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Demi-linear analysis III
—demi-distributions with compact support

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Abstract. A series of detailed quantitative results is established for the family of demi-distributions which is a large extension of the family of usual distributions.

§1 Introduction

In [1] we show that there is an entirely original generalization of the basic theory of usual distributions.

As was shown in [1] and [2], the family of demi-distributions is a large extension of the family of usual distributions, that is, the family of demi-distributions includes nonlinear functionals as many as usual distributions, at least.

The theory of demi-distributions not only contains the theory of usual distributions as a special case but causes a series of essential changes in the distribution theory. For instance, in the case of usual distributions the constant distributions are only solutions of the equation $y' = 0$ but in the case of demi-distributions the equation $y' = 0$ has tremendous solutions which are nonlinear functionals, and every constant is of course a solution of $y' = 0$ [1, Th. 2.3]. Moreover, the family of demi-distributions is closed with respect to extremely many of nonlinear transformations such as $|f(\cdot)|$, $|f(\cdot)|^{2/3}$, $\sin|f(\cdot)|$, $\mathrm{e}^{|f(\cdot)|-1}$, etc.

In this paper we carry out a detailed quantitative analysis for demi-distributions. Our vivid quantitative results show that the demi-linear mapping introduced in [2] is a very important object and, indeed, since the basic principles such as the equicontinuity theorem and the uniform boundedness principle hold for the family of demi-linear mappings [2, Th. 3.1, Th. 3.2, Th. 3.3 , Th. 4.1] and a nice duality theory has established for demi-linear dual pairs [3, Th. 3.4, Th. 3.12, Th. 3.14, Th. 3.22, Th. 3.24], it is trivial that the family of demi-linear mappings is an important extension of the family of linear operators.

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Fix an $n \in \mathbb{N}$ and a nonempty open set $\Omega \subset \mathbb{R}^n$. Let

$$C^\infty(\Omega) = \big\{\xi \in \mathbb{C}^\Omega : \xi \text{ is infinitely differentiable in } \Omega\big\},$$
$$C_0^\infty(\Omega) = \big\{\xi \in C^\infty(\Omega) : supp\,\xi \text{ is compact}\big\},$$

where $supp\,\xi = \overline{\{x \in \Omega : \xi(x) \neq 0\}} \cap \Omega$ for every $\xi \in \mathbb{C}^\Omega$ and so if $\xi \in C_0^\infty(\Omega)$ then the compact $supp\,\xi = \overline{\{x \in \Omega : \xi(x) \neq 0\}} \cap \Omega = \overline{\{x \in \Omega : \xi(x) \neq 0\}} \subset \Omega$. For every $M \subset \Omega$, $C^\infty(M) = \{\xi \in C^\infty(\Omega) : supp\,\xi \subset M\}$ and $C_0^\infty(M) = \{\xi \in C_0^\infty(\Omega) : supp\,\xi \subset M\}$ [4, p.14].

Let K be a compact subset of Ω, that is, K is bounded and closed in $\mathbb{R}^n$ and $K \subset \Omega$, and $k \in \{0, 1, 2, 3, \cdots\}$. Then

$$\|\xi\|_{K,k} = \sum_{|\alpha| \leq k} \sup_K |\partial^\alpha \xi|, \ \xi \in C^\infty(\Omega)$$

defines a seminorm on $C^\infty(\Omega)$, and the family $\big\{\|\cdot\|_{K,k} : K \text{ is compact}, K \subset \Omega, \ k \in \{0\} \cup \mathbb{N}\big\}$ gives a locally convex Fréchet topology for $C^\infty(\Omega)$, and $C^\infty(\Omega)$ has the Montel property, i.e., bounded sets in $C^\infty(\Omega)$ are relatively compact [5, 2.1].

For a compact $K \subset \Omega$ the sequence $\big\{\|\cdot\|_{K,k}\big\}_{k=0}^\infty$ gives a locally convex Fréchet topology for $C^\infty(K)$. Since $\Omega = \bigcup_{j=1}^\infty K_j$ where each K_j is compact and $K_1 \subset K_2 \subset \cdots$, with the inductive topology using the inclusion maps, $C_0^\infty(\Omega) = \bigcup_{j=1}^\infty C^\infty(K_j)$ is a (LF) space which are both barrelled and bornological. Then $C_0^\infty(\Omega)$ also has the montel property, and the inclusion map $I : C_0^\infty(\Omega) \to C^\infty(\Omega)$ is continuous [5, 2.2].

A distribution f in Ω is a continuous linear functional on $C_0^\infty(\Omega)$, that is, $f : C_0^\infty(\Omega) \to \mathbb{C}$ is linear and for every compact $K \subset \Omega$ there exist $C > 0$ and $k \in \{0\} \cup \mathbb{N}$ such that

$$\big|f(\xi)\big| \leq C \sum_{|\alpha| \leq k} \sup_K \big|\partial^\alpha \xi\big|, \quad \xi \in C_0^\infty(K) = C^\infty(K) \tag{1.1}$$

[4, Def. 2.1.1, Th. 2.1.4].

Let $C(0) = \big\{\gamma \in \mathbb{C}^\mathbb{C} : \lim_{t \to 0} \gamma(t) = \gamma(0) = 0, \ |\gamma(t)| \geq |t| \text{ if } |t| \leq 1\big\}$. For a topological vector space X, $\mathcal{N}(X)$ denotes the family of neighborhoods of $0 \in X$.

Definition 1.1. *([2, Def. 2.1]) Let X, Y be topological vector spaces over the scalar field $\mathbb{K}$. A mapping $f : X \to Y$ is said to be demi-linear if $f(0) = 0$ and there exist $\gamma \in C(0)$ and $U \in \mathcal{N}(X)$ such that every $x \in X$, $u \in U$ and $t \in \big\{t \in \mathbb{K} : |t| \leq 1\big\}$ yield r, $s \in \mathbb{K}$ for which $|r - 1| \leq |\gamma(t)|$, $|s| \leq |\gamma(t)|$ and $f(x + tu) = rf(x) + sf(u)$.*

Let $\mathscr{L}_{\gamma,U}(X, Y)$ be the family of demi-linear mappings related to $\gamma \in C(0)$ and $U \in \mathcal{N}(X)$, and let

$$\mathscr{K}_{\gamma,U}(X, Y) = \big\{f \in \mathscr{L}_{\gamma,U}(X, Y) : \text{if } x \in X, \ u \in U \text{ and } |t| \leq 1, \text{ then } f(x + tu) = f(x) + sf(u) \text{ for some } s \text{ with } |s| \leq |\gamma(t)|\big\}.$$

If $\gamma(t) = Mt$ with $M \geq 1$, then we write that $\mathscr{L}_{\gamma,U}(X, Y) = \mathscr{L}_{M,U}(X, Y)$ and $\mathscr{K}_{\gamma,U}(X, Y) = \mathscr{K}_{M,U}(X, Y)$. Moreover, if X is normed and $U = \big\{x \in X : \|x\| \leq \varepsilon\big\}$ then $\mathscr{L}_{\gamma,\varepsilon}(X, Y) = \mathscr{L}_{\gamma,U}(X, Y)$ and $\mathscr{K}_{\gamma,\varepsilon}(X, Y) = \mathscr{K}_{\gamma,U}(X, Y)$. Thus, both $\mathscr{L}_{M,\varepsilon}(\mathbb{R}, \mathbb{R})$ and $\mathscr{K}_{M,\varepsilon}(\mathbb{R}, \mathbb{R})$ are families of demi-linear functions in $\mathbb{R}^\mathbb{R}$.

Definition 1.2. *([1, Def. 1.1]) A function $f : C_0^\infty(\Omega) \to \mathbb{C}$ is called a demi-distribution if f is continuous and $f \in \mathscr{L}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$ for some $\gamma \in C(0)$ and $U \in \mathcal{N}(C_0^\infty(\Omega))$.*

Let $C_0^\infty(\Omega)^{(\gamma,U)}$ (resp., $C_0^\infty(\Omega)^{[\gamma,U]}$) be the family of demi-distributions which are functionals in $\mathscr{L}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$ (resp., $\mathscr{K}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$).

Let $C_0^\infty(\Omega)'$ be the family of usual distributions. Then

$$\mathscr{D}'(\Omega) = C_0^\infty(\Omega)' \subset C_0^\infty(\Omega)^{[\gamma,U]} \subset C_0^\infty(\Omega)^{(\gamma,U)}, \quad \forall\, \gamma \in C(0),\ U \in \mathcal{N}(C_0^\infty(\Omega))$$

and, in general, $C_0^\infty(\Omega)^{[\gamma,U]} \backslash C_0^\infty(\Omega)'$ includes nonlinear functionals as many as usual distributions, at least (see [1-3]).

