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Propagation of traveling wave solutions to the Vakhnenko-Parkes dynamical equation via modified mathematical methods

Aly R. Seadawy[1,*] Asghar Ali[2]
Wafaa A. Albarakati[3] Dumitru Baleanu[4]

Abstract. In this paper, we investigate some new traveling wave solutions to Vakhnenko-Parkes equation via three modified mathematical methods. The derived solutions have been obtained including periodic and solitons solutions in the form of trigonometric, hyperbolic, and rational function solutions. The graphical representations of some solutions by assigning particular values to the parameters under prescribed conditions in each solutions and comparing of solutions with those gained by other authors indicate that these employed techniques are more effective, efficient and applicable mathematical tools for solving nonlinear problems in applied science.

§1 Introduction

In different areas of applied science, the explorations of exact travelling wave solutions have been played a vital role for demonstrating wave the character of nonlinear problems. For plentiful expression of dilemmas in mathematical physics several nonlinear wave systems have discussed such as the phenomena flow of heat, plasma physics and optical fibers, biology, oceanology, solid state physics, chemical physics and geometry. The physical phenomena and processes which take place in nature normally have complicated nonlinear features. This escorts to nonlinear mathematical models for the real processes. There is a lot interest in the practical matters involved, as well as the progress of methods to investigate the associated nonlinear mathematical problems including nonlinear wave propagation. In recent years several powerful and efficient methods have been discovered for finding analytic solutions of nonlinear equations [1–10]. These methods include Homogeneous balance method, Extended tanh-function method, Jacobi elliptic function expansion method [11], Simple equation method [12–14], (G/G')-expansion method [15], Hirotas bilinear method [16], Exp func-

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*Corresponding author.

tion method [17], general projective Riccati equation method [18], Modified simple equation method [19–21], The extended direct algebraic method [22, 23] and auxiliary method [24]. The learning analytical solutions and further properties of soliton in [25–30].

Purpose of this article is investigating the soliton solutions of Vakhnenko-Parkes equation (VPE) by employing the three modified mathematical methods This equation having fruitful applications like Vakhnenko equation to handle the frequency waves process in relaxing medium.In previous different authors [32–35] used different methodolgy for constructing the solitary solutions of to Vakhnenko-Parkes equation with the assistance of simplest equation method, improved (G'/G)- expansion method, the ansatz methods and inverse scattering transform (IST) method respectively. Our work is giving concentration to find the analytical solutions of the Vakhnenko-Parkes equation by generalizing direct algebraic, extended simple equation and modifying F-expansion methods. The graphical demonstration 2D and 3D of some derived solutions and the discuss results segment proves valid capacity our proposed techniques. Hence these solutions are productive tools for solving numerous problems in the applied sciences.

This article is arranged as: description of proposed methods in Section 2, the applications of the methods in details in Section 3, results and discussion in Section 4, summary and conclusion in Section 5.

§2 Description proposed methods

We will consider the generalized partial differential equations as following

$$Q_1\left(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...\right) = 0, \tag{1}$$

suppose that

$$u = V(\xi), \qquad \xi = kx + \mu t, \tag{2}$$

put (2) in (1),

$$Q_2\left(V, V', V'', ...\right) = 0, \tag{3}$$

2.1 Generalized Direct Algebraic Method

Let solution of (3) has the form

$$V = \sum_{i=0}^{n} A_i \Psi^i + \sum_{i=-1}^{-n} B_{-i} \Psi^i + \sum_{i=2}^{n} C_i \Psi^{i-2} \Psi' + \sum_{i=1}^{n} D_i \left(\frac{\Psi'}{\Psi}\right)^i. \tag{4}$$

Suppose Ψ satisfies the following,

$$\Psi' = \sqrt{q_1 \Psi^2 + q_2 \Psi^3 + q_3 \Psi^4}, \tag{5}$$

where q_1, q_2, q_3 are arbitrary constants.

Put (4) with (5) in (3), attained system of collection containing k, μ, q_i. Putting these values of parameters along with solution of Ψ in (4), we achieved the require destination of (1).

