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Numerical scheme to solve a class of variable–order Hilfer–Prabhakar fractional differential equations with Jacobi wavelets polynomials

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Abstract. In this paper, we introduced a numerical approach for solving the fractional differential equations with a type of variable-order Hilfer-Prabhakar derivative of order $\mu(t)$ and $\nu(t)$. The proposed method is based on the Jacobi wavelet collocation method. According to this method, an operational matrix is constructed. We use this operational matrix of the fractional derivative of variable-order to reduce the solution of the linear fractional equations to the system of algebraic equations. Theoretical considerations are discussed. Finally, some numerical examples are presented to demonstrate the accuracy of the proposed method.

§1 Introduction

Differential equations as one of the branches of mathematics with variable-order (fixed) fractional derivatives have significant applications in mathematical fields, for example, Physics [8, 10], electromagnetic and mathematical science [11, 12] and engineering [19, 28]. Recently, due to important applications wavelets as a basis functions and properties of this functions have demonstrated in several areas of physics and mathematics and engineering. Obtain exact solutions some fractional differential equations are not convenient and for this aim we have to refer to numerical solutions which used in some articles. Here some numerical methods is suggested, for instance, Odibat derived a solution for the VO-fractional differential equations using variational iteration scheme [26], El-Kalla and Hosseini derived a solution for the VO-fractional differential equations using Adomian decomposition scheme [13, 16], Momani et al. derived a solution for the VO-fractional differential equations using generalized differential transform scheme [22], Mahmoud et al. derived a new operational matrix based on Jacobi wavelets for a class of variable-order fractional differential equation [21], Sun et al. derived a numerical algorithm to solve a class of variable order fractional integral-differential equation

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based on Chebyshev polynomials [33], Chen et al. derived a solution for the VO-fractional differential equations using Wavelet scheme [9], Rong and Chang derived a Jacobi wavelet operational matrix of fractional integration for solving fractional integro-differential equation [30], Secer and Cinar derived a Jacobi wavelet collocation method for fractional fisher's equation in time [34], Xu and Ertürk derived a solution for the VO-fractional differential equations using finite difference scheme [36], Zayernouri and Karniadakis derived a solution for the VO-fractional differential equations using collocation scheme [37] and Mohebbi and Saffarian derived a solution for the VO-fractional differential equations using implicit RBF meshless scheme [23] and wavelets construct a connection with efficient numerical algorithms [3, 7, 18] other methods as Chebyshev wavelet method [2], Bernstein polynomials [35], first integral method [4], functional variable method [5], Chebyshev cardinal functions [15], shifted Legendre polynomials [1], hybrid functions approach [20], analytical method [25], efficient chebyshev semi-iterative method [32] and Laplace transform [6]. What we are going to discuss in this paper is given by replacing Riemann-Liouville fractional integrals with Prabhakar integrals in the definition of Hilfer derivatives that the so-called Hilfer-Prabhakar fractional derivative is in fact a very convenient way to generalize both definitions of derivatives as it actually interpolates them by introducing only one additional real parameter. In this paper, we focus on the solution of a class of variable order fractional integral-differential equation of the form:

$$y^{(n)}(t) = {}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+} y(t) + f(t, y(t)), \tag{1}$$

with initial conditions

$$y^{(r)}(0) = y_0^r,\ r = 0, 1, 2, \ldots, n-1. \tag{2}$$

where $0 < \mu(t) \leq 1$ is the variable order fractional derivate, $\rho > 0,\ \gamma > 0,\ \omega > 0$ and ${}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+}$ is fractional derivative of Hilfer-Prabhakar sense. In this study, we focus on extending the concept of the Jacobi Polynomial to Jacobi Wavelet. In this paper, we present a numerical method for the variable-order fractional equation by combining some properties of Jacobi Wavelet polynomials and the variable-order fractional derivatives effectively and the method is based on reducing the equation to a system of algebraic equations and we apply matlab to solve the equation. For this purpose, the structure of the paper is as follows: in Section 2 we recall some definitions and lemmas in generalized fractional calculus. In Section 3, we introduce some important properties of the Jacobi Wavelet polynomials and the Jacobi Wavelet polynomials also in this section we show function approximation and we obtain numerical solutions for differential equation (1),(2). In Section 5 we show applications of the operational matrix of fractional derivative. In Section 6, we present several numerical examples and show the effectiveness and accuracy of the given method.

§2 Preliminaries

In this section, we express some important definitions and lemmas which are applied in this paper to prove the main results.

2.1 Fractional calculus

We describe the definitions of fractional integral and derivative of order $\mu > 0$ as follows [14, 17, 27, 29, 31]:

Definition 2.1. *Let $0 < \alpha \leq 1$ and $f \in L^1[a,b]$, $0 < t < b \leq \infty$. Then the Riemann-Liouville fractional integrals of order α is given by:*

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \tag{3}$$

$$I^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau-t)^{\alpha-1} d\tau, \tag{4}$$

where $I^{\alpha}_{a^+}$ is the left-sided and Riemann-Liouville fractional integrals and $I^{\alpha}_{b^-}$ is the right-sided Riemann-Liouville fractional integrals. Also, the left-sided and the right-sided Riemann-Liouville fractional derivatives of order α are given by:

$$D^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)^{-\alpha} d\tau, \tag{5}$$

$$D^{\alpha}_{b^-} f(t) = -\frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_t^b f(\tau)(\tau-t)^{-\alpha} d\tau. \tag{6}$$

Also, for the function $f(t)$ which is absolutely continuous, the left-sided and the right-sided Caputo fractional derivatives of order α are defined as follows:

$$\begin{aligned} {}^{C}D^{\alpha}_{a^+} f(t) &= I^{1-\alpha}_{a^+} \frac{d}{dt} f(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau, \end{aligned} \tag{7}$$

$$\begin{aligned} {}^{C}D^{\alpha}_{b^-} f(t) &= -I^{1-\alpha}_{b^-} \frac{d}{dt} f(t) \\ &= -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau. \end{aligned} \tag{8}$$

Definition 2.2. *Let $f \in L^1[0,b]$, $-\infty \leq a < t < b \leq \infty$. The Hilfer fractional derivative of order μ and type ν is defined by:*

$$\begin{aligned} \mathbb{D}^{\mu,\nu}_{a^+} f(t) &= \Big(I^{\nu(1-\mu)}_{a^+} \frac{d}{dt} \big(I^{(1-\nu)(1-\mu)}_{a^+} f\big)\Big)(t), \\ & m-1 < \mu \leq m,\, m-1 < \nu \leq m. \end{aligned} \tag{9}$$

Also, for $\nu = 0$ Eq.(9) changes in to the Riemann- Liouville derivative and for $\nu = 1$ changes in to the Caputo fractional derivative.