Notice that the notations $\mathscr{L}_{\gamma,U}(X, Y)$, $\mathscr{K}_{\gamma,U}(X, Y)$, $C_0^\infty(\Omega)^{(\gamma,U)}$ and $C_0^\infty(\Omega)^{[\gamma,U]}$ always mean that $\gamma \in C(0)$ and $U \in \mathcal{N}(X)$ (resp., $\mathcal{N}(C_0^\infty(\Omega))$), automatically. We also have similar understanding for $\mathscr{L}_{M,U}(X, Y)$, $\mathscr{K}_{M,\varepsilon}(\mathbb{C}, \mathbb{C})$, etc.

§2 Continuity of Demi-distributions

Throughout this paper, $n \in \mathbb{N}$ and Ω is a nonempty open set in $\mathbb{R}^n$.

Definition 2.1. $\mathcal{S} \subset \mathbb{C}^\Omega$, $\mathcal{S} \neq \emptyset$. *For $\xi \in \mathcal{S}$ and $f : \mathcal{S} \to \mathbb{C}$, let*

$$supp\,\xi = \overline{\{x \in \Omega : \xi(x) \neq 0\}} \cap \Omega,$$

$$supp\, f = \{x \in \Omega : \forall \text{ open } G \subset \Omega \text{ with } x \in G\, \exists\, \xi \in \mathcal{S} \text{ with } supp\,\xi \subset G \text{ such that } f(\xi) \neq 0\}.$$

Lemma 2.1. *Let $\mathcal{S} \subset \mathbb{C}^\Omega$ with $\mathcal{S} \neq \emptyset$. For $\xi \in \mathcal{S}$ and $f \in \mathbb{C}^{\mathcal{S}}$, both $supp\,\xi$ and $supp\, f$ are closed in Ω and so both $\Omega \backslash supp\,\xi$ and $\Omega \backslash supp\, f$ are open in $\mathbb{R}^n$.*

Proof. *Let $x_k \in supp\, f$ and $x_k \to x \in \Omega$. If $x \notin supp\, f$ then there is an open $G \subset \Omega$ with $x \in G$ such that $f(\xi) = 0$ for every $\xi \in \mathcal{S}$ with $supp\,\xi \subset G$. But $x_k \in G$ eventually and so $x_k \notin supp\, f$ eventually. This contradiction shows that $x \in supp\, f$.* $\square$

Lemma 2.2. *Let $\xi \in C_0^\infty(\Omega)$ and $f \in \mathscr{L}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$. If $supp\,\xi \cap supp\, f = \emptyset$, then $f(\xi) = 0$.*

Proof. *If $\xi = 0$ then $f(\xi) = f(0) = 0$ by Def. 1.1.*

Let $\xi \neq 0$. Since $supp\,\xi \neq \emptyset$ and $supp\, f \cap supp\,\xi = \emptyset$, $supp\, f \subsetneqq \Omega$. By Lemma 2.1, for every $x \in supp\,\xi$ there is an open $G_x \subset \Omega \backslash supp\, f$ such that $x \in G_x$ and $f(\eta) = 0$ for all $\eta \in C_0^\infty(G_x)$. Since $supp\,\xi$ is compact, there exist $x_1, \cdots, x_m \in supp\,\xi$ such that $supp\,\xi \subset \bigcup_{j=1}^m G_{x_j}$ and so $\xi \in C_0^\infty(\bigcup_{j=1}^m G_{x_j})$. By Th. 1.4.4 of [4], $\xi = \sum_{j=1}^m \xi_j$ where $\xi_j \in C_0^\infty(G_{x_j})$, $j = 1, 2, \cdots, m$.

Pick a $p \in \mathbb{N}$ for which $\frac{1}{p}\xi_j \in U$, $j = 1, 2, \cdots, m$. But each $\frac{1}{p}\xi_j \in C_0^\infty(G_{x_j})$ so $f(\frac{1}{p}\xi_j) = 0$,

$j = 1, 2, \cdots, m$. *Therefore,*

$$
\begin{aligned}
f(\xi) = f\Big(\sum_{j=1}^{m}\xi_j\Big) &= f\Big(\sum_{j=1}^{m-1}\xi_j + (p-1)\frac{1}{p}\xi_m + \frac{1}{p}\xi_m\Big) \\
&= r_1 f\Big(\sum_{j=1}^{m-1}\xi_j + (p-1)\frac{1}{p}\xi_m\Big) + s_1 f\Big(\frac{1}{p}\xi_m\Big) \\
&= r_1 f\Big(\sum_{j=1}^{m-1}\xi_j + (p-2)\frac{1}{p}\xi_m + \frac{1}{p}\xi_m\Big) \\
&= r_1 r_2 \cdots r_p f\Big(\sum_{j=1}^{m-1}\xi_j\Big) = \cdots = r_1 r_2 \cdots r_{mp-1} f\Big(\frac{1}{p}\xi_1\Big) = 0. \quad \square
\end{aligned}
$$

A linear functional $f : C_0^\infty(\Omega) \to \mathbb{C}$ is continuous if and only if the condition (1.1) holds for f [4, Th. 2.1.4]. However, for demi-linear functionals in $\mathscr{L}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$ the relation between continuity and the condition (1.1) is quite complicated. First, we show that many demi-distributions satisfy the condition (1.1).

Example 2.1. *(1) Let $n = 1$ and $f(\xi) = \int_{-1}^{1} |\sin|\xi(x)||\, dx$, $\xi \in C_0^\infty(\mathbb{R})$. It is easy to see that f is not linear but $f \in C_0^\infty(\mathbb{R})^{[\gamma,U]}$ where $\gamma(t) = \frac{\pi}{2}t$ for $t \in \mathbb{C}$ and $U = \big\{\xi \in C_0^\infty(\mathbb{R}) : \sup_{|x|\le 1} |\xi(x)| \le 1\big\}$. For every compact $K \subset \mathbb{R}$ and $\xi \in C_0^\infty(K)$,*

$$
\big|f(\xi)\big| = \int_{-1}^{1} \big|\sin|\xi(x)|\big|\, dx \le \int_{-1}^{1} \big|\xi(x)\big|\, dx \le 2\sup_{x\in K}\big|\xi(x)\big| = 2\sup_{K}\big|\partial^0\xi\big|.
$$

Thus, f is a demi-distribution of order 0. Moreover, the constant $C = 2$ in (1.1) is available for all compact $K \subset \mathbb{R}$.

(2) Pick a $f \in L^1_{loc}(\mathbb{R}^n)$ with $\sup_{x\in\mathbb{R}^n} |f(x)| \le M < +\infty$ and let

$$
\big[f\big](\xi) = \int_{\mathbb{R}^n} \big|f(x)\xi(x)\big|\, dx, \quad \xi \in C_0^\infty(\mathbb{R}^n) = \mathscr{D}.
$$

Then $[f]$ is not linear but $[f] \in \mathscr{D}^{[\gamma,\mathscr{D}]}$ for every $\gamma \in C(0)$ [1, Exam. 1.1(1)].

Let K be a compact set in $\mathbb{R}^n$. Pick a cube $L \supset K$ for which $|L| = \int_L 1\, dx < +\infty$. Then

$$
\big|\big[f\big](\xi)\big| = \int_{\mathbb{R}^n} \big|f(x)\xi(x)\big|\, dx \le M \int_L \big|\xi(x)\big|\, dx \le M|L|\sup_K\big|\partial^0\xi\big|, \quad \forall\, \xi \in C_0^\infty(K).
$$

Thus, the condition (1.1) holds for the demi-distribution $[f]$. If $f(x) = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$ for all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, then $supp\,[f] = \mathbb{R}^n$ is not compact but the condition (1.1) holds for $[f]$ and $[f]$ is of order 0.

For $\gamma(t) = Mt$ where $M \ge 1$ we will show that if $f \in C_0^\infty(\Omega)^{[\gamma,U]}$ and *supp* f is compact, then the condition (1.1) holds for f and, in fact, f has a more strong property (see § 4, Th. 4.2), and Exam. 2.1(2) is interesting because this example shows that the condition (1.1) can not imply compactness of support. Moreover, the condition (1.1) fails to hold for some $f \in C_0^\infty(\Omega)^{(\gamma,U)} \backslash C_0^\infty(\Omega)^{[\gamma,U]}$, though *supp* f is compact. We show that (1.1) can be false even if *supp* $f = \{x_0\}$ is a singleton.

