively, and $C_2(x)$, $C_3(x) < 0$ in Ω. If there exist y^0, $y^1 \in \Sigma_\lambda$ such that $U_\lambda(y^0) > 0$ and $V_\lambda(y^1) > 0$, then

$$U_\lambda(x),\ V_\lambda(x) \geq 0, \quad x \in \Omega \tag{2.2}$$

for sufficiently small δ. Moreover, if $U_\lambda(x) = 0$ or $V_\lambda(x) = 0$ at some point in Ω, then

$$U_\lambda(x) = V_\lambda(x) \equiv 0 \quad \textit{almost everywhere in } \mathbb{R}^N. \tag{2.3}$$

The above conclusions are valid for an unbounded narrow region Ω if we further suppose that

$$\varliminf_{|x|\to\infty} U_\lambda(x),\ V_\lambda(x) \geq 0.$$

Remark 2.2. Compared with the narrow region principle for the Schrödinger system with fractional Laplace equations in [21], here we need to impose the extra assumption that there exist y^0, $y^1 \in \Sigma_\lambda$ such that $U_\lambda(y^0) > 0$ and $V_\lambda(y^1) > 0$ to overcome the difficulties caused by the nonlinearity of the fractional p-Laplacian. As a matter of fact, this condition is automatically satisfied for (1.6), (1.9), (1.12) and (1.13), which can be easily seen from the proof of our main results in the later section.

Proof of Theorem 2.1. The proof goes by contradiction. Without loss of generality, we assume that there exists $x^0 \in \Omega$ such that

$$U_\lambda(x^0) = \min_{\Omega} U_\lambda < 0.$$

Otherwise, the same arguments as follows can also yield a contradiction for the case that there exists $x^1 \in \Omega$ such that $V_\lambda(x^1) = \min\limits_{\Omega} V_\lambda < 0$.

By a direct calculation, we obtain

$$\begin{aligned} & (-\Delta)_p^s u_\lambda(x^0) - (-\Delta)_p^s u(x^0) \\ = \;& C_{N,sp}\, PV \int_{\mathbb{R}^N} \frac{G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u(y))}{|x^0 - y|^{N+sp}} dy \\ = \;& C_{N,sp}\, PV \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^0) - u_\lambda(y))}{|x^0 - y|^{N+sp}} + \frac{G(u_\lambda(x^0) - u(y))}{|x^0 - y^\lambda|^{N+sp}} dy \\ & - C_{N,sp}\, PV \int_{\Sigma_\lambda} \frac{G(u(x^0) - u(y))}{|x^0 - y|^{N+sp}} + \frac{G(u(x^0) - u_\lambda(y))}{|x^0 - y^\lambda|^{N+sp}} dy \\ = \;& C_{N,sp} \int_{\Sigma_\lambda} \frac{\left[G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u_\lambda(y))\right]}{|x^0 - y^\lambda|^{N+sp}} dy \\ & + C_{N,sp} \int_{\Sigma_\lambda} \frac{\left[G(u_\lambda(x^0) - u(y)) - G(u(x^0) - u(y))\right]}{|x^0 - y^\lambda|^{N+sp}} dy \\ & + C_{N,sp}\, PV \int_{\Sigma_\lambda} \left[\frac{1}{|x^0 - y|^{N+sp}} - \frac{1}{|x^0 - y^\lambda|^{N+sp}}\right] \end{aligned}$$

$$\begin{aligned} & \times \left[G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u(y))\right] dy \\ := \quad & C_{N,sp} \left(I_1 + I_2\right). \end{aligned} \tag{2.4}$$

We start by estimating I_1. It follows from mean value theorem and the monotonicity of G that

$$I_1 = U_\lambda(x^0) \int_{\Sigma_\lambda} \frac{G'\left(\zeta(y)\right) + G'\left(\eta(y)\right)}{|x^0 - y^\lambda|^{N+sp}} dy \leq 0, \tag{2.5}$$

where $\zeta(y) \in \left(u_\lambda(x^0) - u_\lambda(y),\, u(x^0) - u_\lambda(y)\right)$ and $\eta(y) \in \left(u_\lambda(x^0) - u(y),\, u(x^0) - u(y)\right)$.

Now we turn our attention to I_2. Let $\delta_{x^0} := \operatorname{dist}\left\{x^0,\, T_\lambda\right\}$, it is not difficult to verify that $\delta_{x^0} = \lambda - x_1^0$. Then applying mean value theorem again, we compute

$$\frac{1}{|x^0 - y|^{n+sp}} - \frac{1}{|x^0 - y^\lambda|^{N+sp}} = \frac{2\left(N + sp\right)\left(\lambda - y_1\right)}{|x^0 - \varsigma|^{N+sp+2}} \delta_{x^0}, \tag{2.6}$$

where ς is a point on the line segment between y and y^λ. Thus,

$$\begin{aligned} I_2 \quad = \quad & \delta_{x^0} \int_{\Sigma_\lambda} \frac{2\left(N + sp\right)\left(\lambda - y_1\right)}{|x^0 - \varsigma|^{N+sp+2}} \left[G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u(y))\right] dy \\ := \quad & \delta_{x^0} F(x^0). \end{aligned} \tag{2.7}$$

Before estimating further, we claim that there exists a positive constant c_1 such that

$$F(x^0) \leq -\frac{c_1}{2} \tag{2.8}$$

for sufficiently small δ_{x^0}. In doing so, we first show that

$$F(x^0) < 0. \tag{2.9}$$

Applying the monotonicity of G, we derive

$$G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u(y)) \leq 0,$$

which is not identically zero in Σ_λ. Hence, we conclude (2.9) by virtue of the continuity of u and

$$\frac{2\left(N + sp\right)\left(\lambda - y_1\right)}{|x^0 - \varsigma|^{N+sp+2}} = \frac{1}{\delta_{x^0}} \left[\frac{1}{|x^0 - y|^{N+sp}} - \frac{1}{|x^0 - y^\lambda|^{N+sp}}\right] > 0.$$

Next, we continue to prove (2.8). If not, then

$$F(x^0) \to 0 \;\text{ as }\; \delta_{x^0} \to 0.$$

It is revealed that if $\delta_{x^0} \to 0$, then

$$G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u(y)) \to 0 \;\text{ for }\; \forall\, y \in \Sigma_\lambda.$$

Utilizing the monotonicity of G and the continuity of u again, we obtain

$$U_\lambda(x^0) - U_\lambda(y) \to 0 \;\text{ for }\; \forall\, y \in \Sigma_\lambda.$$

Note that $U_\lambda(x^0) \to 0$ as $\delta_{x^0} \to 0$, then we derive

$$U_\lambda(y) \equiv 0 \;\text{ for }\; \forall\, y \in \Sigma_\lambda,$$

which contradicts with the condition that there exists $y^0 \in \Sigma_\lambda$ such that $U_\lambda(y^0) > 0$. Thus, we can deduce there exists a positive constant c_2 such that

$$F(x^0) \to -c_2 \;\text{ as }\; \delta_{x^0} \to 0.$$

Hence, we conclude the assertion (2.8) from the continuity of $F(x^0)$ with respect to x^0.