Definition 2.3. . *For $m-1 < \Re(\mu) \leq m$ and $f \in L^1[0,b]$, $0 < t < b \leq \infty$, the left-sided and the right-sided Prabhakar fractional integrals are defined as follows*

$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a^+} f)(t) = \int_a^t (t-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-\tau)^{\rho}) f(\tau) d\tau, \tag{10}$$

$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,b^-} f)(t) = \int_t^b (\tau-t)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\tau-t)^{\rho}) f(\tau) d\tau, \tag{11}$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}$ and $E^{\gamma}_{\rho,\mu}$ is introduced by Prabhakar and it is given by.

$$E^{\gamma}_{\rho,\mu}(z) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)} z^n, \ \mu, \gamma, \rho \in \mathbb{C}, \ \Re(\rho) > 0. \tag{12}$$

Definition 2.4. . *Let $f \in L^1[0,b]$. Then the left-sided and the right-sided Prabhakar fractional derivatives are defined as:*

$$(D^{\gamma}_{\rho,\mu,\omega,a^+} f)(t) = \frac{d^m}{dt^m} \mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,a^+} f(t), \tag{13}$$

$$(D^{\gamma}_{\rho,\mu,\omega,b^-} f)(t) = (-1)^m \frac{d^m}{dt^m} \mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,b^-} f(t), \tag{14}$$

where $m-1 < \Re(\mu) \leq m$ and $\rho, \omega, \gamma \in \mathbb{C}$. For the absolutely continuous function f, the left-sided and the right-sided Caputo-Prabhakar fractional derivatives are also defined as follows:

$${}^{C}D^{\gamma}_{\rho,\mu,\omega,a^+} f(t) = \mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,a^+} \frac{d^m}{dt^m} f(t), \tag{15}$$

$${}^{C}D^{\gamma}_{\rho,\mu,\omega,b^-} f(t) = (-1)^m \mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,b^-} \frac{d^m}{dt^m} f(t). \tag{16}$$

Definition 2.5. *Let $f \in L^1[0,b]$, $-\infty \leq a < t < b \leq \infty$. The Hilfer-Prabhakar derivative of order μ and ν are given as:*

$$\mathfrak{D}^{\gamma,\mu,\nu}_{\rho,\omega,a^+} f(t) = \Big(\mathbf{E}^{-\gamma\nu}_{\rho,\nu(m-\mu),\omega,a^+} \frac{d}{dt} \big(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(m-\nu)(m-\mu),\omega,a^+} f\big)\Big)(t), \tag{17}$$

where $0 < \mu \leq 1$, $0 < \nu \leq 1, \rho > 0$, γ, $\omega \in \mathbb{C}$ and $\mathbf{E}^{0}_{\rho,0,\omega,a^+} = f$. Also, for any the absolutely continuous function as f we define the regularized type of (17) as follows:

$$\begin{aligned} {}^{C}\mathrm{D}^{\gamma,\mu}_{\rho,\omega,a^+} f(t) &= \Big(\mathbf{E}^{-\gamma\nu}_{\rho,\nu(m-\mu),\omega,a^+} \mathbf{E}^{-\gamma(1-\nu)}_{\rho,(m-\nu)(m-\mu),\omega,a^+} \frac{d}{dt} f\Big)(t) \\ &= \Big(\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,a^+} \frac{d}{dt} f\Big)(t). \end{aligned} \tag{18}$$

2.2 Variable-order fractional calculus

This subsection defines the Hilfer-Prabhakar derivative and the regularized Hilfer derivative of variable-order as $\mu(t)$, $\nu(t)$ that this definitions is similar to the classical fractional calculus.

Definition 2.6. *For an absolutely integrable function as $f(t)$ of order $\mu(t)$ and $\nu(t)$, The Hilfer-Prabhakar fractional derivative are given by:*

$$\begin{aligned} &\mathcal{D}^{\gamma,\mu(t),\nu(t)}_{\rho,\omega,a^+} f(t) \\ &= \Big(\mathbf{E}^{-\gamma\nu}_{\rho,\nu(t)(m-\mu(t)),\omega,a^+} \frac{d}{dt} \big(\mathbf{E}^{-\gamma(1-\nu(t))}_{\rho,(m-\nu(t))(m-\mu(t)),\omega,a^+} f\big)\Big)(t), \end{aligned} \tag{19}$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}$.