Example 2.2. *For $a > 0$ let $H_a(x) = 1/a$ when $0 < x < a$ and $H_a(x) = 0$ otherwise. Let $1 \geq a_0 > a_1 > a_2 > \cdots$ be a positive sequence with $\sum_{j=0}^{\infty} a_j = a < +\infty$ and $u = \lim_k (H_{a_0} * \cdots * H_{a_k})$. By Th. 1.3.5 of [4], $u \in C_0^\infty(\mathbb{R})$, supp $u \subset [0, a]$, $\int u\,dx = 1$ and*

$$\left|u^{(k)}(x)\right| \leq \frac{1}{2}\int \left|u^{(k+1)}(x)\right| dx \leq 2^k/(a_0 \cdots a_k), \quad x \in \mathbb{R},\ k = 0, 1, 2 \cdots.$$

Pick an $x_0 \in (0, a)$ for which $u(x_0) = \sup_{x\in\mathbb{R}} u(x) > 0$ and define a continuous $f : C_0^\infty(\mathbb{R}) \to \mathbb{R}$ by $f(\xi) = \mathrm{e}^{|\xi(x_0)|} - 1$, $\xi \in C_0^\infty(\mathbb{R})$. Letting $\gamma(t) = \mathrm{e}t$ for $t \in \mathbb{C}$ and $U = \{\xi \in C_0^\infty(\mathbb{R}) : \sup_{0\leq x\leq a} |\xi(x)| \leq 1\}$, it is easy to see that $f \in \mathscr{L}_{\gamma,U}(C_0^\infty(\mathbb{R}), \mathbb{R})$ and so f is a demi-distribution in $C_0^\infty(\mathbb{R})^{(\gamma,U)}$. Clearly, supp f is compact and, in fact, supp $f = \{x_0\}$.

Observe that $u \in C_0^\infty(\mathbb{R})$. Then $mu \in C_0^\infty(\mathbb{R})$ and supp $(mu) =$ supp $u \subset [0, a]$ for all $m \in \mathbb{N}$, and $f(mu) = \mathrm{e}^{|\mathrm{mu}(\mathrm{x}_0)|} - 1 = \mathrm{e}^{\mathrm{mu}(\mathrm{x}_0)} - 1 = \mathrm{e}^{\mathrm{c_m}}\mathrm{mu}(\mathrm{x}_0)$ where $\lim_{\mathrm{m}} \mathrm{e}^{\mathrm{c_m}} = +\infty$. Now let $C > 0$ and $k \in \mathbb{N} \cup \{0\}$. There is an $m_0 \in \mathbb{N}$ such that

$$\mathrm{e}^{\mathrm{c_m}} > \mathrm{C}(\mathrm{k}+1)\frac{2^{\mathrm{k}}}{\mathrm{u}(\mathrm{x}_0)\mathrm{a}_0\mathrm{a}_1\cdots\mathrm{a}_\mathrm{k}},\ \forall\,\mathrm{m} \geq \mathrm{m}_0.$$

Then $|f(mu)| = \mathrm{e}^{\mathrm{c_m}}\mathrm{mu}(\mathrm{x}_0) > \mathrm{C}(\mathrm{k}+1)\mathrm{m}\frac{2^{\mathrm{k}}}{\mathrm{a}_0\mathrm{a}_1\cdots\mathrm{a}_\mathrm{k}} > \mathrm{C}\sum_{\mathrm{j}=0}^{\mathrm{k}} \mathrm{m}\frac{2^{\mathrm{j}}}{\mathrm{a}_0\mathrm{a}_1\cdots\mathrm{a}_\mathrm{j}} \geq \mathrm{C}\sum_{\mathrm{j}=0}^{\mathrm{k}} |(\mathrm{mu})^{(\mathrm{j})}(\mathrm{x})|$, $\forall\,\mathrm{m} \geq \mathrm{m}_0,\ \mathrm{x} \in \mathbb{R}$.

Thus, for every $C > 0$ and $k \in \mathbb{N} \cup \{0\}$ there exists $m_0 \in \mathbb{N}$ such that $mu \in C_0^\infty([0, a])$ for all $m \geq m_0$ and $\left|f(mu)\right| > C\sum_{j=0}^{k} \sup_{x\in[0,a]} \left|(mu)^{(j)}(x)\right|$, $\forall\, m \geq m_0$, that is, the condition (1.1) fails to hold for f.

However, for demi-distributions there is a simple condition impling (1.1).

Theorem 2.1. *Let $f \in C_0^\infty(\Omega)^{(\gamma,U)}$. If there is an $\varepsilon > 0$ such that*

$$\varepsilon\left|tf(\xi)\right| \leq \left|f(t\xi)\right|, \quad \forall\, t > 0,\ \xi \in C_0^\infty(\Omega), \tag{2.1}$$

then the condition (1.1) holds for f.

Proof. *If the conclusion fails, there is a compact $K \subset \Omega$ such that*

$$\forall\, j \in \mathbb{N}\, \exists\, \xi_j \in C_0^\infty(K) \text{ such that } \left|f(\xi_j)\right| > j \sum_{|\alpha|\leq j} \sup_K \left|\partial^\alpha \xi_j\right|. \tag{2.2}$$

Then $|f(\xi_j)| > 0$, $\xi_j \neq 0$, $\xi_j(x_j) \neq 0$ for some $x_j \in K$ and

$$\sum_{|\alpha|\leq j} \sup_K \left|\partial^\alpha \xi_j\right| \geq \sup_K \left|\partial^0 \xi_j\right| \geq \left|\xi_j(x_j)\right| > 0, \quad j = 1, 2, 3, \cdots.$$

It follows from (2.1) and (2.2) that

$$\varepsilon < \varepsilon\left|\frac{f(\xi_j)}{j\sum_{|\alpha|\leq j} \sup_K |\partial^\alpha \xi_j|}\right| \leq \left|f\big(\frac{\xi_j}{j\sum_{|\alpha|\leq j} \sup_K |\partial^\alpha \xi_j|}\big)\right|,\ j = 1, 2, 3, \cdots. \tag{2.3}$$

Let β be a multi-index. Then $\sup_{x\in K} \left|\partial^\beta\big(\frac{\xi_j}{j\sum_{|\alpha|\leq j} \sup_K |\partial^\alpha \xi_j|}\big)(x)\right| \leq \frac{1}{j}$, $\forall\, j \geq |\beta|$,

$$\lim_{j\to\infty} \sup_K \left|\partial^\beta\big(\frac{\xi_j}{j\sum_{|\alpha|\leq j} \sup_K |\partial^\alpha \xi_j|}\big)\right| = 0, \quad \forall \text{ multi-index } \beta. \tag{2.4}$$

Observing $\{\xi_j/j\sum_{|\alpha|\leq j} \sup_K |\partial^\alpha \xi_j|\} \subset C_0^\infty(K)$, it follows from (2.4) that $\frac{\xi_j}{j\sum_{|\alpha|\leq j} \sup_K |\partial^\alpha \xi_j|} \to 0$ in $C_0^\infty(\Omega)$. Since $f \in C_0^\infty(\Omega)^{(\gamma,U)}$ is continuous, $f\big(\frac{\xi_j}{j\sum_{|\alpha|\leq j} \sup_K |\partial^\alpha \xi_j|}\big) \to 0$ but this con-

tradicts (2.3). $\square$

Theorem 2.2. *Every nonzero usual distribution $f \in \mathscr{D}'(\Omega)$ produces uncountably many of nonlinear demi-distributions satisfying the conditions (2.1) and (1.1).*

Proof. *Let $f \in \mathscr{D}'(\Omega)$, $f \neq 0$. There is $U \in \mathcal{N}(C_0^\infty(\Omega))$ such that $|f(\eta)| \leq 1$ for all $\eta \in U$.*

Pick a nonlinear continuous $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = x$ when $|x| \leq 1$, $\frac{1}{2} \leq \frac{h(b)-h(a)}{b-a} \leq 1$ when $a < b$. Clearly, $\mathbb{R}^\mathbb{R}$ includes uncountably many of this kind functions. For every $\varepsilon \in (0, \frac{1}{2}]$ and $x \in \mathbb{R}$ we have $\varepsilon|x| \leq \big|h(x)\big| \leq |x|$, $\varepsilon|x| \leq \big|h(|x|)\big| = h\big(|x|\big) \leq |x|$.