Inserting (2.8) into (2.7), we obtain

$$I_2 \leq -\frac{c_2}{2}\delta_{x^0}. \tag{2.10}$$

Then a combination of (2.4), (2.5) and (2.10) yields that

$$(-\Delta)_p^s u_\lambda(x^0) - (-\Delta)_p^s u(x^0) \leq -C\delta_{x^0}. \tag{2.11}$$

Thus, applying the first inequality in (2.1) and $C_1(x) \geq C_1$, we derive

$$\begin{aligned} -C_2(x^0)V_\lambda(x^0) &\leq -C\delta_{x^0} + C_1(x^0)U_\lambda(x^0) \\ &\leq -C\delta_{x^0} + C_1 U_\lambda(x^0). \end{aligned} \tag{2.12}$$

Note that since

$$\nabla U_\lambda(x^0) = 0,$$

we get

$$0 = U_\lambda(x^2) = U_\lambda(x^0) + \nabla U_\lambda(x^0)\left(x^2 - x^0\right) + o\left(|x^2 - x^0|\right)$$

by Taylor expansion, where $x^2 = \left(\lambda, x_2^0, ..., x_N^0\right) \in T_\lambda$. Hence, it means that

$$U_\lambda(x^0) = o(1)\delta_{x^0} \tag{2.13}$$

for sufficiently small δ_{x^0}. Substituting (2.13) into (2.12), we have

$$-C_2(x^0)V_\lambda(x^0) \leq \delta_{x^0}\left(-C + C_1 o(1)\right) < 0$$

for small enough δ_{x^0}. Then it follows from $C_2(x) < 0$ that $V_\lambda(x^0) < 0$. Hence, the lower semi-continuity of V_λ on $\overline{\Omega}$ implies there exists $x^1 \in \Omega$ such that

$$V_\lambda(x^1) = \min_{\Omega} V_\lambda < 0.$$

In analogy with (2.11) and (2.13), we can deduce

$$(-\Delta)_p^t v_\lambda(x^1) - (-\Delta)_p^t v(x^1) \leq -C\delta_{x^1} \tag{2.14}$$

and

$$V_\lambda(x^1) = o(1)\delta_{x^1} \tag{2.15}$$

for sufficiently small δ_{x^1}, respectively, where $\delta_{x^1} := \text{dist}\left\{x^1, T_\lambda\right\} = \lambda - x_1^1$.

In terms of the assumptions imposed on $C_3(x)$ and $C_4(x)$ in Theorem 2.1 , and combining the second inequality in (2.1), (2.14), (2.13) with (2.15), we can conclude that

$$\begin{aligned} 0 &\leq (-\Delta)_p^t v_\lambda(x^1) - (-\Delta)_p^t v(x^1) + C_3(x^1)U_\lambda(x^1) + C_4(x^1)V_\lambda(x^1) \\ &\leq -C\delta_{x^1} + C_3 U_\lambda(x^0) + C_4 V_\lambda(x^1) \\ &= -C\delta_{x^1} + C_3\, o(1)\delta_{x^0} + C_4\, o(1)\delta_{x^1} < 0 \end{aligned}$$

for sufficiently small δ, which deduces a contradiction. Thus, (2.2) is proved.

Subsequently, in order to prove (2.3), we assume that there exists a point $\widetilde{x} \in \Omega$ such that

$$U_\lambda(\widetilde{x}) = \min_{\Sigma_\lambda} U_\lambda = 0.$$

Now we claim that

$$U_\lambda(x) \equiv 0, \text{ a.e. in } \Sigma_\lambda. \tag{2.16}$$

If not, then

$$(-\Delta)_p^s u_\lambda(\widetilde{x}) - (-\Delta)_p^s u(\widetilde{x})$$

$$
\begin{aligned}
&= C_{N,sp}\, PV \int_{\mathbb{R}^N} \frac{G(u_\lambda(\widetilde{x}) - u_\lambda(y)) - G(u(\widetilde{x}) - u(y))}{|\widetilde{x} - y|^{N+sp}} dy \\
&= C_{N,sp}\, PV \int_{\mathbb{R}^N} \frac{G(u(\widetilde{x}) - u_\lambda(y)) - G(u(\widetilde{x}) - u(y))}{|\widetilde{x} - y|^{N+sp}} dy \\
&= C_{N,sp}\, PV \int_{\Sigma_\lambda} \left[\frac{1}{|\widetilde{x} - y|^{N+sp}} - \frac{1}{|\widetilde{x} - y^\lambda|^{N+sp}}\right] [G(u(\widetilde{x}) - u_\lambda(y)) - G(u(\widetilde{x}) - u(y))]\, dy \\
&< 0. \qquad (2.17)
\end{aligned}
$$

Combining the above inequality with (2.1) and $C_2(x) < 0$, we derive

$$V_\lambda(\widetilde{x}) < 0,$$

which is contradictive with (2.2). Thus, it follows from (2.16) and the anti-symmetry of $U_\lambda(x)$ that

$$U_\lambda(x) \equiv 0 \quad \text{a.e. in } \mathbb{R}^N. \qquad (2.18)$$

Applying (2.18), (2.1) and (2.2), we obtain

$$V_\lambda(x) \equiv 0 \quad \text{ in } \Omega.$$

It remains to be proved $V_\lambda(x) \equiv 0$ for almost everywhere $x \in \Sigma_\lambda \backslash \Omega$. If not, the same argument as (2.17) deduces that

$$(-\Delta)_p^t v_\lambda(x) - (-\Delta)_p^t v(x) < 0$$

for $x \in \Omega$, which is contradictive with the second inequality in (2.1). A combination of $V_\lambda(x) \equiv 0$ for almost everywhere $x \in \Sigma_\lambda$ and the anti-symmetry of $V_\lambda(x)$ yields that

$$V_\lambda(x) \equiv 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Similarly, one can show that if $V_\lambda(x) = 0$ at some point in Ω, then both $U_\lambda(x)$ and $V_\lambda(x)$ are identically zero almost everywhere in $\mathbb{R}^N$.

For the unbounded narrow region Ω, the condition

$$\varliminf_{|x| \to \infty} U_\lambda(x),\, V_\lambda(x) \geq 0$$

guarantees that the negative minimum of U_λ and V_λ must be attained at some point x^0 and x^1, respectively, then we can derive the similar contradictions as above.

This completes the proof of Theorem 2.1 . □

Furthermore, in order to carry on the direct method of moving planes in $\mathbb{R}^N$, we also need to construct the decay at infinity theorem. We proceed by introducing the following useful technical lemma.