2.2 Extended Simple Equation Method

Let (3) has solution as

$$V = \sum_{i=-n}^{n} a_i \Psi^i. \tag{6}$$

Let Ψ gratify

$$\Psi' = c_0 + c_1\Psi + c_2\Psi^2 + c_3\Psi^3 \tag{7}$$

Substituting (6) along with (7) into (3). After solving obtained values of the parameters, substitute these values and solution of Ψ into (7). We obtained solution of (1).

2.3 Modified F-expansion Method

Step 1: Let us suppose that (3) has solution as:

$$V = a_0 + \sum_{i=1}^{n} a_i F^i(\xi) + \sum_{i=1}^{n} b_i F^{-i}(\xi). \tag{8}$$

Let F gratifies,

$$F' = A + BF + CF^2. \tag{9}$$

Step 2: Put (10) along (11) in (3), solving for require values of the parameters.
Step 3: Selecting A, B, C and F from Table 1 in [31] and substitute a_i b_i into Eq.(5), completed for solution (1).

§3 Applications

3.1 Application of Generalized Direct Algebraic Method

Consider the general form of the Vakhnenko-Parkes Equation [32–35] as

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0. \tag{10}$$

Let

$$u = V, \quad \xi = x - \mu t, \tag{11}$$

Putting (11) in (10), yields

$$V^3 - 3\left(V'\right)^2 + 3\mathrm{VV}'' = 0 \tag{12}$$

Let (12) has solution,

$$V(\xi) = A_0 + A_1\Psi + A_2\Psi^2 + \frac{B_1}{\Psi} + \frac{B_2}{\Psi^2} + C_2\Psi' + D_1\frac{\Psi'}{\Psi} + D_2\left(\frac{\Psi'}{\Psi}\right)^2 \tag{13}$$

Put (13) along with (5) in (12), after solving, we have

$$\begin{aligned} &A_0 = -q_1 D_2, \quad A_1 = \frac{1}{2}\left(-3q_2 - 2q_2 D_2\right), \quad A_2 = q_3\left(-D_2\right) - 3q_3, \\ &B_1 = 0, \quad B_2 = 0, \quad C_2 = \pm 3\sqrt{q_3}, \quad D_1 = 0, \end{aligned} \tag{14}$$

Put (14) in (13) , we have
Case- I

$$V_1 = -\left(D_2q_3+3q_3\right)\left(-\frac{q_1\left(\epsilon\coth\left(\frac{1}{2}\left(\xi+\xi_0\right)\sqrt{q_1}\right)+1\right)}{q_2}\right)^2+$$
$$\frac{\left(-2D_2q_2-3q_2\right)\left(-q_1\left(\epsilon\coth\left(\frac{1}{2}\left(\xi+\xi_0\right)\sqrt{q_1}\right)+1\right)\right)}{2q_2}-$$
$$D_2q_1+D_2\left(\frac{q_1^{3/2}\epsilon\operatorname{csch}^2\left(\frac{1}{2}\left(\xi+\xi_0\right)\sqrt{q_1}\right)}{\frac{(2q_2)\left(-q_1\left(\epsilon\coth\left(\frac{1}{2}(\xi+\xi_0)\sqrt{q_1}\right)+1\right)\right)}{q_2}}\right)^2+\frac{+3\sqrt{q_3}\left(q_1^{3/2}\epsilon\operatorname{csch}^2\left(\frac{1}{2}\left(\xi+\xi_0\right)\sqrt{q_1}\right)\right)}{2q_2},$$
$$q_1>0,\quad q_2^2-4q_1q_3=0. \tag{15}$$