By Th. 1.1 of [2], $h \in \mathscr{K}_{1,1}(\mathbb{R}, \mathbb{R})$, that is, for $x \in \mathbb{R}$ and $u, t \in [-1, 1]$ we have that $h(x+tu) = h(x)+sh(u)$ where $|s| \leq |t|$. Then for $\xi \in C_0^\infty(\Omega)$, $\eta \in U$ and $|t| \leq 1$, $h(|f(\xi+t\eta)|) = h(|f(\xi) + tf(\eta)|) = h(|f(\xi)| + s|f(\eta)|)$ where $|s| \leq |t| \leq 1$ and, therefore, $h\big(|f(\xi + t\eta)|\big) = h\big(|f(\xi)|\big) + s'h\big(|f(\eta)|\big)$, $|s'| \leq |s| \leq |t| \leq |\gamma(t)|$, $\forall\, \gamma \in C(0)$. This shows that $h(|f(\cdot)|) \in C_0^\infty(\Omega)^{[\gamma,U]}\backslash\mathscr{D}'(\Omega)$, $\forall\, \gamma \in C(0)$.

Let $0 < \varepsilon \leq \frac{1}{2}$. Then $\varepsilon\big|th\big(|f(\xi)|\big)\big| \leq \varepsilon\big|tf(\xi)\big| = \varepsilon\big|f(t\xi)\big| \leq \big|h\big(|f(t\xi)|\big)\big|$, $\forall\, t \in \mathbb{R}$, $\xi \in C_0^\infty(\Omega)$. So (2.1) holds for $h(|f(\cdot)|)$. By Th. 2.1, $h(|f(\cdot)|)$ satisfies the condition (1.1), and the usual distribution f produces uncountably many of this kind nonlinear demi-distributions. $\square$

We also are interested in the converse implications.

Theorem 2.3. *Let $f \in \mathscr{L}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$. If the condition (1.1) holds for f, then f is sequentially continuous.*

Proof. *Suppose $\xi_j \to \xi$ in $C_0^\infty(\Omega)$. Then $\xi_j - \xi \to 0$ and there is a compact $K \subset \Omega$ such that supp $(\xi_j - \xi) \subset K$ for all j [4, p.35]. Moreover, there exist sequences $t_j \to 0$ in $\mathbb{C}$ and $\eta_j \to 0$ in $C_0^\infty(\Omega)$ such that $\xi_j - \xi = t_j\eta_j$ for all j [6, Exam. 2]. We may assume that $|t_j| \leq 1$ and $\eta_j \in U$ for all j. Then*

$$f(\xi_j) - f(\xi) = f(\xi + \xi_j - \xi) - f(\xi) = f(\xi + t_j\eta_j) - f(\xi) = (r_j - 1)f(\xi) + s_jf(\eta_j),$$

where $|r_j - 1| \leq |\gamma(t_j)| \to 0$ and $|s_j| \leq |\gamma(t_j)| \to 0$.

If $t_j = 0$ then $f(\xi_j) = f(\xi + t_j\eta_j) = f(\xi)$ so we may assume that $t_j \neq 0$ for all j. Then supp η_j = supp $(t_j\eta_j)$ = supp $(\xi_j - \xi) \subset K$ for all j and by the condition (1.1) there exist $C > 0$ and $k \in \mathbb{N} \cup \{0\}$ such that $\big|f(\eta_j)\big| \leq C \sum_{|\alpha|\leq k} \sup_K \big|\partial^\alpha\eta_j\big| \to 0$ as $j \to \infty$ since $\eta_j \to 0$ in $C_0^\infty(\Omega)$, so $f(\eta_j) \to 0$. Thus, $f(\xi_j) - f(\xi) = (r_j - 1)f(\xi) + s_jf(\eta_j) \to 0$. $\square$

Let $\gamma_0 \in C(0)$, $\gamma_0(t) = t$ for $t \in \mathbb{C}$. For every $U \in \mathcal{N}(C_0^\infty(\Omega))$ the family $\mathscr{K}_{1,U}(C_0^\infty(\Omega), \mathbb{C}) = \mathscr{K}_{\gamma_0,U}(C_0^\infty(\Omega), \mathbb{C})$ includes all linear functionals and much more nonlinear functionals, e.g., for every nonzero linear $f : C_0^\infty(\Omega) \to \mathbb{C}$, $|f(\cdot)|$ is nonlinear but $|f(\cdot)| \in \mathscr{K}_{1,U}(C_0^\infty(\Omega), \mathbb{C})$, $\forall\, U \in \mathcal{N}(C_0^\infty(\Omega))$.

Theorem 2.4. *If $f \in \mathscr{K}_{1,U}(C_0^\infty(\Omega), \mathbb{C})$ and the condition (1.1) holds for f, then f is continuous and so f is a demi-distribution in $C_0^\infty(\Omega)^{[\gamma_0,U]}$.*

Proof. *By Th. 2.3, f is sequentially continuous. Since $C_0^\infty(\Omega)$ is bornological, $C_0^\infty(\Omega)$ is C-sequential and f is continuous by Th. 1.1 of [1].* $\square$

In § 4 we will improve this result (see Cor. 4.2).

§3 Extensions of Demi-distributions

For every $C \geq 1$ and $\varepsilon > 0$, $\mathscr{K}_{C,\varepsilon}(\mathbb{C},\mathbb{C})$ includes uncountably many nonlinear functionals, and $h \circ f$ is a demi-distribution in Ω for every $h \in \mathscr{L}_{\gamma,\varepsilon}(\mathbb{C},\mathbb{C})$ and $f \in \mathscr{D}'(\Omega)$ (see [1, Th. 1.5, Cor. 1.1]).

Theorem 3.1. *If $h \in \mathscr{L}_{\gamma,\varepsilon}(\mathbb{C},\mathbb{C})$, $h \neq 0$ and $f \in \mathscr{D}'(\Omega)$, then $h \circ f$ is a demi-distribution in Ω and*

$$\mathit{supp}\,\big(h \circ f\big) = \mathit{supp}\, f.$$

Proof. *If $x \in \Omega \backslash \mathit{supp}\, f$ then there is an open $N_x \subset \Omega$ such that $x \in N_x$ and $f(\eta) = 0$ for all $\eta \in C_0^\infty(N_x)$. Then $(h\circ f)(\eta) = h(f(\eta)) = h(0) = 0$ when $\eta \in C_0^\infty(N_x)$ and so $x \notin \mathit{supp}\,(h \circ f)$. Thus $\mathit{supp}\,(h\circ f) \subset \mathit{supp}\, f$.*

If $u \in \mathbb{C}$ such that $0 < |u| < \varepsilon$ and $h(u) = 0$, then for every $z \in \mathbb{C}$ there is a $p \in \mathbb{N}$ such that $\frac{1}{p}|\frac{z}{u}| \leq 1$ and $h(z) = h(p\frac{z}{pu}u) = h((p-1)\frac{z}{pu}u + \frac{z}{pu}u) = r_1 h((p-1)\frac{z}{pu}u) + s_1 h(u) = r_1 h((p-1)\frac{z}{pu}u) = \cdots = r_1 r_2 \cdots r_{p-1} s_p h(u) = 0$. This contradicts that $h \neq 0$. Hence $h(u) \neq 0$ when $0 < |u| < \varepsilon$.

Let $x \in \mathit{supp}\, f$ and N_x an open neighborhood of x such that $N_x \subset \Omega$. Then $f(\eta) \neq 0$ for some $\eta \in C_0^\infty(N_x)$ and $0 < |\frac{1}{p}f(\eta)| < \varepsilon$ for some $p \in \mathbb{N}$. Observing f is a usual distribution, $\frac{1}{p}\eta \in C_0^\infty(N_x)$ and $(h\circ f)(\frac{1}{p}\eta) = h[f(\frac{1}{p}\eta)] = h[\frac{1}{p}f(\eta)] \neq 0$. This shows that $x \in \mathit{supp}\,(h\circ f)$ so $\mathit{supp}\, f \subset \mathit{supp}\,(h \circ f)$. $\square$

For $M \subset \Omega$ let $C_0(M) = \big\{\xi \in \mathbb{C}^\Omega : \xi \text{ is continuous}, \mathit{supp}\,\xi \text{ is compact}\big\}$. We have an analogue of Th. 1.4.4 of [4] as follows.