Lemma 2.3. (cf. [12]) *For $G(w) = |w|^{p-2}w$, it follows from mean value theorem that*

$$G(w_2) - G(w_1) = G'(\zeta)(w_2 - w_1).$$

Then there exists a positive constant c_0 such that

$$|\zeta| \geq c_0 \max\{|w_1|, |w_2|\}. \qquad (2.19)$$

Now we turn to establish the decay at infinity theorem for the fractional p-Laplacian systems, which is important for the proof of Theorem 1.1 .

Theorem 2.4. ***(Decay at infinity)*** *Let Ω be an unbounded region in Σ_λ. Assume that $u \in \mathcal{L}_{sp} \cap C^{1,1}_{loc}(\Omega)$ and $v \in \mathcal{L}_{tp} \cap C^{1,1}_{loc}(\Omega)$, U_λ, V_λ satisfy (v) in Theorem 1.1 and*

$$\begin{cases} (-\Delta)^s_p u_\lambda(x) - (-\Delta)^s_p u(x) + C_1(x)U_\lambda(x) + C_2(x)V_\lambda(x) \geq 0, & x \in \Omega\,, \\ (-\Delta)^t_p v_\lambda(x) - (-\Delta)^t_p v(x) + C_3(x)U_\lambda(x) + C_4(x)V_\lambda(x) \geq 0, & x \in \Omega\,, \\ U_\lambda(x) \geq 0,\, V_\lambda(x) \geq 0, & x \in \Sigma_\lambda \backslash \Omega\,, \\ U_\lambda(x^\lambda) = -U_\lambda(x),\, V_\lambda(x^\lambda) = -V_\lambda(x), & x \in \Sigma_\lambda\,, \end{cases} \tag{2.20}$$

where $C_1(x)$, $C_4(x) \geq 0$ and $C_2(x)$, $C_3(x) < 0$ on Ω such that

$$\varliminf_{|x|\to\infty} C_2(x)|x|^{\gamma(p-2)+sp} = 0, \ \varliminf_{|x|\to\infty} C_3(x)|x|^{\tau(p-2)+tp} = 0, \tag{2.21}$$

where γ and τ given in Theorem 1.1. Then there exists a positive constant R_0 such that if

$$U_\lambda(x^0) = \min_\Omega U_\lambda < 0, \quad V_\lambda(x^1) = \min_\Omega V_\lambda < 0, \tag{2.22}$$

then at least one of x^0 and x^1 satisfies

$$|x| \leq R_0. \tag{2.23}$$

Proof. The proof is carried out by contradiction. If (2.23) is violated, then by the monotonicity of G and mean value theorem, we can compute

$$\begin{aligned} & (-\Delta)^s_p u_\lambda(x^0) - (-\Delta)^s_p u(x^0) \\ = \ & C_{N,sp}\, PV \int_{\Sigma_\lambda} \left[\frac{1}{|x^0 - y|^{N+sp}} - \frac{1}{|x^0 - y^\lambda|^{N+sp}} \right] \\ & \qquad \times \left[G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u(y)) \right] dy \\ & + C_{N,sp} \int_{\Sigma_\lambda} \frac{\left[G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u_\lambda(y)) \right]}{|x^0 - y^\lambda|^{N+sp}} dy \\ & + C_{N,sp} \int_{\Sigma_\lambda} \frac{\left[G(u_\lambda(x^0) - u(y)) - G(u(x^0) - u(y)) \right]}{|x^0 - y^\lambda|^{N+sp}} dy \\ \leq \ & C_{N,sp} \int_{\Sigma_\lambda} \frac{\left[G(u_\lambda(x^0) - u_\lambda(y)) - G(u(x^0) - u_\lambda(y)) \right]}{|x^0 - y^\lambda|^{N+sp}} dy \\ & + C_{N,sp} \int_{\Sigma_\lambda} \frac{\left[G(u_\lambda(x^0) - u(y)) - G(u(x^0) - u(y)) \right]}{|x^0 - y^\lambda|^{N+sp}} dy \\ = \ & C_{N,sp}\, U_\lambda(x^0) \int_{\Sigma_\lambda} \frac{G'\left(\zeta(y)\right) + G'\left(\eta(y)\right)}{|x^0 - y^\lambda|^{N+sp}} dy, \end{aligned} \tag{2.24}$$

where $\zeta(y) \in \big(u_\lambda(x^0) - u_\lambda(y),\, u(x^0) - u_\lambda(y)\big)$ and $\eta(y) \in \big(u_\lambda(x^0) - u(y),\, u(x^0) - u(y)\big)$. Let $R = |x^0|$ and $x^\lambda_R = (x^0_1 + (M+1)|x^0|, x^0_2, ..., x^0_N)$, then $|x^\lambda_R| \geq MR$. Here M is a sufficiently large number such that

$$B_R(x_R) \subset \Sigma_\lambda \,\text{and}\, B_R(x^\lambda_R) \subset \Sigma^C_\lambda$$

for fixed λ. Moreover, the M guarantees that

$$u(y) \leq \frac{C}{M^\gamma R^\gamma} \leq \frac{c}{R^\gamma} \leq u(x^0) \tag{2.25}$$

for any $y \in B_R(x^\lambda_R)$ by (v) in Theorem 1.1. Hence, a combination of (2.24), (2.25) and Lemma 2.3 yields that

$$\begin{aligned}
& C_{N,sp}\, U_\lambda(x^0) \int_{\Sigma_\lambda} \frac{G'\left(\zeta(y)\right) + G'\left(\eta(y)\right)}{|x^0 - y^\lambda|^{N+sp}} dy \\
\leq\ & C_{N,sp}\, U_\lambda(x^0) \int_{B_R(x_R)} \frac{G'\left(\zeta(y)\right)}{|x^0 - y^\lambda|^{N+sp}} dy \\
=\ & C_{N,sp}\, (p-1) U_\lambda(x^0) \int_{B_R(x_R)} \frac{|\zeta(y)|^{p-2}}{|x^0 - y^\lambda|^{N+sp}} dy \\
\leq\ & C_{N,sp}\, c_0^{p-2} (p-1) U_\lambda(x^0) \int_{B_R(x_R)} \frac{\left|u(x^0) - u_\lambda(y)\right|^{p-2}}{|x^0 - y^\lambda|^{N+sp}} dy \\
\leq\ & C_{N,sp}\, c_0^{p-2} (p-1) U_\lambda(x^0) \int_{B_R(x_R^\lambda)} \frac{\left|u(x^0) - \frac{C}{M^\gamma c} u(x^0)\right|^{p-2}}{|x^0 - y|^{N+sp}} dy \\
\leq\ & C\, U_\lambda(x^0) \int_{B_R(x_R^\lambda)} \frac{u^{p-2}(x^0)}{|x^0 - y|^{N+sp}} dy \\
\leq\ & C\, \frac{U_\lambda(x^0)}{R^{\gamma(p-2)+sp}}.
\end{aligned}$$

That is to say,

$$(-\Delta)_p^s u_\lambda(x^0) - (-\Delta)_p^s u(x^0) \leq C\, \frac{U_\lambda(x^0)}{|x^0|^{\gamma(p-2)+sp}}. \tag{2.26}$$