Case- II

$$V_2 = D_2\left(-\frac{\sqrt{\frac{q_1}{q_3}}\left(\frac{\sqrt{q_1}\epsilon\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}{\eta+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}-\frac{\sqrt{q_1}\epsilon\sinh^2\left((\xi+\xi_0)\sqrt{q_1}\right)}{\left(\eta+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)\right)^2}\right)}{2\left(-\sqrt{\frac{q_1}{4q_3}}\left(\frac{\epsilon\sinh\left((\xi+\xi_0)\sqrt{q_1}\right)}{\eta+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}+1\right)\right)}\right)^2+$$
$$\frac{1}{2}\sqrt{\frac{q_1}{4q_3}}\left(3q_2-2D_2q_2\right)\left(\frac{\epsilon\sinh\left(\left(\xi+\xi_0\right)\sqrt{q_1}\right)}{\eta+\cosh\left(\left(\xi+\xi_0\right)\sqrt{q_1}\right)}+1\right)+$$
$$\frac{q_1+\left(-D_2q_3-3q_3\right)\left(\frac{\epsilon\sinh\left((\xi+\xi_0)\sqrt{q_1}\right)}{\eta+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}+1\right)^2}{4q_3}-$$
$$\frac{1}{2}\sqrt{q_1}\left(\frac{\sqrt{q_1}\epsilon\cosh\left(\left(\xi+\xi_0\right)\sqrt{q_1}\right)}{\eta+\cosh\left(\left(\xi+\xi_0\right)\sqrt{q_1}\right)}-\frac{\sqrt{q_1}\epsilon\sinh^2\left(\left(\xi+\xi_0\right)\sqrt{q_1}\right)}{\left(\eta+\cosh\left(\left(\xi+\xi_0\right)\sqrt{q_1}\right)\right)^2}\right)$$
$$-D_2q_1,\quad q_1>0,\quad q_3>0,\quad q_2=\sqrt{4q_1q_3} \tag{16}$$

Case- III

$$V_3 = D_2\left(-\frac{q_1\left(\frac{\sqrt{q_1}\epsilon\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}{\eta\sqrt{p^2+1}+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}-\frac{\sqrt{q_1}\epsilon\sinh\left((\xi+\xi_0)\sqrt{q_1}\right)\left(p+\sinh\left((\xi+\xi_0)\sqrt{q_1}\right)\right)}{\left(\eta\sqrt{p^2+1}+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)\right)^2}\right)}{\frac{q_2\left(-q_1\left(\frac{\epsilon\left(p+\sinh\left((\xi+\xi_0)\sqrt{q_1}\right)\right)}{\eta\sqrt{p^2+1}+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}+1\right)\right)}{q_2}}\right)^2-$$
$$D_2q_1+\left(\frac{q_1}{2q_2}\right)\left(2D_2q_2+3q_2\right)\left(\frac{\epsilon\left(p+\sinh\left(\left(\xi+\xi_0\right)\sqrt{q_1}\right)\right)}{\eta\sqrt{p^2+1}+\cosh\left(\left(\xi+\xi_0\right)\sqrt{q_1}\right)}+1\right)+$$
$$\frac{q_1^2\left(-D_2q_3-3q_3\right)\left(\frac{\epsilon\left(p+\sinh\left((\xi+\xi_0)\sqrt{q_1}\right)\right)}{\eta\sqrt{p^2+1}+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}+1\right)^2}{q_2^2}$$
$$\frac{+3\sqrt{q_3}\left(-\left(q_1\left(\frac{\sqrt{q_1}\epsilon\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}{\eta\sqrt{p^2+1}+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)}-\frac{\sqrt{q_1}\epsilon\sinh\left((\xi+\xi_0)\sqrt{q_1}\right)\left(p+\sinh\left((\xi+\xi_0)\sqrt{q_1}\right)\right)}{\left(\eta\sqrt{p^2+1}+\cosh\left((\xi+\xi_0)\sqrt{q_1}\right)\right)^2}\right)\right)\right)}{q_2},$$
$$q_1>0 \tag{17}$$

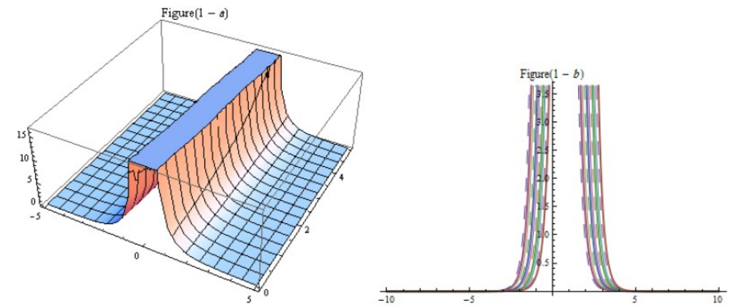

Figure 1. Traveling waves of solution of Eq.(15).