Lemma 3.1. *Let $\Omega_1, \cdots, \Omega_k$ be open sets in Ω and let $\xi \in C_0(\bigcup_1^k \Omega_j)$. Then one can find $\xi_j \in C_0(\Omega_j)$, $j = 1, 2, \cdots, k$, such that $\xi = \sum_1^k \xi_j$. If $\xi \geq 0$ one can take all $\xi_j \geq 0$.*

Proof. *If $x \in \mathit{supp}\,\xi$ then $x \in \Omega_j$ for some $j \in \{1, 2, \cdots, k\}$ and there is a compact neighborhood of x contained in Ω_j. Since $\mathit{supp}\,\xi$ is compact, a finite number of such neighborhoods can be chosen which cover all of $\mathit{supp}\,\xi$. Hence $\mathit{supp}\,\xi \subset \bigcup_1^k K_j$ where each K_j is compact and $K_j \subset \Omega_j$.*

By Th. 1.4.1 of [4], there is $\mathcal{X}_j \in C_0^\infty(\Omega_j)$ such that $0 \leq \mathcal{X}_j \leq 1$ and $\mathcal{X}_j = 1$ in a neighborhood of K_j, $j = 1, 2, \cdots, k$. Let

$$\xi_1 = \xi\mathcal{X}_1,\ \xi_2 = \xi\mathcal{X}_2(1-\mathcal{X}_1),\ \cdots,\ \xi_k = \xi\mathcal{X}_k(1-\mathcal{X}_1)\cdots(1-\mathcal{X}_{k-1}),$$

then each $\mathit{supp}\,\xi_j \subset \mathit{supp}\,\mathcal{X}_j \subset \Omega_j$ and $\xi = \sum_1^k \xi_j$. $\square$

Let $\mathcal{S} \subset C(\Omega)$, $M \subset \Omega$ and $\mathcal{S}(M) = \big\{\xi \in \mathcal{S} : \mathit{supp}\,\xi \subset M\big\}$. For a function $f : \mathcal{S} \to \mathbb{C}$ define $f_M : \mathcal{S}(M) \to \mathbb{C}$ by $f_M(\xi) = f(\xi)$, $\forall\, \xi \in \mathcal{S}(M)$, and f_M is called the restriction of f to M.

We now improve Th. 2.2.1 of [4].

Theorem 3.2. *Let $f \in \mathscr{L}_{\gamma,U}(C_0(\Omega),\mathbb{C})$. If every point in Ω has a neighborhood to which the restriction of f is 0, then $f=0$. The same fact is valid for $f \in \mathscr{L}_{\gamma,U}(C_0^\infty(\Omega),\mathbb{C})$.*

Proof. Let $\xi \in C_0(\Omega)$. Since $supp\,\xi$ is compact, there exist $x_1, \cdots, x_m \in supp\,\xi$ such that $supp\,\xi \subset \bigcup_1^m N_j$ where N_j is an open neighborhood of x_j such that $f_{N_j}=0$, $1 \le j \le m$. Then $\xi \in C_0(\bigcup_1^m N_j)$ and $\xi = \sum_1^m \xi_j$ where each $\xi_j \in C_0(N_j)$ by Lemma 3.1.

Pick a $p \in \mathbb{N}$ such that $\frac{1}{p}\xi$, $\frac{1}{p}\xi_j \in U$, $j=1,2,\cdots,m$. Then $\frac{1}{p}\xi = \sum_1^m \frac{1}{p}\xi_j$ and $f(\frac{1}{p}\xi) = f(\sum_1^m \frac{1}{p}\xi_j) = r_1 f(\sum_1^{m-1}\frac{1}{p}\xi_j) + s_1 f(\frac{1}{p}\xi_m) = r_1 f(\sum_1^{m-1}\frac{1}{p}\xi_j) = \cdots = r_1 r_2 \cdots r_{m-1} f(\frac{1}{p}\xi_1) = 0$ since each $\frac{1}{p}\xi_j \in C_0(N_j)$ and $f(\frac{1}{p}\xi_j) = f_{N_j}(\frac{1}{p}\xi_j) = 0$. Thus, $f(\xi) = f(p\frac{1}{p}\xi) = t_1 f((p-1)\frac{1}{p}\xi) = \cdots = t_1 t_2 \cdots t_{p-1} f(\frac{1}{p}\xi) = 0$.

The same conclusion can be obtained for $f \in \mathscr{L}_{\gamma,U}(C_0^\infty(\Omega),\mathbb{C})$ using Th. 1.4.4 of [4] instead of Lemma 3.1. □

Theorem 3.3. *If $f \in \mathscr{L}_{\gamma,U}(C_0(\Omega),\mathbb{C})$ (resp., $\mathscr{L}_{\gamma,U}(C_0^\infty(\Omega),\mathbb{C})$) and $\xi \in C_0(\Omega)$ (resp., $C_0^\infty(\Omega)$) such that $supp\,f \cap supp\,\xi = \emptyset$, then $f(\xi)=0$.*

Proof. Let $x \in supp\,\xi$. Since $x \notin supp\,f$ and $supp\,f$ is closed in Ω by Lemma 2.1, there is an open neighborhood N_x of x such that $N_x \subset \Omega \backslash supp\,f$ and the restriction $f_{N_x}=0$. Since $supp\,\xi$ is compact, there exist $x_1,\cdots,x_m \in supp\,\xi$ such that $supp\,\xi \subset \bigcup_1^m N_{x_j}$ and $f_{N_{x_j}}=0$, $j=1,2,\cdots,m$. Then $\xi \in C_0(\bigcup_1^m N_{x_j})$ (resp., $C_0^\infty(\bigcup_1^m N_{x_j})$) and $\xi = \sum_1^m \xi_j$ where $\xi_j \in C_0(N_{x_j})$ (resp., $C_0^\infty(N_{x_j})$) by Lemma 3.1 (resp., Th. 1.4.4 of [4]), $j=1,2,\cdots,m$.

Now $f(\xi)=0$ as in the proof of Th. 3.2. □

Note that Th. 3.3 is not a consequence of Th. 3.2 because for $f \ne 0$ and $x \in supp\,f$ the restriction $f_{N_x} \ne 0$ when N_x is a neighborhood of x.

Definition 3.1. *Let $f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega),\mathbb{C})$ and $\xi \in C^\infty(\Omega)$. We say that $\xi = \xi_0 + \xi_1$ is a f-decomposition of ξ if $\xi_0 \in C_0^\infty(\Omega)$ and $supp\,\xi_1 \cap supp\,f = \emptyset$.*

Observe that for every compact $K \subset \Omega$ there is a $\mathcal{X} \in C_0^\infty(\Omega)$ such that $0 \le \mathcal{X} \le 1$ and $\mathcal{X}=1$ in a neighborhood of K.

Lemma 3.2. *Let $f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega),\mathbb{C})$ and $\xi \in C^\infty(\Omega)$ such that $supp\,\xi \cap supp\,f$ is compact. If K is a compact subset of Ω such that $supp\,\xi \cap supp\,f \subseteq K$ and $\mathcal{X} \in C_0^\infty(\Omega)$ for which $\mathcal{X}=1$ in a neighborhood of K, then $\xi = \mathcal{X}\xi + (1-\mathcal{X})\xi$ is a f-decomposition of ξ: $\mathcal{X}\xi \in C_0^\infty(\Omega)$, $supp\,[(1-\mathcal{X})\xi] \cap supp\,f = \emptyset$.*

Proof. Since $supp\,(\mathcal{X}\xi) \subset supp\,\mathcal{X}$, $\mathcal{X}\xi \in C_0^\infty(\Omega)$. There is an open $G \subset \Omega$ such that $K \subset G$ and $\mathcal{X}=1$ in G. If $[(1-\mathcal{X})\xi](x) = (1-\mathcal{X}(x))\xi(x) \ne 0$, then $x \in supp\,\xi \cap (\Omega \backslash G)$ and so $supp\,[(1-\mathcal{X})\xi] \subset supp\,\xi \cap (\Omega\backslash G) \subset supp\,\xi \cap (\Omega \backslash K) \subset supp\,\xi \cap [\Omega \backslash (supp\,\xi \cap supp\,f)] \subset \Omega \backslash supp\,f$, i.e., $supp\,[(1-\mathcal{X})\xi] \cap supp\,f = \emptyset$. □

Corollary 3.1. *Let $f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega),\mathbb{C})$ and $\xi \in C_0^\infty(\Omega)$. If K is a compact subset of Ω such that $supp\,\xi \cap supp\,f \subseteq K$ and $\mathcal{X} \in C_0^\infty(\Omega)$ for which $\mathcal{X}=1$ in a neighborhood of K, then $f(\xi) = f(\mathcal{X}\xi)$.*

Theorem 3.4. *Let $f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$ and $\xi \in C^\infty(\Omega)$. If both $\xi = \xi_0 + \xi_1$ and $\xi = \eta_0 + \eta_1$ are f-decompositions of ξ, where ξ_0, $\eta_0 \in C_0^\infty(\Omega)$ and $supp\, \xi_1 \cap supp\, f = supp\, \eta_1 \cap supp\, f = \emptyset$, then $f(\xi_0) = f(\eta_0)$.*

Proof. Let $\mathcal{X} = \xi_0 - \eta_0$. Then $supp\, \mathcal{X} \subset supp\, \xi_0 \cup supp\, \eta_0$ and so $\mathcal{X} \in C_0^\infty(\Omega)$. Since $\eta_1 - \xi_1 = \xi_0 - \eta_0 = \mathcal{X}$ so $supp\, \mathcal{X} \subset supp\, \xi_1 \cup supp\, \eta_1$, $supp\, \mathcal{X} \cap supp\, f \subset (supp\, \xi_1 \cap supp\, f) \bigcup (supp\, \eta_1 \cap supp\, f) = \emptyset$.