Lemma 2.7. *Let $\rho, \gamma, \mu(t), \varsigma, \omega \in \mathbb{C}$. Then for any $\Re(\rho), \Re(\mu(t)), \Re(\varsigma) > 0$ the following relation is hold:*

$$\int_0^t (t-u)^{\mu(t)-1} E^{\gamma}_{\rho,\mu(t)}(\omega(t-u)^{\rho}) u^{\varsigma-1} du = \Gamma(\varsigma) t^{\mu(t)+\varsigma-1} E^{\gamma}_{\rho,\mu(t)+\varsigma}(\omega t^{\rho}). \tag{20}$$

Proof. The use of (12), we obtain:

$$\int_0^t (t-\tau)^{\mu(t)-1} E^{\gamma}_{\rho,\mu(t)}\Big(\omega(t-\tau)^{\rho}\Big)\tau^{\varsigma-1}d\tau$$
$$=\frac{1}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)\omega^k}{k!\Gamma(\rho k+\mu(t))}\int_0^t (t-\tau)^{\rho k+\mu(t)-1}\tau^{\varsigma-1}d\tau. \tag{21}$$

Now, employing $\int_0^t (t-\tau)^{\rho k+\mu(t)-1}\tau^{\varsigma-1}d\tau=\Gamma(\rho k+\mu(t))\Big(\mathbb{I}_t^{\rho k+\mu(t)}t^{\varsigma-1}\Big)$, we have:

$$\int_0^t (t-\tau)^{\mu(t)-1} E^{\gamma}_{\rho,\mu(t)}\Big(\omega(t-\tau)^{\rho}\Big)\tau^{\varsigma-1}d\tau$$
$$=\tfrac{1}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\tfrac{\Gamma(\gamma+k)\omega^k}{k!\Gamma(\rho k+\mu(t))}\Big(\Gamma(\rho k+\mu(t))\Big(\mathbb{I}_t^{\rho k+\mu(t)}t^{\varsigma-1}\Big)\Big)$$
$$=\tfrac{\Gamma(\varsigma)}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\tfrac{\Gamma(\gamma+k)\omega^k}{k!\Gamma(\rho k+\mu(t)+\varsigma)}t^{\varsigma+\rho k+\mu(t)-1}$$
$$=\Gamma(\varsigma)t^{\varsigma+\mu(t)-1}\tfrac{1}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\tfrac{\Gamma(\gamma+k)\omega^k t^{\rho k}}{k!\Gamma(\rho k+\mu(t)+\varsigma)}$$
$$=\Gamma(\varsigma)t^{\varsigma+\mu(t)-1}E^{\gamma}_{\rho,\mu(t)+\varsigma}(\omega t^{\rho}). \tag{22}$$

This completes the proof. □

Definition 2.8. *For the absolutely continuous function f, the regularized Hilfer derivative is defined as follows:*

$${}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,a^+}f(t)$$
$$=\Big(\mathbf{E}^{-\gamma\nu(t)}_{\rho,\nu(t)(m-\mu(t)),\omega,a^+}\mathbf{E}^{-\gamma(1-\nu(t))}_{\rho,(m-\nu(t))(m-\mu(t)),\omega,a^+}\frac{d}{dt}f\Big)(t)$$
$$=\Big(\mathbf{E}^{-\gamma}_{\rho,m-\mu(t),\omega,a^+}\frac{d}{dt}f\Big)(t), \tag{23}$$

where $\rho,\omega,\gamma\in\mathbb{C}$ and $m-1<\mu(t)\le m$. Here in relation (23) there is no dependence on the interpolating parameter $\nu(t)$.
So according to the above definition, ${}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,a^+}t^{\sigma}$ is calculated as follows:

$${}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+}t^{\sigma}=\Big(\mathbf{E}^{-\gamma}_{\rho,m-\mu(t),\omega,a^+}\frac{d}{dt}t^{\sigma}\Big)(t)$$
$$=\sigma\Big(\mathbf{E}^{-\gamma}_{\rho,m-\mu(t),\omega,a^+}t^{\sigma-1}\Big)(t), \tag{24}$$

applying Eqs.(20),(11) on Eq.(24), we obtain

$${}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+}t^{\sigma}$$
$$=\begin{cases}\Gamma(\sigma+1)t^{\sigma-\mu(t)}E^{-\gamma}_{\rho,\sigma-\mu(t)+1}(\omega t^{\rho}), & m=1,\\ 0, & m=0.\end{cases} \tag{25}$$

§3 Jacobi wavelets polynomials

On the interval $x\in[-1,1]$ the Jacobi wavelets polynomials of degree n with a recurrence relation is defined as follows [8]:

$$\mathrm{P}^{(\xi,\zeta)}_{n+1}(t)=\Big(a^{(\xi,\zeta)}_n-b^{(\xi,\zeta)}_n\Big)\mathrm{P}^{(\xi,\zeta)}_n(t)-c^{(\xi,\zeta)}_n\mathrm{P}^{(\xi,\zeta)}_{n-1}(t),\ n\ge 1, \tag{26}$$

where $\mathrm{P}_0^{(\xi,\zeta)}(t)=1$, $\mathrm{P}_1^{(\xi,\zeta)}=\frac{1}{2}(\xi+\zeta+2)+\frac{1}{2}(\xi-\zeta)$ and the other parameters of this recursive sequence are defined as follows:

$$\begin{aligned} a_n^{\xi,\zeta} &= \frac{(2n+\xi+\zeta+1)(2n+\xi+\zeta+2)}{2(n+1)(n+\xi+\zeta+1)}, \\ b_n^{\xi,\zeta} &= \frac{(2n+\xi+\zeta+1)(-\xi^2+\zeta^2)}{2(n+1)(n+\xi+\zeta+1)(2n+\xi+\zeta)}, \\ c_n^{\xi,\zeta} &= \frac{(2n+\xi+\zeta+1)(n+\xi)(n+\zeta)}{(n+1)(n+\xi+\zeta+1)(2n+\xi+\zeta)}. \end{aligned} \tag{27}$$

The Jacobi wavelets polynomials $\mathrm{P}_{n+1}^{(\xi,\zeta)}(t)$ of degree n may be displayed as an infinite series as:

$$\begin{aligned} &\mathrm{P}_i^{(\xi,\zeta)}(t) \\ &= \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+1+\zeta)\Gamma(i+k+1+\xi+\zeta)(t+1)^k}{\Gamma(k+1+\zeta)\Gamma(i+1+\xi+\zeta)(i-k)!k!(2)^k}. \end{aligned} \tag{28}$$