Applying the first inequality in (2.20) and $C_1(x) \geq 0$, we derive

$$U_\lambda(x^0) \geq -C C_2(x^0) |x^0|^{\gamma(p-2)+sp} V_\lambda(x^0). \tag{2.27}$$

Then it follows from $C_2(x) < 0$ that

$$V_\lambda(x^0) < 0. \tag{2.28}$$

In terms of (v) in Theorem 1.1 , (2.28) and the lower semi-continuity of V_λ on $\overline{\Omega}$, we can show that there exists $x^1 \in \Omega$ such that

$$V_\lambda(x^1) = \min_{\Omega} V_\lambda < 0$$

for sufficiently large $|x^1|$. By proceeding similarly as (2.26), we have

$$(-\Delta)_p^t v_\lambda(x^1) - (-\Delta)_p^t v(x^1) \leq C\, \frac{V_\lambda(x^1)}{|x^1|^{\tau(p-2)+tp}}. \tag{2.29}$$

Finally, utilizing the second inequality in (2.20), (2.29), $C_2(x)$, $C_3(x) < 0$, $C_4(x) \geq 0$, (2.27) and (2.21), we can conclude a contradiction as follows

$$\begin{aligned}
0 \ \leq\ & (-\Delta)_p^t\, v_\lambda(x^1) - (-\Delta)_p^t\, v(x^1) + C_3(x^1) U_\lambda(x^1) + C_4(x^1) V_\lambda(x^1) \\
\leq\ & C\, \frac{V_\lambda(x^1)}{|x^1|^{\tau(p-2)+tp}} + C_3(x^1) U_\lambda(x^0) \\
\leq\ & C\, \frac{V_\lambda(x^1)}{|x^1|^{\tau(p-2)+tp}} - C C_3(x^1) C_2(x^0) |x^0|^{\gamma(p-2)+sp} V_\lambda(x^1) \\
=\ & \frac{V_\lambda(x^1)}{|x^1|^{\tau(p-2)+tp}} \Big(C - C C_2(x^0) |x^0|^{\gamma(p-2)+sp} C_3(x^1) |x^1|^{\tau(p-2)+tp} \Big) \\
<\ & 0
\end{aligned}$$

for sufficiently large $|x^0|$ and $|x^1|$. Hence, the relation (2.23) must be valid for at least one of x^0 and x^1. The proof of Theorem 2.4 is completed. □

Remark 2.5. We believe that Theorem 2.1 , 2.4 and the arguments behind the proof will be applied to other nonlinear nonlocal systems with fractional p-Laplacian.

§3 Radial Symmetry of Positive Solutions

In this section, we establish the radial symmetry of positive solutions to (1.1) in the whole space and the unit ball (i.e, Theorem 1.1 and 1.3) based on the direct method of moving planes. We start by proving Theorem 1.1 .

Proof of Theorem 1.1. Choosing a direction to be x_1-direction and keeping the notations T_λ, Σ_λ, x_λ, U_λ and V_λ defined in Section 2 , we divide the proof into two steps.

Step 1. Start moving the plane T_λ from $-\infty$ to the right along the x_1-axis. We first argue that the assertion

$$U_\lambda(x),\, V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \tag{3.1}$$

is true for sufficiently negative λ.

If (3.1) is violated, without loss of generality, we assume that there exists an $x^0 \in \Sigma_\lambda$ such that

$$U_\lambda(x^0) = \min_{\Sigma_\lambda} U_\lambda < 0.$$

By proceeding similarly as (2.26), we get

$$(-\Delta)_p^s u_\lambda(x^0) - (-\Delta)_p^s u(x^0) \leq C\, \frac{U_\lambda(x^0)}{|x^0|^{\gamma(p-2)+sp}}. \tag{3.2}$$

Now we show that

$$V_\lambda(x^0) < 0. \tag{3.3}$$

If not, applying the assumptions $a > 0$, (i), (ii), (iv), (v) in Theorem 1.1 and combining with mean value theorem, we obtain

$$\begin{aligned}
& (-\Delta)_p^s u_\lambda(x^0) - (-\Delta)_p^s u(x^0) \\
= \ & f\left(u_\lambda(x^0), v_\lambda(x^0)\right) - a u_\lambda^{p-1}(x^0) - \left(f\left(u(x^0), v(x^0)\right) - a u^{p-1}(x^0)\right) \\
= \ & \left(\frac{\partial f}{\partial u}(\xi_1, v(x^0)) - a(p-1)\xi_1^{p-2}\right) U_\lambda(x^0) + \left(f\left(u_\lambda(x^0), v_\lambda(x^0)\right) - f\left(u_\lambda(x^0), v(x^0)\right)\right) \\
\geq \ & \left(\frac{\partial f}{\partial u}(u(x^0), v(x^0)) - a(p-1)u^{p-2}(x^0)\right) U_\lambda(x^0) \\
\geq \ & u^{m-1}(x^0) v^n(x^0) U_\lambda(x^0) \\
\geq \ & \frac{C}{|x^0|^{\gamma(m-1)+\tau n}} U_\lambda(x^0)
\end{aligned} \tag{3.4}$$

for sufficiently negative λ, where $\xi_1 \in \left(u_\lambda(x^0),\, u(x^0)\right)$. Note that (3.4) contradicts with (3.2), which is ensured by $\gamma(m-1) + \tau n > \gamma(p-2) + sp$. Thus, (3.3) holds. In terms of (3.3) and (v), we can conclude there exists an $x^1 \in \Sigma_\lambda$ such that

$$V_\lambda(x^1) = \min_{\Sigma_\lambda} V_\lambda < 0.$$

In analogy with the above argument, then (i), (iii), (iv), (v) and $\tau(r-1) + \gamma q > \tau(p-2) + tp$ are necessary to guarantee the validity of $U_\lambda(x^1) < 0$.

Thus, in terms of the above estimates, mean value theorem, (ii), (iii) and (iv), we derive

$$\begin{aligned}
&(-\Delta)_p^s u_\lambda(x^0) - (-\Delta)_p^s u(x^0) \\
= \;& \left(\frac{\partial f}{\partial u}(\xi_1, v(x^0)) - a(p-1)\xi_1^{p-2}\right) U_\lambda(x^0) + \frac{\partial f}{\partial v}(u_\lambda(x^0), \eta_1) V_\lambda(x^0) \\
\geq \;& \left(u^{m-1}(x^0)v^n(x^0) - a(p-1)u^{p-2}(x^0)\right) U_\lambda(x^0) + u^m(x^0)v^{n-1}(x^0)V_\lambda(x^0),
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&(-\Delta)_p^t v_\lambda(x^1) - (-\Delta)_p^t v(x^1) \\
= \;& \frac{\partial g}{\partial u}(\xi_2, v_\lambda(x^1)) U_\lambda(x^1) + \left(\frac{\partial g}{\partial v}(u(x^1), \eta_2) - b(p-1)\eta_2^{p-2}\right) V_\lambda(x^1) \\
\geq \;& u^{q-1}(x^1)v^r(x^1)U_\lambda(x^1) + \left(u^q(x^1)v^{r-1}(x^1) - b(p-1)v^{p-2}(x^1)\right) V_\lambda(x^1),
\end{aligned} \tag{3.6}$$