Pick a $p \in \mathbb{N}$ for which $\frac{1}{p}\mathcal{X} \in U$. Then $supp\,(\frac{1}{p}\mathcal{X}) \cap (supp\, f) = \emptyset$ and so $f(\frac{1}{p}\mathcal{X}) = 0$ by Lemma 2.2. Then $\xi_0 = \eta_0 + \mathcal{X}$ and

$$f(\xi_0) = f\big(\eta_0 + p\frac{1}{p}\mathcal{X}\big) = f\big(\eta_0 + (p-1)\frac{1}{p}\mathcal{X}\big) = \cdots = f\big(\eta_0 + \frac{1}{p}\mathcal{X}\big) = f(\eta_0). \quad \square$$

By Lemma 3.2 and Th. 3.4 we have

Corollary 3.2. *Let $f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$ and $\xi, \eta \in C_0^\infty(\Omega)$. If $supp\,(\xi - \eta) \cap supp\, f = \emptyset$, then $f(\xi) = f(\eta)$.*

Definition 3.2. *For $f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$ let*

$$\mathcal{S}(f) = \big\{\xi \in C^\infty(\Omega) : supp\, \xi \cap supp\, f \text{ is compact}\big\},$$

and define $\widetilde{f} : \mathcal{S}(f) \to \mathbb{C}$ by

$$\widetilde{f}(\xi) = f(\xi_0) \text{ when } \xi = \xi_0 + \xi_1 \text{ is a } f\text{-decomposition of } \xi \in \mathcal{S}(f).$$

We say that $\widetilde{f}$ is the canonical extension of f. If $supp\, f$ is compact then $\mathcal{S}(f) = C^\infty(\Omega)$ and $\widetilde{f}$ is defined on $C^\infty(\Omega)$.

Theorem 3.5. *Let $f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$. Then $\mathcal{S}(f)$ is a vector subspace of $C^\infty(\Omega)$ and $C_0^\infty(\Omega) \subset \mathcal{S}(f)$. Moreover, $\widetilde{f}(\xi) = f(\xi)$, $\xi \in C_0^\infty(\Omega)$, and $\widetilde{f}(\xi) = 0$ when $\xi \in C^\infty(\Omega)$ but $supp\, \xi \cap supp\, f = \emptyset$.*

Proof. If $\xi, \eta \in \mathcal{S}(f)$ and $t \in \mathbb{C}$, then $(supp\, \xi \cap supp\, f) \bigcup (supp\, \eta \cap supp\, f)$ is compact and $supp\,(\xi + t\eta) \cap supp\, f \subset (supp\, \xi \cup supp\, \eta) \bigcap supp\, f$. This shows that $\xi + t\eta \in \mathcal{S}(f)$. If $\xi \in C_0^\infty(\Omega)$ then $supp\, \xi \cap supp\, f \subset supp\, \xi$ so $\xi \in \mathcal{S}(f)$, and $\widetilde{f} = f(\xi)$ since $\xi = \xi + 0$ is a f-decomposition of ξ.

If $\xi \in C^\infty(\Omega)$ but $supp\, \xi \cap supp\, f = \emptyset$, then $\xi = 0 + \xi$ is a f-decomposition of ξ and $\widetilde{f}(\xi) = f(0) = 0$ by Def. 1.1. $\square$

Recall that $C^\infty(\Omega)$ is a Fréchet space.

Lemma 3.3. *Let $\eta \in C^\infty(\Omega)$ and $T_\eta(\xi) = \eta\xi$ for $\xi \in C^\infty(\Omega)$. Then $T_\eta : C^\infty(\Omega) \to C^\infty(\Omega)$ is a continuous linear operator.*

Proof. Let $\xi_v \to 0$ in $C^\infty(\Omega)$. For every compact $K \subset \Omega$ and $k \in \mathbb{N} \cup \{0\}$, it follows from *the Leibniz formula that*

$$\left\|\eta\xi_v\right\|_{K,k} = \sum_{|\alpha|\le k} \sup_K \left|\partial^\alpha(\eta\xi_v)\right| \le C \sum_{|\alpha|+|\beta|\le k} \sup_K \left|\partial^\alpha \xi_v\right|\left|\partial^\beta \eta\right|$$
$$\le C \max_{|\beta|\le k} \sup_K \left|\partial^\beta \eta\right| \sum_{|\alpha|\le k} \sup_K \left|\partial^\alpha \xi_v\right| \to 0. \quad \square$$

The topology of the inductive limit $C_0^\infty(\Omega)$ is strictly stronger than the topology of the subspace $C_0^\infty(\Omega)$ of $C^\infty(\Omega)$. So the following fact is interesting and useful for further discussions.

Lemma 3.4. *Let $\mathcal{X} \in C_0^\infty(\Omega)$ and $T_\mathcal{X}(\xi) = \mathcal{X}\xi$ for $\xi \in C^\infty(\Omega)$. Then $T_\mathcal{X} : C^\infty(\Omega) \to C_0^\infty(\Omega)$ is a continuous linear operator.*

Proof. *Let $\xi_v \to 0$ in $C^\infty(\Omega)$. Since $\mathcal{X} \in C_0^\infty(\Omega)$, supp $\mathcal{X}$ is compact and supp $(\mathcal{X}\xi_v) \subset$ supp $\mathcal{X}$ for all v. By Lemma 3.3, $\mathcal{X}\xi_v \to 0$ in $C^\infty(\Omega)$ and so for every compact $K \subset \Omega$ and every multi-index α, $\lim_v \sup_K |\partial^\alpha(\mathcal{X}\xi_v)| \le \lim_v \sum_{|\beta|\le|\alpha|} \sup_K |\partial^\beta(\mathcal{X}\xi_v)| = \lim_v \|\mathcal{X}\xi_v\|_{K,|\alpha|} = 0$. Thus $\mathcal{X}\xi_v \to 0$ in $C_0^\infty(\Omega)$ and $T_\mathcal{X} : C^\infty(\Omega) \to C_0^\infty(\Omega)$ is continuous because $C^\infty(\Omega)$ is a Fréchet space.* $\square$

Lemma 3.5. *Let X, Y be topological vector spaces and $f \in \mathscr{L}_{\gamma,U}(X,Y)$. Then f is continuous if and only if f is continuous at $0 \in X$.*

Proof. *Suppose that f is continuous at $0 \in X$. Let $x \in X$ and $V \in \mathcal{N}(Y)$. Pick a balanced $W \in \mathcal{N}(Y)$ for which $W + W \subset V$.*

There is a balanced $U_0 \in \mathcal{N}(X)$ such that $U_0 \subset U$ and $f(U_0) \subset W$. Since $\lim_{t\to 0}\gamma(t) = 0$, there is a $p \in \mathbb{N}$ for which $|\gamma(\frac{1}{p})| < 1$ and $\gamma(\frac{1}{p})f(x) \in W$. If $z \in x + \frac{1}{p}U_0$, then $p(z-x) \in U_0 \subset U$ and $f(z) - f(x) = f(x+z-x) - f(x) = f[x + \frac{1}{p}p(z-x)] - f(x) = rf(x) + sf[p(z-x)] - f(x) = (r-1)f(x) + sf[p(z-x)]$, where $|r-1| \le |\gamma(\frac{1}{p})| < 1$ and $|s| \le |\gamma(\frac{1}{p})| < 1$.