3.2 Applications of Extended Simple Equation Method

Let (12) has solution,

$$V = a_2\Psi^2 + a_1\Psi + \frac{a_{-2}}{\Psi^2} + \frac{a_{-1}}{\Psi} + a_0 \tag{18}$$

Put (18) in (12) along with (7) and after solving obained system of equations, we have

Case I: $c_3 = 0$,

Family-I

$$a_0 = -6c_0c_2, \quad a_{-2} = 0, \quad a_{-1} = 0, \quad a_2 = -6c_2^2, \quad a_1 = -6c_1c_2 \tag{19}$$

Substitute (19) in (18) with (7), then solution of Eq.(10) achieved,

$$V_4 = -6c_0c_2 + \frac{6c_1\left(c_1 - \sqrt{4c_0c_2 - c_1^2}\tan(\frac{\sqrt{4c_0c_2-c_1^2}}{2}(\xi+\xi_0))\right)}{2} - \frac{3\left(c_1 - \sqrt{4c_0c_2 - c_1^2}\tan(\frac{\sqrt{4c_0c_2-c_1^2}}{2}(\xi+\xi_0))\right)^2}{2}, \tag{20}$$

$$4c_0c_2 > c_1^2 \tag{21}$$

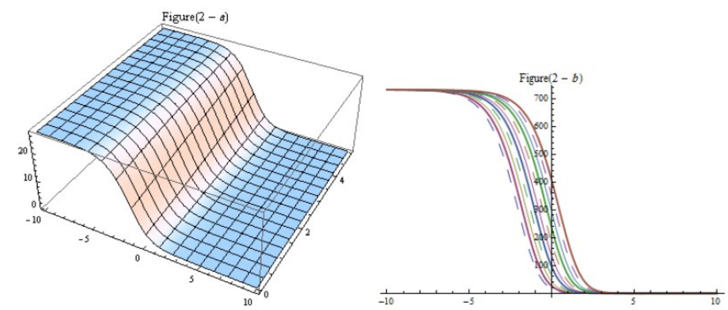

Figure 2. Traveling waves of solution of Eq.(17).

Family-II

$$a_0=-6c_0c_2,\quad a_{-2}=-6c_0^2,\quad a_{-1}=-6c_0c_1,\quad a_2=0,\quad a_1=0 \tag{22}$$

Put (22) in (18),

$$V_5=-6c_0c_2+\frac{12c_2c_0c_1}{\left(c_1-\sqrt{4c_2c_0-c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0-c_1^2}(\xi+\epsilon)\right)\right)}-\\ \frac{24c_2^2c_0^2}{\left(c_1-\sqrt{4c_2c_0-c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0-c_1^2}(\xi+\xi_0)\right)\right)^2} \tag{23}$$

$$4c_0c_2>c_1^2. \tag{24}$$

Case II: $c_0=c_3=0$,

$$a_0=0,\quad a_{-2}=0,\quad a_{-1}=0,\quad a_2=-6c_2^2,\quad a_1=-6c_1c_2 \tag{25}$$

Put (25) in (18),

$$V_6=\frac{-6c_2c_1^2e^{c_1(\xi+\xi_0)}}{(1-c_2e^{c_1(\xi+\xi_0)})}-\frac{6c_2^2c_1^2e^{2c_1(\xi+\xi_0)}}{(1-c_2e^{c_1(\xi+\xi_0)})^2}\qquad c_1>0. \tag{26}$$

$$V_7=\frac{6c_2c_1^2e^{c_1(\xi+\xi_0)}}{(1+c_2e^{c_1(\xi+\xi_0)})}-\frac{6c_2^2c_1^2e^{2c_1(\xi+\xi_0)}}{(1+c_2e^{c_1(\xi+\xi_0)})^2},\qquad c_1<0. \tag{27}$$