The orthogonality condition is

$$\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\zeta}\mathrm{P}_i^{(\xi,\zeta)}(t)\mathrm{P}_j^{(\xi,\zeta)}(t)dt = \begin{cases} \frac{1}{h_i} & i=j \\ 0 & i\neq j \end{cases} \tag{29}$$

where

$$h_i^{(\xi,\zeta)} = \frac{(2i+\xi+\zeta+1)\Gamma(i+1)\Gamma(i+\xi+\zeta+1)}{2^{\xi+\zeta}\Gamma(i+\xi+1)\Gamma(i+\zeta+1)}. \tag{30}$$

3.1 Jacobi wavelets

The Jacobi wavelet functions on the interval $[0,L]$ of degree m and k that k is the evel of resolution are given by:

$$\varphi_{nm}(t) = \begin{cases} \sqrt{h_m^{(\xi,\zeta)}}\frac{2^{\frac{k}{2}}}{\sqrt{L}}\mathrm{P}_m^{(\xi,\zeta)}\Big(\frac{2^{\frac{k+1}{2}}}{L}t-2n+1\Big), \\ \qquad \frac{(n-1)L}{2^k}\leq t\leq \frac{nL}{2^k}, \\ \qquad 0, \qquad\qquad \text{otherwise} \end{cases} \tag{31}$$

where $m=0,1,2,\ldots,M-1$ and $k=1,2,3,\ldots$ and $n=1,2,\ldots,2^k$ also x is the normalized time. Orthogonality properties for the Jacobi wavelets polynomials are given by:

$$\begin{aligned} &\langle\varphi_{nm}(t),\varphi_{n'm'}(t)\rangle \\ &= \int_0^L \varpi_n(t)\varphi_{nm}(t)\varphi_{n'm'}(t) = \begin{cases} 1, & (n,m)=(n',m'), \\ 0, & (n,m)\neq(n',m'), \end{cases} \end{aligned} \tag{32}$$

where $\langle . \rangle$ is the inner product and it defined by$\langle f,g\rangle=\int_0^L f(t)g(t)dt,\ g,h\in L^2[0,L]$ and the weight function $\varpi_n(t)$ is defined by:

$$\varpi_n(t) = \begin{cases} (2n-\frac{2^{k+1}}{L}t)^{\xi}(\frac{2^{k+1}}{L}t-2n+2)^{\zeta}, & \frac{(n-1)L}{2^k}\leq t\leq\frac{nL}{2^k}, \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

A function $f(t)\in L^2[0,L]$ can be approximate as

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}a_{nm}\varphi_{nm}(t), \tag{34}$$

where $\varphi_{nm}(t)$ is the Jacobi wavelets polynomials. By multiplying the sides Eq.(34) in $\varphi_{n'm'}(t)$ we obtain

$$f(t)\varphi_{n'm'}(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}\varphi_{nm}(t)\varphi_{n'm'}(t), \tag{35}$$

now, we take the integral from Eq.(35), we have:

$$\begin{aligned}&\int_0^L \varpi_n(t)\varphi_{nm}(t)\varphi_{n'm'}(t)\\ &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}\int_0^L \varpi_n(t)\varphi_{nm}(t)\varphi_{n'm'}(t).\end{aligned} \tag{36}$$

According to the orthogonal conditions $\varphi_{nm}(t)$, we get:

$$a_{nm} = \int_0^L \varpi_n(t)\varphi_{nm}(t)f(t). \tag{37}$$

Considering a finite number of series sentences Eq.(34) as follows:

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\varphi_{nm}(t) = \mathbf{A}^T\Psi(t), \tag{38}$$

where $\mathbf{A}, \psi(t)$ are given by:

$$\begin{aligned}\mathbf{A} &= [a_{10}, a_{11}, \dots, a_{1M-1}, a_{20}, a_{21}, \dots, a_{2M-1}, \dots, a_{2^{k-1}0}\\ &, a_{2^{k-1}1}, \dots, a_{2^kM-1}]^T,\\ \psi(t) &= [\varphi_{10}(t), \varphi_{11}(t), \dots, \varphi_{1M-1}(t), \varphi_{20}(t), \varphi_{21}(t),\\ &\dots, \varphi_{2M-1}(t), \dots,\\ &\dots, \varphi_{2^k0}(t), \varphi_{2^k1}(t), \dots, \varphi_{2^kM-1}(t)]^T.\end{aligned}$$

In this case, we can for simplicity Eq. (38) rewritten as follows:

$$f(t) = \sum_{l=1}^{\hat{m}} a_l\varphi_l(t) = \mathbf{A}^T\Psi(t), \tag{39}$$

where $\hat{m} = 2^kM, l = M(n-1)+m+1$ and the vectors $\mathbf{A}$, $\Psi(t)$ are defined by:

$$\begin{aligned}\mathbf{A}^T &= [a_1, a_2, \dots, a_l],\\ \Psi(t) &= [\varphi_1, \varphi_2, \dots, \varphi_l]^T,\end{aligned} \tag{40}$$

where $a_l = a_{nm}$, $\varphi_l = \varphi_{nm}$ and

$$\varphi_l = \varphi_{nm} = \begin{cases} t^m, & \frac{(n-1)L}{2^k} \le t \le \frac{nL}{2^k}\\ 0, & \text{otherwise.}\end{cases} \tag{41}$$

Theorem 3.1. *Suppose $y_N(t)$ and $y(t)$ are the numerical and exact solutions of equation*(1) *and* (2) *such that*

$$\Lambda_{N+1}(t) = y_{N+1}^{(n)}(t) - {}^C\mathfrak{H}_{\rho,\omega,0^+}^{\gamma,\mu(t)} y_{N+1}(t) + f(t, y_{N+1}(t)), \tag{42}$$

where here $\Lambda_{N+1}(t)$ is the residual function. If there exists a constant $\beta_N \in (0,1)$ such that $\| \Lambda_{N+1}(t) \|_{L^2[0,1]} \le \beta_N \| \Lambda_N(t) \|_{L^2[0,1]}$, $n \in N\bigcup\{0\}$. In this case $\| y_N(t) - y(t) \| \to 0$ as $N \to \infty$.