If $\gamma(\frac{1}{p}) = 0$ then $r - 1 = s = 0$ so $f(z) - f(x) = 0 \in V$. If $\gamma(\frac{1}{p}) \ne 0$, then $(r-1)f(x) = \frac{r-1}{\gamma(1/p)}\gamma(1/p)f(x) \in \frac{r-1}{\gamma(1/p)}W \subset W$ and $sf[p(z-x)] \in sf(U_0) \subset sW \subset W$. So $f(z) - f(x) \in W + W \subset V$. Thus, $\frac{1}{p}U_0 \in \mathcal{N}(X)$ and $f(x + \frac{1}{p}U_0) \subset f(x) + V$, i.e., f is continuous at x. $\square$

Recall that $C_0^\infty(\Omega)^{[\gamma,U]} = \{f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C}) : f \text{ is continuous}\}$ is the family of demi-distributions, and $C_0^\infty(\Omega)^{[\gamma,U]}$ is a large extension of the family $\mathscr{D}'(\Omega)(= C_0^\infty(\Omega)')$ of usual distributions.

Theorem 3.6. *Let $f \in C_0^\infty(\Omega)^{[\gamma,U]}$ such that supp f is compact. Then there is a $V \in \mathcal{N}(C^\infty(\Omega))$ such that the canonical extension $\widetilde{f} \in C^\infty(\Omega)^{[\gamma,V]}$ and supp $\widetilde{f}$ = supp f.*

Proof. *Since supp f is compact, $\mathcal{S}(f) = C^\infty(\Omega)$ and the canonical extension $\widetilde{f}$ of f is defined on $C^\infty(\Omega)$. Pick a $\mathcal{X} \in C_0^\infty(\Omega)$ such that $\mathcal{X} = 1$ in a neighborhood of supp f. By Lemma 3.2 and Def. 3.2, $\widetilde{f}(\xi) = f(\mathcal{X}\xi)$, $\forall\, \xi \in C^\infty(\Omega)$.*

By Lemma 3.4, $V = \{\xi \in C^\infty(\Omega) : \mathcal{X}\xi \in U\} \in \mathcal{N}(C^\infty(\Omega))$. If $\xi \in C^\infty(\Omega)$, $\eta \in V$ and $|t| \le 1$, then $\widetilde{f}(\xi + t\eta) = f(\mathcal{X}\xi + t\mathcal{X}\eta) = f(\mathcal{X}\xi) + sf(\mathcal{X}\eta) = \widetilde{f}(\xi) + s\widetilde{f}(\eta)$ where $|s| \le |\gamma(t)|$. Thus $\widetilde{f} \in \mathscr{K}_{\gamma,V}(C^\infty(\Omega), \mathbb{C})$.

Let $\xi_v \to 0$ in $C^\infty(\Omega)$. By Lemma 3.4, $\mathcal{X}\xi_v \to 0$ in $C_0^\infty(\Omega)$ and so $\widetilde{f}(\xi_v) = f(\mathcal{X}\xi_v) \to f(0) = 0 = \widetilde{f}(0)$. This shows that $\widetilde{f}$ is continuous at $0 \in C^\infty(\Omega)$ since $C^\infty(\Omega)$ is a Fréchet space. Thus $\widetilde{f}$ is continuous by Lemma 3.5.

Let $x \in \Omega\backslash supp\, f$. There is an open $N_x \subset \Omega\backslash supp\, f$ such that $x \in N_x$ and $f(\eta) = 0,\ \forall\, \eta \in C_0^\infty(N_x)$. If $\xi \in C^\infty(N_x)$, then $supp\,(\mathcal{X}\xi) \subset supp\,\xi \subset N_x$ so $\widetilde{f}(\xi) = f(\mathcal{X}\xi) = 0$. Thus, $x \notin supp\,\widetilde{f}$ and so $supp\,\widetilde{f} \subset supp\, f$. Conversely, if $x \in \Omega\backslash supp\,\widetilde{f}$ then there is an open $N_x \subset supp\,\widetilde{f}$ such that $\widetilde{f}(\xi) = 0$ for all $\xi \in C^\infty(N_x)$ so $f(\eta) = \widetilde{f}(\eta) = 0,\ \forall\,\eta \in C_0^\infty(N_x)$. Then $x \notin supp\, f$ and so $supp\, f \subset supp\,\widetilde{f}$. □

Theorem 3.7. *Let $f \in C^\infty(\Omega)^{[\gamma,V]}$ and define $f_0 : C_0^\infty(\Omega) \to \mathbb{C}$ by $f_0(\xi) = f(\xi)$ for $\xi \in C_0^\infty(\Omega)$. Then $U = \{\eta \in C_0^\infty(\Omega) : \eta \in V\} \in \mathcal{N}(C_0^\infty(\Omega))$ and $f_0 \in C_0^\infty(\Omega)^{[\gamma,U]}$.*

Proof. Let $I : C_0^\infty(\Omega) \to C^\infty(\Omega)$, $I(\xi) = \xi$ for $\xi \in C_0^\infty(\Omega)$. Then I is a continuous linear operator. Hence $U = I^{-1}(V) \in \mathcal{N}(C_0^\infty(\Omega))$.

Let $\xi \in C_0^\infty(\Omega)$, $\eta \in U$ and $|t| \le 1$. Then $\xi \in C^\infty(\Omega)$ and $\eta = I(\eta) \in V$ so $f_0(\xi + t\eta) = f(\xi + t\eta) = f(\xi) + sf(\eta) = f_0(\xi) + sf_0(\eta)$ where $|s| \le |\gamma(t)|$. Thus $f_0 \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$.

If $(\xi_\lambda)_{\lambda\in\Delta}$ is a net in $C_0^\infty(\Omega)$ such that $\xi_\lambda \to \xi \in C_0^\infty(\Omega)$. Then $\xi_\lambda = I(\xi_\lambda) \to I(\xi) = \xi$ in $C^\infty(\Omega)$ and so $f_0(\xi_\lambda) = f(\xi_\lambda) \to f(\xi) = f_0(\xi)$. This shows that $f_0 : C_0^\infty(\Omega) \to \mathbb{C}$ is continuous, i.e., $f_0 \in C_0^\infty(\Omega)^{[\gamma,U]}$. □

§4 Demi-distributions with Compact Support

Definition 4.1. *Let $f \in \mathscr{L}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$ (resp., $\mathscr{L}_{\gamma,V}(C^\infty(\Omega), \mathbb{C})$) and $k \in \mathbb{N} \cup \{0\}$. If for every compact $K \subset \Omega$ there is a $C > 0$ such that*

$$|f(\xi)| \le C \sum_{|\alpha|\le k} \sup_{x\in K} |\partial^\alpha \xi(x)|,\ \forall\,\xi \in C_0^\infty(K)\ (\textit{resp., } \xi \in C^\infty(K)), \tag{1.1}$$

then we say that f is of order $\le k$.

Let $M \ge 1$ and $\gamma(t) = Mt,\ \forall\, t \in \mathbb{C}$. Then $\gamma \in C(0)$ and for every $U \in \mathcal{N}(C_0^\infty(\Omega))$ the family of demi-distributions $C_0^\infty(\Omega)^{[\gamma,U]}$ is a very large extension of $\mathscr{D}'(\Omega)$ $(= C_0^\infty(\Omega)')$, the family of usual distributions (see [2, Th. 1.1, Th. 2.1]; [1, Th. 1.5, Cor. 1.3]). By Th. 3.6, if $f \in C_0^\infty(\Omega)^{[\gamma,U]}$ has compact support, then f has an extension $\widetilde{f} \in C^\infty(\Omega)^{[\gamma,V]}$ and, conversely, every $f \in C^\infty(\Omega)^{[\gamma,V]}$ has the restriction $f|_{C_0^\infty(\Omega)} \in C_0^\infty(\Omega)^{[\gamma,U]}$, where the relations between U and V are very simple.

For $C^\infty(\Omega)^{[\gamma,U]}$ we have a very nice result as follows.

Theorem 4.1. *Let $M \ge 1$, $\gamma(t) = Mt,\ \forall\, t \in \mathbb{C},\ V \in \mathcal{N}(C^\infty(\Omega))$. Then for every $f \in C^\infty(\Omega)^{[\gamma,V]}$ there exist compact $L \subset \Omega$, $C > 0$ and $k \in \mathbb{N} \cup \{0\}$ such that*

$$|f(\xi)| \le C \sum_{|\alpha|\le k} \sup_L |\partial^\alpha \xi|,\quad \forall\,\xi \in C^\infty(\Omega). \tag{4.1}$$

Thus, $supp\, f$ is compact, the condition (1.1) holds for f, and f is of order $\le k$.