Case III: $\alpha_1=\alpha_3=0$,

Family-I

$$a_0=-6c_0c_2,\quad a_{-2}=0,\quad a_{-1}=0,\quad a_2=-6c_2^2,\quad a_1=0 \tag{28}$$

Put (28) in (18),

$$V_8=-6c_0c_2-6c_0c_2\left(\tan\sqrt{c_0c_2}(\xi+\xi_0)\right)^2,\quad c_2c_0>0. \tag{29}$$

$$V_9=-6c_0c_2+6c_0c_2\left(\tanh\sqrt{-c_0c_2}(\xi+\xi_0)\right)^2,\quad c_2c_0<0. \tag{30}$$

Family-II

$$a_0=-6c_0c_2,\quad a_{-2}=-6c_0^2,\quad a_{-1}=0,\quad a_2=0,\quad a_1=0 \tag{31}$$

Put (31) in (18),

$$V_{10}=-6c_0c_2-\frac{6c_0c_2}{\left(\tan\sqrt{c_0c_2}(\xi+\xi_0)\right)^2},\qquad c_0c_2>0, \tag{32}$$

$$V_{11}=-6c_0c_2+\frac{6c_0c_2}{\left(\tanh\sqrt{-c_0c_2}(\xi+\xi_0)\right)^2},\quad c_0c_2<0, \tag{33}$$

Family-III

$$a_0=-12c_0c_2,\quad a_{-2}=-6c_0^2,\quad a_{-1}=0,\quad a_2=-6c_2^2,\quad a_1=0 \tag{34}$$

Put (34) in (18),

$$V_{12}=-12c_0c_2-6c_0c_2\left(\left(\tan\sqrt{c_0c_2}(\xi+\xi_0)\right)^2+\frac{1}{\left(\tan\sqrt{c_0c_2}(\xi+\xi_0)\right)^2}\right),\quad c_0c_2>0. \tag{35}$$

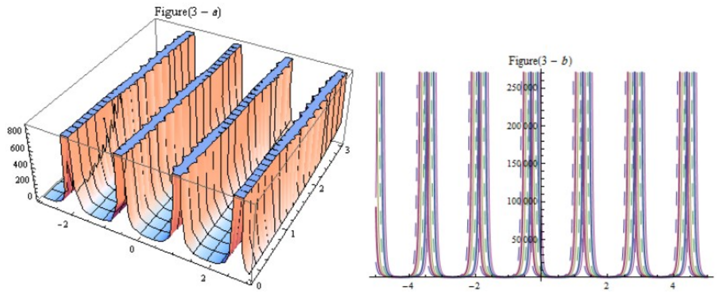

Figure 3. Traveling waves of solution of (35).

$$V_{13} = -12c_0c_2 + 6c_0c_2\left(\left(\tanh\sqrt{-c_0c_2}(\xi+\xi_0)\right)^2 + \frac{1}{\left(\tanh\sqrt{-c_0c_2}(\xi+\xi_0)\right)^2}\right), \quad c_0c_2 < 0. \tag{36}$$

3.3 Applications of Modified F-expansion Method

Suppose solution of (14) is;

$$V = a_0 + a_1F + a_2F^2 + \frac{b_1}{F} + \frac{b_2}{F^2} \tag{37}$$

Substitute (37) in (14) with (11), solved for solution of Eq.(10).

For $A = 0$, $B = 1$, $C = -1$, we have,

$$a_0 = 0, \quad a_2 = -6, \quad a_1 = 6, \quad b_1 = 0, \quad b_2 = 0 \tag{38}$$

Put (38) in (37),

$$V_{14} = 3\left(1 + \tanh(\frac{1}{2}\xi)\right) - \frac{3}{2}\left(1 + \tanh(\frac{1}{2}\xi)\right)^2 \tag{39}$$

When $A = 0$, $B = -1$, $C = 1$, then we have,

$$a_0 = 0, \quad a_2 = -6, \quad a_1 = 6, \quad b_1 = 0, \quad b_2 = 0 \tag{40}$$