Proof. For this purpose we define the residual function such as follows:

$$\Lambda_{N+1}(t)=y^{(n)}_{N+1}(t)-{}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+}y_{N+1}(t)+f(t,y_{N+1}(t))=\sum_{j=0}^{N+1}\frac{y^j(t)}{j!}t^j. \tag{43}$$

$$\Lambda(t)=y^{(n)}(t)-{}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+}y(t)+f(t,y(t)). \tag{44}$$

Now we show $\Lambda(t)$ is a Cauchy sequence in $L^2[0,1]$, for this aim, for any M,N such that $N\geq M$, we have:

$$\begin{aligned}&\parallel \Lambda_N(t)-\Lambda_M(t)\parallel_{L^2[0,1]}=\parallel(\Lambda_N(t)-\Lambda_{N-1}(t))+(\Lambda_{N-1}(t)-\Lambda_{N-2}(t))+\dots\\&+(\Lambda_{N+1}(t)-\Lambda_N(t))\parallel_{L^2[0,1]}\\&=\parallel \Lambda_N(t)-\Lambda_{N-1}(t)\parallel_{L^2[0,1]}+\parallel \Lambda_{N-1}(t)-\Lambda_{N-2}(t)\parallel_{L^2[0,1]}+\dots\\&+\parallel \Lambda_{M+1}(t)-\Lambda_M(t)\parallel_{L^2[0,1]},\end{aligned} \tag{45}$$

from $\parallel \Lambda_{N+1}(t)\parallel_{L^2[0,1]}\leq\beta_N\parallel\Lambda_N(t)\parallel_{L^2[0,1]}$, $0<\beta_N<1$ is used, we obtain

$$\begin{aligned}&\parallel \Lambda_N(t)-\Lambda_M(t)\parallel_{L^2[0,1]}\leq(\beta_N-1)^N\parallel\Lambda_0(t)\parallel_{L^2[0,1]}\\&+(\beta_N-1)^{N-1}\parallel\Lambda_0(t)\parallel_{L^2[0,1]}\dots+(\beta_N-1)^{N+1}\parallel\Lambda_0(t)\parallel_{L^2[0,1]}\\&=\Pi_{N,M}\parallel\Lambda_0(t)\parallel_{L^2[0,1]}.\end{aligned} \tag{46}$$

where $\Pi_{N,M}=\frac{\left(1-(\beta_N-1)^{N-M}\right)(\beta_N-1)^{M+1}}{2-\beta_N}$. Therefore when $M\to\infty,N\to\infty$ we have $\parallel \Lambda_N(t)-\Lambda_M(t)\parallel_{L^2[0,1]}\to 0$, then $\Lambda_N(t)$ is a Cauchy sequence in $L^2[0,1]$. Because it is complete, so it is converges. So the proof of this Theorem is obtained. □

In this section, to obtain the numerical solutions of Eqs. (1) and (2), we propose transforms in the form of matrix. The connection between these functions and Jacobi wavelets can be demonstrated as follows:

$$\varphi_i(t)=\sum_{j=1}^{2^kM}p_{ij}\Psi_j(t),\ i=1,2,\dots,2^kM, \tag{47}$$

where $\Psi(t)=[\Psi_1(t),\Psi_2(t),\dots,\Psi_{2^kM}(t)]^T$, $\Phi(t)=[\varphi_1,\varphi_2,\dots,\varphi_{\hat m}]^T$. Then, the following result is obtained:

$$\Phi(t)=\mathbf{P}\Psi(t), \tag{48}$$

$p_{ij}=\langle\Psi_i(t),\varphi_j(t)\rangle$ the matrix entries of $\mathbf{P}$. Also, $\mathbf{P}$ is invertible, then we have:

$$\mathbf{P}^{-1}\Phi(t)=\Psi(t). \tag{49}$$

The derivative of the vector $\Psi(t)$ can be expressed by:

$$\frac{d}{dt}\Psi(t)=\mathbf{P}^{-1}\frac{d}{dt}\Phi(t)=\mathbf{P}^{-1}\Delta_M\Phi(t), \tag{50}$$

where Δ_M is the $\hat m\times\hat m$ operational matrix of derivative of $\Phi(t)$ and can be obtained from:

$$\Delta_M=(e_{ij})=\begin{cases} j+1 & i=j+1,\ j=0,1,2,\dots,M-1,\\ 0, & \text{otherwise,}\end{cases} \tag{51}$$

Now, using (50) and (51), then it is easy to write:

$$\frac{d}{dt}\Psi(t)=\mathbf{P}^{-1}\frac{d}{dt}\Phi(t)=\mathbf{P}^{-1}\Delta_M\Phi(t)=\mathbf{P}^{-1}\Delta_M\mathbf{P}\Psi(t)$$

$$= \Lambda^{(1)}\Psi(t), \tag{52}$$

where $\Lambda^{(1)} = \mathbf{P}^{-1}\Delta_M\mathbf{P}$.

Theorem 3.2. *Let $\Psi(t)$ be Jacobi wavelets vector presented in (40) and $\mu(t) > 0$ then*

$$^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+}\Psi(t) = \Omega_{\mu(t)}\Psi(t), \tag{53}$$

where $\Omega_{\mu(t)}$ operational matrix of order $\mu(t)$ including the Hilfer sense which is given by:

$$\Omega_{\mu(t)} = \boldsymbol{P}^{-1}\Upsilon\boldsymbol{P}, \tag{54}$$

where Υ is a $M \times M$ matrix and its elements k_{ij}, $0 \le i, j \le M-1$ are given by:

$$k_{ij} = \begin{cases} \Gamma(i+1)t^{-\mu(t)}E^{-\gamma}_{\rho,i-\mu(t)+1}(\omega t^{\rho}), \\ i = j,\ j = n, n+1, \ldots, M-1, \\ \quad 0, \qquad\qquad \textit{otherwise}. \end{cases} \tag{55}$$

Proof. Direct calculations give

$$\begin{aligned} {}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+}\Psi(t) &= \mathbf{P}^{-1}\,{}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+}\Phi(t) = \mathbf{P}^{-1}\Upsilon\Phi(t) \\ &= \mathbf{P}^{-1}\Upsilon\mathbf{P}\Psi(t) = \Omega_{\mu(t)}\Psi(t). \end{aligned} \tag{56}$$

□

§4 Applications of the operational matrix of fractional derivative

In this section, we first state a Theorem about the absolute error and then using the matrix operators given in section four we solve the system introduced in Eqs. (1) and (2).