where $\xi_1 \in \big(u_\lambda(x^0),\, u(x^0)\big)$, $\eta_1 \in \big(v_\lambda(x^0),\, v(x^0)\big)$, $\xi_2 \in \big(u_\lambda(x^1),\, u(x^1)\big)$ and $\eta_2 \in \big(v_\lambda(x^1),\, v(x^1)\big)$, respectively. Let

$$\begin{aligned}
C_1(x^0) &= a(p-1)u^{p-2}(x^0) - u^{m-1}(x^0)v^n(x^0) \\
&\sim \frac{a(p-1)}{|x^0|^{\gamma(p-2)}} - \frac{1}{|x^0|^{\gamma(m-1)+\tau n}} \geq 0, \\
0 > C_2(x^0) &= -u^m(x^0)v^{n-1}(x^0) \\
&\sim \frac{1}{|x^0|^{\gamma m + \tau(n-1)}}, \\
0 > C_3(x^1) &= -u^{q-1}(x^1)v^r(x^1) \\
&\sim \frac{1}{|x^1|^{\gamma(q-1)+\tau r}},
\end{aligned}$$

and

$$\begin{aligned}
C_4(x^1) &= b(p-1)v^{p-2}(x^1) - u^q(x^1)v^{r-1}(x^1) \\
&\sim \frac{b(p-1)}{|x^1|^{\tau(p-2)}} - \frac{1}{|x^1|^{\gamma q + \tau(r-1)}} \geq 0,
\end{aligned}$$

for sufficiently negative λ, which are ensured by a, $b > 0$, $p > 2$, (v), (1.7) and (1.8). Moreover, we have

$$\varliminf_{|x^0|\to\infty} C_2(x^0)|x^0|^{\gamma(p-2)+sp} = 0 \text{ and } \varliminf_{|x^1|\to\infty} C_3(x^1)|x^1|^{\tau(p-2)+tp} = 0.$$

Then by virtue of the proof of Theorem 2.4, (1.7) and (1.8), it implies that one of $U_\lambda(x)$ and $V_\lambda(x)$ must be nonnegative in Σ_λ for sufficiently negative λ. Without loss of generality, we can suppose that

$$U_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.7}$$

To show that (3.7) also holds for $V_\lambda(x)$, we argue by contradiction again. If $V_\lambda(x)$ is negative at some point in Σ_λ, then (v) guarantees there exists an $x^1 \in \Sigma_\lambda$ such that

$$V_\lambda(x^1) = \min_{\Sigma_\lambda} V_\lambda < 0.$$

From (2.29) and the similar argument as (3.4), we derive

$$\frac{CV_\lambda(x^1)}{|x^1|^{\tau(p-2)+tp}} \geq (-\Delta)_p^t v_\lambda(x^1) - (-\Delta)_p^t v(x^1) \geq \frac{CV_\lambda(x^1)}{|x^1|^{\gamma q + \tau(r-1)}},$$

then $\tau(p-2) + tp < \gamma q + \tau(r-1)$ deduces a contradiction for sufficiently negative λ. Hence, (3.1) is true, which provides a starting point to move the plane T_λ.

Step 2. Continue to move the plane T_λ to the right along the x_1-axis as long as (3.1) holds to its limiting position . More precisely, let

$$\lambda_0 := \sup\{\lambda \mid U_\mu(x) \geq 0,\ V_\mu(x) \geq 0,\ x \in \Sigma_\mu,\ \mu \leq \lambda\},$$

then the behavior of u and v at infinity guarantee $\lambda_0 < \infty$.

Next, we claim that u is symmetric about the limiting plane T_{λ_0}, that is to say

$$U_{\lambda_0}(x) = V_{\lambda_0}(x) \equiv 0, \quad x \in \mathbb{R}^N. \tag{3.8}$$

By the definition of λ_0, we first show that either

$$U_{\lambda_0}(x) = V_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0},$$

or

$$U_{\lambda_0}(x),\ V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}.$$

To prove this, without loss of generality, we assume there a point $\widetilde{x} \in \Sigma_{\lambda_0}$ such that

$$U_{\lambda_0}(\widetilde{x}) = \min_{\Sigma_{\lambda_0}} U_{\lambda_0} = 0,$$

then it must be revealed that

$$U_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}.$$

If not, on one hand

$$\begin{aligned}
&(-\Delta)_p^s u_{\lambda_0}(\widetilde{x}) - (-\Delta)_p^s u(\widetilde{x}) \\
= \;& C_{N,sp}\, PV \int_{\mathbb{R}^N} \frac{G(u_{\lambda_0}(\widetilde{x}) - u_{\lambda_0}(y)) - G(u(\widetilde{x}) - u(y))}{|\widetilde{x} - y|^{N+sp}} dy \\
= \;& C_{N,sp}\, PV \int_{\mathbb{R}^N} \frac{G(u(\widetilde{x}) - u_{\lambda_0}(y)) - G(u(\widetilde{x}) - u(y))}{|\widetilde{x} - y|^{N+sp}} dy \\
= \;& C_{N,sp}\, PV \int_{\Sigma_{\lambda_0}} \left[\frac{1}{|\widetilde{x} - y|^{N+sp}} - \frac{1}{|\widetilde{x} - y^{\lambda_0}|^{N+sp}}\right] [G(u(\widetilde{x}) - u_{\lambda_0}(y)) - G(u(\widetilde{x}) - u(y))]\, dy \\
< \;& 0.
\end{aligned}$$

On the other hand,

$$(-\Delta)_p^s u_{\lambda_0}(\widetilde{x}) - (-\Delta)_p^s u(\widetilde{x}) = \frac{\partial f}{\partial v}(u_{\lambda_0}(\widetilde{x}), \eta_1) V_{\lambda_0}(\widetilde{x}) \geq 0,$$

which deduces a contradiction. Then it follows from the anti-symmetry of U_λ that

$$U_{\lambda_0}(x) \equiv 0, \quad x \in \mathbb{R}^n,$$

which can deduce $V_{\lambda_0}(\widetilde{x}) = 0$. In analogy with the above estimates, we can also derive

$$V_{\lambda_0}(x) \equiv 0, \quad x \in \mathbb{R}^n.$$

Therefore, if (3.8) is violated, then we only have the case that

$$U_{\lambda_0}(x),\ V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}. \tag{3.9}$$

In the sequel, we prove that the plane can still move further in this case. To be more rigorous, there exists $\varepsilon > 0$ such that

$$U_\lambda(x),\ V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \tag{3.10}$$

for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$. This is a contradiction with the definition of λ_0, then (3.8) holds.