Substitute (40) into (37),

$$V_{15} = 3\left(1 - \coth(\frac{1}{2}\xi)\right) - \frac{3}{2}\left(1 - \coth(\frac{1}{2}\xi)\right)^2 \tag{41}$$

For $A = \frac{1}{2}$, $B = 0$, $C = -\frac{1}{2}$, then we have,

Family-I

$$a_0 = \frac{3}{2}, \quad a_2 = 0, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = -\frac{3}{2} \tag{42}$$

Put (42) in (37),

$$V_{16} = \frac{3}{2} - \frac{3}{2}\left(\frac{1}{\coth(\xi) \pm csch(\xi)}\right)^2 \tag{43}$$

Family-II

$$a_0 = \frac{3}{2}, \quad a_2 = -\frac{3}{2}, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = 0 \tag{44}$$

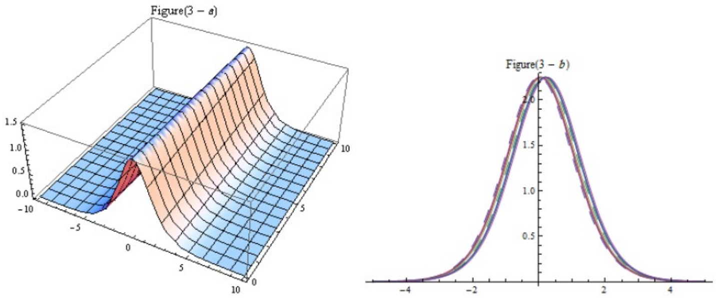

Figure 4. Traveling waves of solution of Eq.(39).

Put (44) in (37),

$$V_{17} = \frac{3}{2} - \frac{3}{2}\left(\pm csch(\xi) + \coth(\xi)\right)^2 \tag{45}$$

Family-III

$$a_0 = 3, \quad a_2 = -\frac{3}{2}, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = -\frac{3}{2} \tag{46}$$

Put (46) in (37),

$$V_{18} = 3 - \frac{3}{2}\left(\frac{1}{(\pm csch(\xi) + \coth(\xi))}\right)^2 - \frac{3}{2}\left(\pm csch(\xi) + \coth(\xi)\right)^2 \tag{47}$$

For $C = -1$, $B = 0$, $A = 1$,

Family-I

$$a_0 = 6, \quad a_2 = 0, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = -6 \tag{48}$$

Put (48) in (37),

$$V_{19} = 6 - 6\left(\frac{1}{\tanh(\xi)}\right)^2, \quad or \quad 6 - 6\left(\frac{1}{\coth(\xi)}\right)^2 \tag{49}$$

Family-II

$$a_0 = 6, \quad a_2 = -6, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = 0 \tag{50}$$

Put (50) in (37),

$$V_{20}(\xi) = 6 - 6\left(\tanh(\xi)\right)^2 \quad or \quad 6 - 6\left(\coth(\xi)\right)^2 \tag{51}$$

Family-III

$$a_0 = 12, \quad a_2 = -6, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = -6 \tag{52}$$

Put (52) in (26),

$$V_{21} = 12 - 6\left(\tanh(\xi) + \frac{1}{\tanh(\xi)}\right)^2, \quad or \quad 12 - 6\left(\coth(\xi) + \frac{1}{\coth(\xi)}\right)^2 \tag{53}$$

When $A = \frac{1}{2}$, $C = \frac{1}{2}$, $B = 0$,

Family-I

$$a_0 = -\frac{3}{2}, \quad a_2 = -\frac{3}{2}, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = 0 \tag{54}$$

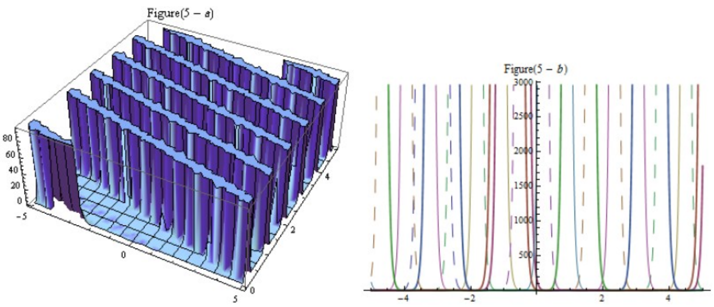

Figure 5. Traveling waves of solution of Eq.(63).