Theorem 4.1. *Let $y(t) \in \mathbb{H}^{\lambda}(0,1)$ be the exact solution of system (1) that $\mathbb{H}^{\lambda}(0,1)$ is introduced in [24] and $y_{k,N}$ be the approximate solution of system (1), $y_{k,N} = \sum_{k_1=0}^{2^k-1}\sum_{k_2=0}^{N-1} a_{k_1k_2}\varphi_{k_1k_2}(t)$ and $Y_{k,N} = \sum_{k_1=0}^{2^k-1}\sum_{k_2=0}^{N-1} \tilde{a}_{k_1k_2}\varphi_{k_1k_2}(t)$ be the best approximation of system (1). Then the following relation is hold:*

$$\begin{aligned} \| y(t) - y_{k,N}(t) \|_{L^2(0,1)} &\le C_y(N-1)^{-\lambda}|y(t)|_{\mathbb{H}^{0,\lambda,N-1,k}(0,1)} \\ &+ N^{\frac{1}{2}}2^{\frac{k-1}{2}} \| A_1 - A_2 \|_2, \end{aligned} \tag{57}$$

where C_y is positive which is depends on y and $A_1 = [a_{1,0}, a_{1,1}, \ldots, a_{2^k-1,N-1}]$, $A_2 = [\tilde{a}_{1,0}, \tilde{a}_{1,1}, \ldots, \tilde{a}_{2^k-1,N-1}]$.

Proof. Let $E_{k,N} = y(t) - y_{k,N}(t)$ be the absolute error, then we have:

$$\begin{aligned} \| y(t) - y_{k,N}(t) \|_{L^2(0,1)} &= \| y(t) - Y_{k,N} + Y_{k,N} - y_{k,N}(t) \|_{L^2(0,1)} \\ &\le \| y(t) - Y_{k,N} \|_{L^2(0,1)} + \| Y_{k,N} - y_{k,N}(t) \|_{L^2(0,1)} . \end{aligned} \tag{58}$$

From the Theorem 2 in [20] is used for $Y_{k,N}$ we obtain:

$$\| y(t) - Y_{k,N} \|_{L^2(0,1)} \le C_y(N-1)^{-\lambda}|y(t)|_{\mathbb{H}^{0,\lambda,N-1,k}(0,1)}. \tag{59}$$

Then, using Eq.(59) on Eq.(58), we obtain:

$$\begin{aligned} \| y(t) - y_{k,N}(t) \|_{L^2(0,1)} &\le C_y(N-1)^{-\lambda}|y(t)|_{\mathbb{H}^{0,\lambda,N-1,k}(0,1)} \\ &+ \| Y_{k,N} - y_{k,N}(t) \|_{L^2(0,1)} . \end{aligned} \tag{60}$$

For calculation $\| Y_{k,N} - y_{k,N}(t) \|_{L^2(0,1)}$, we have:

$$\| Y_{k,N} - y_{k,N}(t) \|^2_{L^2(0,1)} = \| \sum_{k_1=0}^{2^k-1} \sum_{k_2=0}^{N-1} a_{k_1k_2}\varphi_{k_1k_2}(t) - \sum_{k_1=0}^{2^k-1} \sum_{k_2=0}^{N-1} \tilde{a}_{k_1k_2}\varphi_{k_1k_2}(t) - \|^2_{L^2(0,1)} = \sum_{k_1=0}^{2^k-1} \sum_{k_2=0}^{N-1} \int_0^1 \Big[(a_{k_1k_2} - \tilde{a}_{k_1k_2}) \varphi_{k_1k_2}(t)\Big]^2 dt, \tag{61}$$

using the Schwarz's inequality, in this case for Eq.(61), we get:

$$\begin{aligned}
&\| Y_{k,N} - y_{k,N}(t) \|^2_{L^2(0,1)} \\
&\leq \int_0^1 \Big(\sum_{k_1=0}^{2^k-1} \sum_{k_2=0}^{N-1} |a_{k_1k_2} - \tilde{a}_{k_1k_2}|^2 \Big) \Big(\sum_{k_1=0}^{2^k-1} \sum_{k_2=0}^{N-1} |\varphi_{k_1k_2}(t)|^2 \Big) dt \\
&\leq \Big[\sum_{k_1=0}^{2^k-1} \sum_{k_2=0}^{N-1} |a_{k_1k_2} - \tilde{a}_{k_1k_2}|^2 \Big] \\
&\times \Big[\sum_{k_1=0}^{2^k-1} \sum_{k_2=0}^{N-1} |\varphi_{k_1k_2}(t)|^2 \Big] \\
&\leq \| A_1 - A_2 \|_2^2 \\
&\times \Big[\sum_{k_1=0}^{2^k-1} \sum_{k_2=0}^{N-1} |\varphi_{k_1k_2}(t)|^2 \Big] \\
&= \| A_1 - A_2 \|_2^2 \, N 2^k.
\end{aligned} \tag{62}$$

Then by substituting Eq.(62) in Eq.(60), the result is obtained. □

Consider the variable-order fractional initial value problem

$$y^{(n)}(t) = {}^C\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+} y(t) + f(t, y(t)), \tag{63}$$

with initial conditions

$$y^{(r)}(0) = y_0^r, \ r = 0, 1, 2, \dots, n-1. \tag{64}$$

To solve problem (63) and (64) we approximate $y(t)$ by the Jacobi wavelets polynomials as

$$y(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm}\varphi_{nm}(t) = \mathbf{A}^T \Psi(t). \tag{65}$$