Now we prove the assertion (3.10). From (3.9), we have the following bounded away from zero estimate

$$U_{\lambda_0}(x),\, V_{\lambda_0}(x) \geq C_\delta > 0, \quad x \in \Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)$$

for some $R_0 > 0$. By the continuity of $U_\lambda(x)$ and $V_\lambda(x)$ with respect to λ, there exists a positive constant ε such that

$$U_\lambda(x),\, V_\lambda(x) \geq 0, \quad x \in \Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)$$

for any $\lambda \in (\lambda_0, \lambda_0+\varepsilon)$. Moreover, by virtue of Theorem 2.4 , we know that if

$$U_\lambda(x^0) = \min_{\Sigma_\lambda} U_\lambda < 0 \quad \text{and} \quad V_\lambda(x^1) = \min_{\Sigma_\lambda} V_\lambda < 0,$$

then there exists a positive constant R_0 large enough such that one of x^0 and x^1 must be in $B_{R_0}(0)$. We may as well suppose $|x^0| < R_0$. Thus, we obtain

$$x^0 \in (\Sigma_\lambda \backslash \Sigma_{\lambda_0-\delta}) \cap B_{R_0}(0). \tag{3.11}$$

Next, we show that (3.11) also holds for x^1. If $x^1 \in \Sigma_\lambda \cap B^c_{R_0}(0)$, then by virtue of (2.29), (3.5), (3.6), (i), (iii), (iv), (v), $b>0$, $p>2$, $r \geq p-1$ and (2.11), we have

$$\begin{aligned}
\frac{CV_\lambda(x^1)}{|x^1|^{\tau(p-2)+tp}} &\geq (-\Delta)^t_p v_\lambda(x^1) - (-\Delta)^t_p v(x^1) \\
&= \frac{\partial g}{\partial u}(\xi_2, v_\lambda(x^1)) U_\lambda(x^1) + \left(\frac{\partial g}{\partial v}(u(x^1), \eta_2) - b(p-1)\eta_2^{p-2}\right) V_\lambda(x^1) \\
&\geq \frac{\partial g}{\partial u}(\xi_2, v_\lambda(x^1)) U_\lambda(x^0) + \left(u^q(x^1) v^{r-1}(x^1) - b(p-1) v^{p-2}(x^1)\right) V_\lambda(x^1) \\
&\geq u^{q-1}(x^1) v^r(x^1) U_\lambda(x^0) + \left(\frac{C}{|x^1|^{\gamma q + \tau(r-1)}} - \frac{Cb(p-1)}{|x^1|^{(p-2)\tau}}\right) V_\lambda(x^1) \\
&\geq \frac{C}{|x^1|^{\gamma(q-1)+\tau r}} U_\lambda(x^0)
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
-C\delta_{x^0} &\geq (-\Delta)^s_p u_\lambda(x^0) - (-\Delta)^s_p u(x^0) \\
&= \left(\frac{\partial f}{\partial u}(\xi_1, v(x^0)) - a(p-1)\xi_1^{p-2}\right) U_\lambda(x^0) + \frac{\partial f}{\partial v}(u_\lambda(x^0), \eta) V_\lambda(x^0)
\end{aligned} \tag{3.13}$$

for sufficiently small δ and ε and large R_0, where $\xi_1 \in \big(u_\lambda(x^0),\, u(x^0)\big)$, $\xi_2 \in \big(u_\lambda(x^1),\, u(x^1)\big)$ and $\eta \in \big(v_\lambda(x^0), v(x^0)\big)$. Hence, utilizing the above inequalities, (i) and (2.13), we derive

$$\begin{aligned}
1 &\leq -\frac{C}{\delta_{x^0}} \left[\left(\frac{\partial f}{\partial u}(\xi_1, v(x^0)) - a(p-1)\xi_1^{p-2}\right) U_\lambda(x^0) + \frac{\partial f}{\partial v}(u_\lambda(x^0), \eta) V_\lambda(x^1)\right] \\
&\leq -\frac{C}{\delta_{x^0}} \left[\left(\frac{\partial f}{\partial u}(\xi_1, v(x^0)) - a(p-1)\xi_1^{p-2}\right) U_\lambda(x^0) + C\frac{|x^1|^{\tau(p-2)+tp}}{|x^1|^{\gamma(q-1)+\tau r}} \frac{\partial f}{\partial v}(u_\lambda(x^0), \eta) U_\lambda(x^0)\right] \\
&= -\frac{C}{\delta_{x^0}} U_\lambda(x^0) \left[\left(\frac{\partial f}{\partial u}(\xi_1, v(x^0)) - a(p-1)\xi_1^{p-2}\right) + C\frac{\partial f}{\partial v}(u_\lambda(x^0), \eta) \frac{|x^1|^{\tau(p-2)+tp}}{|x^1|^{\gamma(q-1)+\tau r}}\right] \\
&\leq C\, o(1) \left[\left(\frac{\partial f}{\partial u}(\xi_1, v(x^0)) - a(p-1)\xi_1^{p-2}\right) + \frac{\partial f}{\partial v}(u_\lambda(x^0), \eta) \frac{|x^1|^{\tau(p-2)+tp}}{|x^1|^{\gamma(q-1)+\tau r}}\right]
\end{aligned} \tag{3.14}$$

for sufficiently small δ, ε and large R_0. Note that

$$\left(\frac{\partial f}{\partial u}(\xi_1, v(x^0)) - a(p-1)\xi_1^{p-2}\right) + \frac{\partial f}{\partial v}(u_\lambda(x^0), \eta) \frac{|x^1|^{\tau(p-2)+tp}}{|x^1|^{\gamma(q-1)+\tau r}}$$

is bounded, which is ensured by $\tau r + \gamma(q-1) > \tau(p-2) + tp$, $|x^1| > R_0$, $x^0 \in (\Sigma_\lambda \backslash \Sigma_{\lambda_0-\delta}) \cap B_{R_0}(0)$

and $f \in C^1$. Hence, (3.14) must not be valid for sufficiently small δ and ε. This contradiction deduces that

$$x^1 \in (\Sigma_\lambda \backslash \Sigma_{\lambda_0-\delta}) \cap B_{R_0}(0).$$

In terms of Theorem 2.1 , we can conclude that

$$U_\lambda(x),\, V_\lambda(x) \geq 0, \;\; x \in (\Sigma_\lambda \backslash \Sigma_{\lambda_0-\delta}) \cap B_{R_0}(0)$$

for sufficiently small δ and ε. Hence, (3.10) holds.

Therefore, the above contradiction means that

$$U_{\lambda_0}(x) = V_{\lambda_0}(x) \equiv 0, \;\; x \in \Sigma_{\lambda_0}.$$

Since x_1 direction can be chosen arbitrarily, so we can conclude that the positive solution pair u and v must be radially symmetric and monotone decreasing with respect to some point in $\mathbb{R}^N$. This completes the proof of the Theorem 1.1 . □

We now turn our attention to prove Theorem 1.3 .