Put (54) in (37),

$$V_{22} = -\frac{3}{2} - \frac{3}{2}\left(\sec(\xi) + \tan(\xi)\right)^2 \tag{55}$$

Family-II

$$a_0 = -\frac{3}{2}, \quad a_2 = 0, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = -\frac{3}{2} \tag{56}$$

Put (56) in (37),

$$V_{23} = -\frac{3}{2} - \frac{3}{2}\left(\frac{1}{\tan(\xi) + \sec(\xi) +}\right)^2 \tag{57}$$

Family-III

$$a_0 = -3, \quad a_2 = -\frac{3}{2}, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = -\frac{3}{2} \tag{58}$$

By putting Eq.(58) in (37),

$$V_{24} = -3 - \frac{3}{2}\left(\tan(\xi) + \sec(\xi)\right)^2 - \frac{3}{2}\left(\frac{1}{\tan(\xi) + \sec(\xi)}\right)^2 \tag{59}$$

$A = -\frac{1}{2}$, $B = 0$, $C = -\frac{1}{2}$,

Family-I

$$a_0 = -\frac{3}{2}, \quad a_2 = -\frac{3}{2}, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = 0 \tag{60}$$

Put (60) in (37),

$$V_{25} = -\frac{3}{2} - \frac{3}{2}\left(\sec(\xi) - \tan(\xi)\right)^2 \tag{61}$$

Family-II

$$a_0 = -\frac{3}{2}, \quad b_2 = -\frac{3}{2}, \quad a_1 = 0, \quad b_1 = 0, \quad a_2 = 0 \tag{62}$$

Put (62) in (37),

$$V_{26} = -\frac{3}{2} - \frac{3}{2}\left(\frac{1}{\tan(\xi) - \sec(\xi)}\right)^2 \tag{63}$$

Family-III

$$a_0 = -3, \quad b_2 = -\frac{3}{2}, \quad a_1 = 0, \quad b_1 = 0, \quad a_2 = -\frac{3}{2} \tag{64}$$

Put (64) in (37),

$$V_{27} = -3 - \frac{3}{2}\left(\sec(\xi) - \tan(\xi)\right)^2 + \frac{3}{2}\left(\frac{1}{\tan(\xi) - \sec(\xi)}\right)^2 \tag{65}$$

$C=-1$, $B=0$, $A=-1$,

Family-I

$$a_0=-6, \quad b_2=0, \quad a_1=0, \quad b_1=0, \quad a_2=-6 \tag{66}$$

Put (66) in (37),

$$V_{28}=-6-6\left(\tan(\xi)\cot(\xi)\right)^2 \tag{67}$$

Family-II

$$a_0=-6, \quad a_2=0, \quad a_1=0, \quad b_1=0, \quad b_2=-6 \tag{68}$$

Put (68) in (37),

$$V_{29}=-6-6\left(\frac{1}{(\cot(\xi)\tan(\xi)}\right)^2 \tag{69}$$

Family-III

$$a_0=-12, \quad a_2=-6, \quad a_1=0, \quad b_1=0, \quad b_2=-6 \tag{70}$$

Put (70) in (37),

$$V_{30}(x,t)=-12-6\left(\frac{1}{(\tan(\xi)\cot(\xi)}\right)^2-6\left(\tan(\xi)\cot(\xi)\right)^2 \tag{71}$$