Using Eq.(53), we have:

$$ {}^C\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+} y(t) = \mathbf{A}^{T\,C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+} \Psi(t) = \mathbf{A}^T \Omega_{\mu(t)} \Psi(t). \tag{66}$$

Employing (66) in Eqs. (63) and (64) yields

$$\begin{aligned}
&\mathbf{A}^T \Omega_n \Psi(t) = \mathbf{A}^T \Omega_{\mu(t)} \Psi(t) + f(t, \mathbf{A}^T \Psi(t)), \ m < \mu(t) \leq m-1, \\
&\mathbf{A}^T \Omega_r \Psi(0) = y_0^r, \ r = 0, 1, 2, \dots, n-1.
\end{aligned} \tag{67}$$

Finally, in order to calculate the unknown coefficients in Eq. (67), we apply the collocation-like

Table 1. The absolute error of Example 5.1 for two cases of $\mu(t)$ with $\rho = 0.75,\ \omega = 1,\ \gamma = 1.5$ and different M.

$\mu(t)$	h	M	AE
0.5	0.02	15	$4.334e-8$
	0.01	20	$3.274e-8$
	0.005	25	$2.356e-8$
$\frac{1-\|\cos t\|}{4}$	0.02	15	$4.043e-8$
	0.01	20	$3.255e-8$
	0.005	25	$1.837e-8$

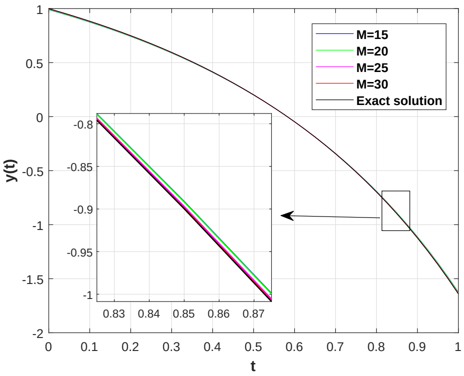

Figure 1. The approximate and exact solutions with several values of M and $\mu(t) = \frac{1-|\cos(t)|}{4},\ \rho = 0.75,\ \omega = 1,\ \gamma = 1.5$ for Example 5.1.

discretization as

$$\begin{aligned}&\mathbf{A}^T\Omega_n\Psi(t_i) = \mathbf{A}^T\Omega_{\mu(t_i)}\Psi(t_i) + f(t_i, \mathbf{A}^T\Psi(t_i)),\ m < \mu(t_i) \leq m-1,\\&\mathbf{A}^T\Omega_r\Psi(0) = y_0^r,\ r = 0, 1, 2, \ldots, n-1,\end{aligned}$$

where $t_i,\ 1 \leq i \leq \tilde{n},\ \tilde{n} = 2^k M$ are the shifted Jacobi collocation nodes of $\mathrm{P}_{\tilde{n}}^{(\xi,\zeta)}(t)$.

Then, we can obtain the unknown coefficients $a_i,\ 1 \leq i \leq \tilde{n}$. Consequently, the approximate solution of $y(t)$ introduced in Eq. (39) can be obtained.

§5 Numerical examples

This section examines some examples using the method proposed in this paper to show the practicability and efficiency it method. For this aim in this section we consider $L = 5$. Also, the absolute error is defined as follows:

$$AE = \sqrt{h\sum_{j=0}^{M}|y_j - y_j^M|^2},$$

where y_j and y_j^M are exact and approximate solutions, respectively.

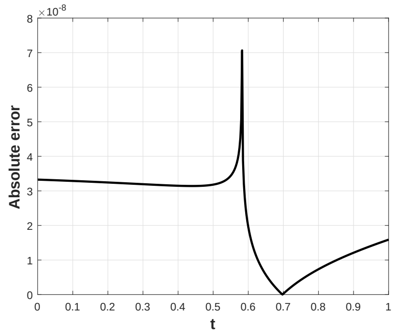

Figure 2. The absolute error function when $\mu(t) = \frac{1-|\cos(t)|}{4}$, $\rho = 0.75$, $\omega = 1$, $\gamma = 1.5$ and $M = 15$ for Example 5.1.

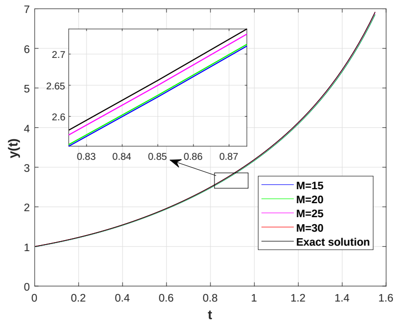

Figure 3. The approximate and exact solutions with several values of M and $\mu(t) = 1 + 0.5|\sin t|$, $\rho = 0.75$, $\omega = 1$, $\gamma = 1.5$ for Example 5.2.

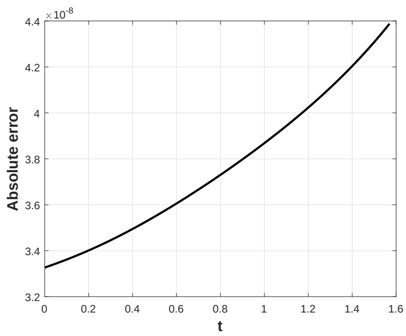

Figure 4. The absolute error function when $\mu(t) = 1 + 0.5|\sin t|$, $\rho = 0.75$, $\omega = 1$, $\gamma = 1.5$ and $M = 15$ for Example 5.2.

Table 2. The absolute error of Example 5.2 for two cases of $\mu(t)$ with $\rho = 0.75,\ \omega = 1,\ \gamma = 1.5$ and different M.