Proof of Theorem 1.3. Choosing a direction to be x_1-direction and start moving the plane T_λ from -1 to the right along the x_1-axis, we proceed in two steps and first argue that the assertion

$$U_\lambda(x),\, V_\lambda(x) \geq 0, \quad x \in \Omega_\lambda \tag{3.15}$$

is true for $\lambda > -1$ sufficiently closing to -1, where $\Omega_\lambda := \{x \in B_1(0) \mid x_1 < \lambda\}$. After a direct calculation, we have

$$(-\Delta)^s_p u_\lambda(x) - (-\Delta)^s_p u(x) + C_1(x)U_\lambda(x) + C_2(x)V_\lambda(x) = 0 \tag{3.16}$$

and

$$(-\Delta)^t_p v_\lambda(x) - (-\Delta)^t_p v(x) + C_3(x)U_\lambda(x) + C_4(x)V_\lambda(x) = 0, \tag{3.17}$$

where

$$\begin{aligned}
C_1(x) &= a(p-1)\xi^{p-2} - \frac{f(u_\lambda(x), v(x)) - f(u(x), v(x))}{u_\lambda(x) - u(x)} \\
&\geq -\frac{f(u_\lambda(x), v(x)) - f(u(x), v(x))}{u_\lambda(x) - u(x)}, \\
C_2(x) &= -\frac{f(u_\lambda(x), v_\lambda(x)) - f(u_\lambda(x), v(x))}{v_\lambda(x) - v(x)}, \\
C_3(x) &= -\frac{g(u_\lambda(x), v_\lambda(x)) - g(u(x), v_\lambda(x))}{u_\lambda(x) - u(x)}, \\
C_4(x) &= b(p-1)\eta^{p-2} - \frac{g(u(x), v_\lambda(x)) - g(u(x), v(x))}{v_\lambda(x) - v(x)} \\
&\geq -\frac{g(u(x), v_\lambda(x)) - g(u(x), v(x))}{v_\lambda(x) - v(x)},
\end{aligned}$$

for $U_\lambda(x),\, V_\lambda(x) \neq 0$. Here ξ is between $u(x)$ and $u_\lambda(x)$, η is between $v(x)$ and $v_\lambda(x)$. Applying the assumptions that f, g are Lipschitz continuous and combining (1.10) with (1.11), we show that $C_1(x)$, $C_2(x)$, $C_3(x)$, $C_4(x)$ have lower bounds and $C_2(x), C_3(x) < 0$ in Ω_λ. Besides, a combination of $u, v > 0$ on $B_1(0)$ and $u, v \equiv 0$ on $\mathbb{R}^N \backslash B_1(0)$ yields that the additional conditions in Theorem 2.1 are automatically satisfied. Hence, in terms of Theorem 2.1 , we conclude the assertion (3.15) holds for $\lambda > -1$ sufficiently closing to -1.

Next, we continue to move the plane T_λ to the right along the x_1-axis until its limiting position as long as (3.15) holds. More precisely, defining

$$\lambda_0 := \sup\{\lambda \le 0 \mid U_\mu(x) \ge 0,\, V_\mu(x) \ge 0,\ x \in \Omega_\mu,\ \mu \le \lambda\}.$$

We now claim that

$$\lambda_0 = 0. \tag{3.18}$$

If not, then we will prove that the plane can still move further such that (3.15) holds. To be more rigorous, there exists $\varepsilon > 0$ such that

$$U_\lambda(x),\, V_\lambda(x) \ge 0, \quad x \in \Omega_\lambda \tag{3.19}$$

for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, which contradicts the definition of λ_0. Since both $U_{\lambda_0}(x)$ and $V_{\lambda_0}(x)$ are not identically zero on Ω_{λ_0}, we utilize the similar argument as in the proof of Theorem 1.1 yields

$$U_{\lambda_0}(x),\, V_{\lambda_0}(x) > 0, \quad x \in \Omega_{\lambda_0}. \tag{3.20}$$

It follows from (3.20) that

$$U_{\lambda_0}(x),\, V_{\lambda_0}(x) \ge C_\delta > 0, \quad x \in \Omega_{\lambda_0 - \delta}.$$

Thus, by the continuity of $U_\lambda(x)$ and $V_\lambda(x)$ with respect to λ, there exists a positive constant ε such that

$$U_\lambda(x),\, V_\lambda(x) \ge 0, \quad x \in \Omega_{\lambda_0 - \delta}$$

for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$. Selecting $\Omega_\lambda \backslash \Omega_{\lambda_0 - \delta}$ as a narrow region, then (3.19) holds for sufficiently small δ and ε by Theorem 2.1 . Hence, the assertion (3.18) is proved.

Finally, we conclude that the positive solution pair u and v are radially symmetric and monotone decreasing about the origin due to x_1 direction can be chosen arbitrarily. This completes the proof of the Theorem 1.3 . □

§4 Monotonicity and Nonexistence of Positive Solutions

In this section, applying the direct method of moving planes to prove Theorem 1.4 and 1.5 , we show that the monotonicity in an unbounded parabolic domain and the nonexistence on the half space for positive solutions to (1.1), respectively. We proceed by proving Theorem 1.4 .

Proof of Theorem 1.4. A direct calculation shows that the coefficients in (3.16) and (3.17) are replaced by

$$\begin{aligned}
C_1(x) &= a(x')(p-1)\xi^{p-2} - \frac{f(u_\lambda(x), v(x)) - f(u(x), v(x))}{u_\lambda(x) - u(x)}, \\
C_2(x) &= -\frac{f(u_\lambda(x), v_\lambda(x)) - f(u_\lambda(x), v(x))}{v_\lambda(x) - v(x)}, \\
C_3(x) &= -\frac{g(u_\lambda(x), v_\lambda(x)) - g(u(x), v_\lambda(x))}{u_\lambda(x) - u(x)}, \\
C_4(x) &= b(x')(p-1)\eta^{p-2} - \frac{g(u(x), v_\lambda(x)) - g(u(x), v(x))}{v_\lambda(x) - v(x)},
\end{aligned}$$

for $U_\lambda(x)$, $V_\lambda(x) \neq 0$, where ξ is between $u(x)$ and $u_\lambda(x)$, η is between $v(x)$ and $v_\lambda(x)$. By virtue of the assumptions in Theorem 1.4 , we can apply Theorem 2.1 to deduce that

$$U_\lambda(x),\, V_\lambda(x) \geq 0, \quad x \in \widehat{\Omega}_\lambda \tag{4.1}$$

for $\lambda > 0$ sufficiently closing to 0, where $\widehat{\Omega}_\lambda := \{x \in \Omega \mid x_N < \lambda\}$ and $x^\lambda := (x', 2\lambda - x_N)$.