When $A=B=0$, $C_3\neq 0$, then we have,

$$a_0=0, \quad a_2=-6C^2, \quad a_1=0, \quad b_1=0, \quad b_2=0 \tag{72}$$

Put (72) in (37),

$$V_{31}=-6C^2\left(\frac{1}{C\xi+\epsilon)}\right)^2 \tag{73}$$

When $B=0$, $C=0$, then we have,

$$a_0=0, \quad a_2=0, \quad b_2=-6A^2, \quad a_1=0, \quad b_1=0, \tag{74}$$

Put (74) in (37),

$$V_{32}=-\frac{6}{\xi^2} \tag{75}$$

When $A\neq 0$, $B\neq 0$, $C=0$, then we have,

$$a_0=0, \quad a_2=0, \quad a_1=0, \quad b_1=-6AB, \quad b_2=-6A^2 \tag{76}$$

Put (76) in (37),

$$V_{33}=-6AB\left(\frac{B}{(\exp(B\xi)-A)}\right)-6A^2\left(\frac{B}{(\exp(B\xi)-A)}\right)^2 \tag{77}$$

§4 Results and Discussion

Different researchers used distinct methodology for the determination of the traveling solutions of Vakhnenko-Parkes equation [32, 33, 35]. But here we have presented novel three modified mathematical methods for construction new wave solutions. By choosing different values of A_i, B_i, C_i and D_i in Eq.(4) and a_i, b_i in Eq.(6) and Eq.(8) respectively due to catching different values, achieved several types solutions. However, some our investigated results are

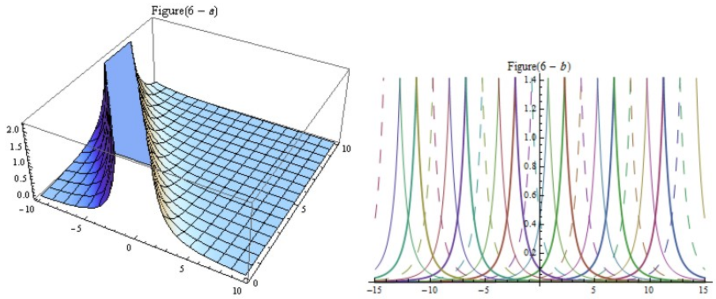

Figure 6. Traveling waves of solution of Eq.(75).

likely similar to with other researchers. Our solution (26) and (27) are in the form of exponential which are also likely similar to the solutions (23) and (24) in [32]. Solution (72) and (73) having exactly same form with involving different parameters to the solution (16) a in [33]. Our solution (30) is approximate similar to solution (92) in [35]. The left behind all our investigated solutions are novel and have a more general as compared to the slutions obtained in [32–35].

Figure(1-6) are plotted after assigning these particular values to the parameters such that, solution V_1 at $D_2 = 0.005$, $\mu = 0.15$, $q_1 = 2$, $q_2 = -4$, $q_3 = 2$, $\xi_0 = -0.07$, $\epsilon = 1$ and solution V_3 at $D_2 = 0.005$, $\eta = 1$, $\mu = -1.25$, $p = 1$, $q_1 = 1$, $q_2 = -2$, $q_3 = 10$, $\xi_0 = -0.07$, $\epsilon = -1$ and V_{12} at $c_0 = 1$ $c_2 = 1$, $\mu = 3.03$), $\xi = 0.5$ and V_{14} at $\mu = 0.05$ and V_{26} at $a_0 = 5$, $\omega = 0.05$ and V_{32} at $\mu = -1.5$ respectively. From the results discussion and graphical representations of some solutions by assigning the particular values with the assistance of Mathematica sofware, we have found that our techniques provide a rich plate form as a mathematical tools for solving nonlinear wave problem in Mathematics, physics and engineerings.

§5 Conclusion

In this work, three modified mathematical methods so called generalized direct algebraic, extended simplest equation and modified F-expansion methods are utilized for the construction of the wave solutions of Vakhnenko-Parkes equation, having a great application to control the frequency waves process in relaxing medium. The achieved solutions more general and provide a basic ground for solving many nonlinear problems.

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[1]Mathematics department, Faculty of science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia.

Email: Aly742001@yahoo.com

[2]Mathematics Department, University of Education, Lahore, Multan Campus, Pakistan.

[3]Mathematics Department, Faculty of science, King Abdulaziz University, Jeddah, Saudi Arabia.

[4]Department of Mathematics, Cankaya University, Ankara, Turkey.