$\mu(t)$	h	M	AE
1.5	0.02	15	$4.0629e-8$
	0.01	20	$3.154e-8$
	0.005	25	$2.344e-8$
$1+0.5\lvert\sin t\rvert$	0.02	15	$4.00e-8$
	0.01	20	$3.216e-8$
	0.005	25	$2.668e-8$

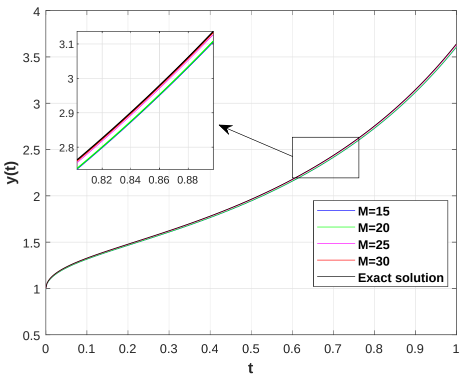

Figure 5. The approximate and exact solutions with several values of M and $\mu(t) = 1 - \frac{1}{2}e^{-t},\ \rho = 0.75,\ \omega = 1,\ \gamma = 1.5$ for Example 5.3.

Example 5.1. *Consider the variable-order fractional Basset equation of order $\mu(t)$ as follows:*

$$
\begin{aligned}
&y'(t) - {}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+} y(t) + 2y(t)\\
&= \tfrac{t^{1-2\mu(t)}}{\Gamma(2-2\mu(t))} - t^{-\mu(t)}\left(E^{\gamma}_{\rho,1-\mu(t)}(\omega t^{\rho}) + 2tE^{\gamma}_{\rho,2-\mu(t)}(\omega t^{\rho})\right) + 2,\\
&y(0) = 1,\ 0 < \mu(t) \le 1,
\end{aligned} \tag{68}
$$

with parameters $\rho > 0,\ \gamma > 0,\ \omega > 0$. The exact solution is $y(t) = 1 - t^{1-\mu(t)}E^{\gamma}_{\rho,2-\mu(t)}(\omega t^{\rho})$. We use the proposed method on the domain $[0,1]$ to this example and demonstrated the numerical results in Fig. 1. By choosing $\mu(t) = \frac{1-|\cos(t)|}{4}$, the approximate solutions described by various values of M together with the exact solution are sketched in Fig. 1. From the obtained numerical results, we see that the approximate solution converges to the exact one by increasing the number of basis functions. The absolute error at some selected points are shown for two constant orders $\mu = 0.5$ and $\mu(t) = \frac{1-|\cos(t)|}{4}$ with different M in Table 1 and the absolute error function depicted in Fig. 2.

Example 5.2. *Consider the variable-order fractional Bagley-Torvik equation including the*

Table 3. The absolute error of Example 5.3 for two cases of $\mu(t)$ with $\rho = 0.75,\ \omega = 1,\ \gamma = 1.5$ and different M.

$\mu(t)$	h	M	AE
0.5	0.02	15	$3.892e-08$
	0.01	20	$3.877e-08$
	0.005	25	$3.556e-08$
$1-\frac{1}{2}e^{-t}$	0.02	15	$3.529e-08$
	0.01	20	$3.5016e-08$
	0.005	25	$3.464e-08$

Hilfer-Prabhakar derivative as follows:

$$\begin{aligned} &y'(t) + {}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+} y(t) + y(t) \\ &= 2 + t + t^{1-\mu(t)}\left(E^{\gamma}_{\rho,2-\mu(t)}(\omega t^{\rho}) + tE^{\gamma}_{\rho,3-\mu(t)}(\omega t^{\rho})\right) + \tfrac{t^{-2\mu(t)}}{\Gamma(3-2\mu(t))}, \\ &y(0) = y'(0) = 1,\ 1 < \mu(t) \le 2, \end{aligned} \tag{69}$$

where $\rho > 0,\ \gamma > 0,\ \omega > 0$. The exact solution is $y(t) = 1 + t + t^{2-\mu(t)}E^{\gamma}_{\rho,3-\mu(t)}(\omega t^{\rho})$. The numerical results described by the proposed method on the domain $[0, \frac{\pi}{2}]$ are plotted in Figure 3 and Table 2. In Figure 3, we display the numerical solutions reported by different values of M and $\mu(t) = 1 + 0.5|\sin t|$. Also, the absolute error at some selected points are depicted in Table 2 for two constant orders $\mu = 1.5$ and $\mu(t) = 1 + 0.5|\sin t|$ with different M and the absolute error function depicted in Fig. 4.

Example 5.3. *Consider the class of variable order fractional integral-differential equation of the form:*

$$\begin{aligned} &y'(t) - {}^{C}\mathfrak{H}^{\gamma,\mu(t)}_{\rho,\omega,0^+} y(t) \\ &= t^{-\mu(t)}E^{\gamma}_{\rho,1-\mu(t)}(\omega t^{\rho}) - \tfrac{t^{1-2\mu(t)}}{\Gamma(2-2\mu(t))}, \\ &y(0) = 1,\ 0 < \mu(t) \le 1, \end{aligned} \tag{70}$$

where $\rho > 0,\ \gamma > 0,\ \omega > 0$. The exact solution is $y(t) = 1 + t^{1-\mu(t)}E^{\gamma}_{\rho,2-\mu(t)}(\omega t^{\rho})$. The numerical results described by the proposed method on the domain $[0, 1]$ are plotted in Figure 5 and Table 3. In Figure 5, we display the numerical solutions reported by different values of M and $\mu(t) = 1 - \frac{1}{2}e^{-t}$.

§6 Conclusion

In this paper, we constructed a numerical method based on the Jacobi wavelet functions of variable-order fractional derivative in terms of Hilfer-Prabhakar sense. For this purpose, the operational matrices of the derivative were derived. The matrix operator is used to approximate the numerical solution of a class of linear fractional differential equations as Basset and Bagley-Torvik equations. The solution obtained using the suggested method shows that this approach can coincidence with the exact solution and the validity of error correction.

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