We continue to move the plane $\widehat{T}_\lambda := \{x \in \Omega \mid x_N = \lambda \text{ for} \lambda \in \mathbb{R}_+\}$ to the right along the x_N-axis as long as (4.1) holds to its limiting position. To be more precise, let

$$\lambda_0 := \sup\{\lambda > 0 \mid U_\mu(x) \geq 0,\, V_\mu(x) \geq 0,\ x \in \widehat{\Omega}_\mu,\ \mu \leq \lambda\}.$$

We now argue the assertion that

$$\lambda_0 = +\infty. \tag{4.2}$$

Otherwise, if $\lambda_0 < +\infty$, we claim that

$$U_{\lambda_0}(x) = V_{\lambda_0}(x) \equiv 0, \ \ x \in \widehat{\Omega}_{\lambda_0}. \tag{4.3}$$

In analogy with the proof of Theorem 1.1 , we can derive either (4.3) or

$$U_{\lambda_0}(x),\, V_{\lambda_0}(x) > 0, \ \ x \in \widehat{\Omega}_{\lambda_0} \tag{4.4}$$

holds. If (4.4) is true, then we will prove that the plane can still move further such that (4.1) holds. To be more precise, there exists $\varepsilon > 0$ such that

$$U_\lambda(x),\, V_\lambda(x) \geq 0, \ \ x \in \widehat{\Omega}_\lambda \tag{4.5}$$

for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, which contradicts the definition of λ_0. It follows from (4.4) that

$$U_{\lambda_0}(x),\, V_{\lambda_0}(x) \geq C_\delta > 0, \ \ x \in \widehat{\Omega}_{\lambda_0 - \delta}$$

for $0 < \delta < \lambda_0$. Thus, by the continuity of $U_\lambda(x)$ and $V_\lambda(x)$ with respect to λ, there exists a positive constant ε such that

$$U_\lambda(x),\, V_\lambda(x) \geq 0, \ \ x \in \widehat{\Omega}_{\lambda_0 - \delta}$$

for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$. We specify $\widehat{\Omega}_\lambda \backslash \widehat{\Omega}_{\lambda_0 - \delta}$ as a narrow region, then (4.5) holds for sufficiently small δ and ε by Theorem 2.1 . Hence, the aforementioned contradiction concludes that (4.3) is valid.

We mention that (4.3) implies

$$u(x_1, x_2, ..., x_{N-1}, 2\lambda_0) = u(x_1, x_2, ..., x_{N-1}, 0) = 0$$

and

$$v(x_1, x_2, ..., x_{N-1}, 2\lambda_0) = v(x_1, x_2, ..., x_{N-1}, 0) = 0,$$

which are contradictive with the fact that $u, v > 0$ on Ω, then the assertion (4.2) holds. Therefore, u and v are strictly increasing with respect to the x_N-axis, which completes the proof of the Theorem 1.4 . □

In the sequel, it remains to be proved Theorem 1.5 .

Proof of Theorem 1.5. We start by proving the assertion that

$$\text{either } u(x),\, v(x) > 0 \text{ or } u(x),\, v(x) \equiv 0 \text{ in } \mathbb{R}^N_+. \tag{4.6}$$

We first show that if there exists $x_0 \in \mathbb{R}^N_+$ such that $u(x_0) = 0$, then

$$u(x),\, v(x) \equiv 0 \text{ in } \mathbb{R}^N_+. \tag{4.7}$$

If $u(x) \not\equiv 0$, on one hand

$$\begin{aligned}(-\Delta)_p^s u(x_0) &= C_{N,sp} PV \int_{\mathbb{R}^N} \frac{|u(x_0)-u(y)|^{p-2}[u(x_0)-u(y)]}{|x_0-y|^{N+sp}} dy \\ &= C_{N,sp} PV \int_{\mathbb{R}^N_+} \frac{-|u(y)|^{p-2}u(y)}{|x_0-y|^{N+sp}} dy \\ &< 0.\end{aligned}$$

On the other hand, it follows from (1.13), (1.14) and (1.10) that

$$(-\Delta)_p^s u(x_0) = f(0, v(x_0)) \geq f(0,0) = 0.$$

This contradiction implies that $u(x) \equiv 0$ in $\mathbb{R}^N_+$. Then we have $f(0,v) = 0$, which is ensured by $u(x) \equiv 0$ and the first equation in (1.13). Now using (1.14) and (1.10) again, we can deduce that $v(x) \equiv 0$ on $\mathbb{R}^N_+$. Indeed, the similar argument as in the proof of (4.7) yields if $v(x)$ attains zero at a point in $\mathbb{R}^N_+$, then $u(x)$, $v(x) \equiv 0$ in $\mathbb{R}^N_+$. Hence, the assertion (4.6) holds.

Now we prove Theorem 1.5 by contradiction. In the sequel, we always assume that

$$u(x),\, v(x) > 0 \text{ in } \mathbb{R}^N_+. \tag{4.8}$$

Adopting the notations

$$T'_\lambda := \{x \in \mathbb{R}^N_+ \mid x_N = \lambda \text{ for } \lambda \in \mathbb{R}_+\},$$
$$\Sigma'_\lambda := \{x \in \mathbb{R}^N_+ \mid x_N < \lambda\},$$

and denoting the reflection of x about the moving plane T'_λ by $x^\lambda := (x_1, x_2, ..., 2\lambda - x_N)$. We proceed in two steps and first argue

$$U_\lambda(x),\, V_\lambda(x) \geq 0, \quad x \in \Sigma'_\lambda \tag{4.9}$$

is valid for $\lambda > 0$ sufficiently closing to 0. A combination of (4.8) and (1.15) yields that

$$\varliminf_{|x|\to\infty} U_\lambda(x), \ \varliminf_{|x|\to\infty} V_\lambda(x) \geq 0. \tag{4.10}$$

Thus, we conclude the assertion (4.9) by Theorem 2.1 .

Next, we continue to move the plane T'_λ to the right along the x_N-axis until its limiting position as long as (4.9) holds. More precisely, let

$$\lambda_0 := \sup\{\lambda > 0 \mid U_\mu(x) \geq 0,\ V_\mu(x) \geq 0,\ x \in \Sigma'_\mu,\ \mu \leq \lambda\}.$$

We show that

$$\lambda_0 = +\infty. \tag{4.11}$$

Otherwise, if $\lambda_0 < +\infty$, combining (4.10) with the similar argument as in the proof of (4.3), we can deduce

$$U_{\lambda_0}(x) = V_{\lambda_0}(x) \equiv 0, \ \ x \in \Sigma'_{\lambda_0}.$$

It reveals that

$$u(x_1, x_2, ..., x_{N-1}, 2\lambda_0) = u(x_1, x_2, ..., x_{N-1}, 0) = 0$$

and

$$v(x_1, x_2, ..., x_{N-1}, 2\lambda_0) = v(x_1, x_2, ..., x_{N-1}, 0) = 0,$$

which are contradictive with the assumption (4.8), then (4.11) holds.

Hence, u and v are increasing with respect to the x_N-axis. In terms of (1.15), we know that it is impossible, and then $u(x)$, $v(x) \equiv 0$ in $\mathbb{R}^N$. This completes the proof of Theorem 1.5 . $\square$

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