Proof. Let $P=\big\{\|\cdot\|_{K,k}: K$ is a compact subset of $\Omega,\ k\in\mathbb{N}\cup\{0\}\big\}$. The topology of $C^\infty(\Omega)$ is just given by the seminorm family P.

There exist $\|\cdot\|_1,\cdots,\|\cdot\|_p\in P$ and $\varepsilon_1,\cdots,\varepsilon_p\in(0,+\infty)$ such that

$$\bigcap_{j=1}^{p}\big\{\xi\in C^\infty(\Omega):\|\xi\|_j\le\varepsilon_j\big\}\subset V.$$

Since f is continuous and $f(0)=0$, there exist $\|\cdot\|_{p+1},\cdots,\|\cdot\|_m\in P$ and $\varepsilon_{p+1},\cdots,\varepsilon_m\in(0,+\infty)$ such that

$$\big|f(\xi)\big|<1,\quad \forall\,\xi\in\bigcap_{j=p+1}^{m}\big\{\eta\in C^\infty(\Omega):\|\eta\|_j\le\varepsilon_j\big\}.$$

Say that $\|\cdot\|_j=\|\cdot\|_{K_j,k_j},\ j=1,2,\cdots,m$, and $\theta=\min_{1\le j\le m}\varepsilon_j$. Then $\theta>0$. Letting $L=\bigcup_{j=1}^m K_j,\ k=\sum_{j=1}^m k_j$ and, simply, $\|\cdot\|=\|\cdot\|_{L,k}$, L is compact and $\|\cdot\|\in P$.

If $\xi\in C^\infty(\Omega)$ such that $\|\xi\|\le\theta$, then

$$\|\xi\|_j=\sum_{|\alpha|\le k_j}\sup_{K_j}|\partial^\alpha\xi|\le\sum_{|\alpha|\le k}\sup_{L}|\partial^\alpha\xi|=\|\xi\|\le\theta\le\varepsilon_j,\ j=1,2,\cdots,m.$$

Thus $W=\big\{\xi\in C^\infty(\Omega):\|\xi\|\le\theta\big\}\subset\bigcap_{j=1}^m\big\{\xi\in C^\infty(\Omega):\|\xi\|_j\le\varepsilon_j\big\}$ and so $W\subset V$, $\big|f(\xi)\big|<1,\ \forall\,\xi\in W$.

If $\xi\in C^\infty(\Omega)$ such that $\|\xi\|=0$, then $\|p\xi\|=p\|\xi\|=0$ for all $p\in\mathbb{N}$ so $p\xi\in W\subset V$ for all $p\in\mathbb{N}$ and $|f(\xi)|=|f(\frac{1}{p}p\xi)|=|s_pf(p\xi)|\le|s_p|\le|\gamma(\frac{1}{p})|=M\frac{1}{p}\to0$ as $p\to+\infty$. Thus, $|f(\xi)|=0\le\frac{M}{\theta}\|\xi\|$.

Let $\xi\in C^\infty(\Omega)$ with $\|\xi\|>0$. Then $0<\frac{\|\xi\|}{p\theta}\le1$ for some $p\in\mathbb{N}$, and $\|\frac{\theta}{\|\xi\|}\xi\|=\theta$, $\frac{\theta}{\|\xi\|}\xi\in W\subset V$ so $|f(\frac{\theta}{\|\xi\|}\xi)|<1$. Hence

$$\begin{aligned}\big|f(\xi)\big|&=\Big|f\big(p\frac{\|\xi\|}{p\theta}\frac{\theta}{\|\xi\|}\xi\big)\Big|=\Big|f\big[(p-1)\frac{\|\xi\|}{p\theta}\big(\frac{\theta}{\|\xi\|}\xi\big)+\frac{\|\xi\|}{p\theta}\big(\frac{\theta}{\|\xi\|}\xi\big)\big]\Big|\\&=\Big|f\big[(p-1)\frac{\|\xi\|}{p\theta}\big(\frac{\theta}{\|\xi\|}\xi\big)\big]+s_1f\big(\frac{\theta}{\|\xi\|}\xi\big)\Big|\\&\qquad\cdots\quad\cdots\\&=\Big|f\big[\frac{\|\xi\|}{p\theta}\big(\frac{\theta}{\|\xi\|}\xi\big)\big]+s_{p-1}f\big(\frac{\theta}{\|\xi\|}\xi\big)+\cdots+s_1f\big(\frac{\theta}{\|\xi\|}\xi\big)\Big|\\&=\Big|\sum_{j=1}^{p}s_jf\big(\frac{\theta}{\|\xi\|}\xi\big)\Big|=\Big|\sum_{j=1}^{p}s_j\Big|\Big|f\big(\frac{\theta}{\|\xi\|}\xi\big)\Big|\\&\le\sum_{j=1}^{p}\big|s_j\big|\le\sum_{j=1}^{p}\big|\gamma\big(\frac{\|\xi\|}{p\theta}\big)\big|=p\big|\gamma\big(\frac{\|\xi\|}{p\theta}\big)\big|=pM\frac{\|\xi\|}{p\theta}=\frac{M}{\theta}\|\xi\|.\end{aligned}$$

Thus we have that

$$\big|f(\xi)\big|\le\frac{M}{\theta}\|\xi\|=\frac{M}{\theta}\sum_{|\alpha|\le k}\sup_{L}\big|\partial^\alpha\xi\big|,\quad\forall\,\xi\in C^\infty(\Omega).$$

If $x\in\Omega\backslash L$, then there is an $\varepsilon>0$ such that

$$N_x=\big\{y\in\Omega:|y-x|=\sqrt{(y_1-x_1)^2+\cdots+(y_n-x_n)^2}\le\varepsilon\big\}\subset\Omega\backslash L$$

and $|f(\xi)| \le \frac{M}{\theta} \sum_{|\alpha|\le k} \sup_L |\partial^\alpha \xi| = 0$ *for all* $\xi \in C^\infty(N_x)$*. Thus* supp $f \subset L$ *and so* supp f *is compact.*

If K *is a compact subset of* Ω*, then for every* $\xi \in C^\infty(K)$ *we have that*

$$\big|f(\xi)\big| \le \frac{M}{\theta} \sum_{|\alpha|\le k} \sup_L \big|\partial^\alpha \xi\big| = \frac{M}{\theta} \sum_{|\alpha|\le k} \sup_{L\cap K} \big|\partial^\alpha \xi\big| \le \frac{M}{\theta} \sum_{|\alpha|\le k} \sup_K \big|\partial^\alpha \xi\big|,$$

i.e., the condition (1.1) holds for f *and* f *is of order* $\le k$*.* □

Now we can obtain many important facts by the help of Th. 4.1.

Theorem 4.2. *Let* $M \ge 1$*,* $\gamma(t) = Mt$ *for* $t \in \mathbb{C}$ *and* $U \in \mathcal{N}(C_0^\infty(\Omega))$*. If* $f \in C_0^\infty(\Omega)^{[\gamma,U]}$ *has compact support, then there exist compact* $L \subset \Omega$*,* $C > 0$ *and* $k \in \mathbb{N} \cup \{0\}$ *such that*

$$\big|f(\xi)\big| \le C \sum_{|\alpha|\le k} \sup_L \big|\partial^\alpha \xi\big|, \quad \forall\, \xi \in C_0^\infty(\Omega). \tag{4.1$'$}$$

Thus, the condition (1.1) holds for f*, and* f *is of order* $\le k$*.*

Proof. By Th. 3.6 there is a $V \in \mathcal{N}(C^\infty(\Omega))$ *such that the canonical extension* $\widetilde{f} \in C^\infty(\Omega)^{[\gamma,V]}$*. By Th. 4.1, there exist compact* $L \subset \Omega$*,* $C > 0$ *and* $k \in \mathbb{N} \cup \{0\}$ *such that*

$$\big|f(\xi)\big| = \big|\widetilde{f}(\xi)\big| \le C \sum_{|\alpha|\le k} \sup_L \big|\partial^\alpha \xi\big|, \quad \forall\, \xi \in C_0^\infty(\Omega).$$

As in the proof of Th. 4.1, (1.1) holds for f*, and* f *is of order* $\le k$*.* □

For $\gamma(t) = \mathrm{e}t \in \mathrm{C}(0)$ and $U = \big\{\xi \in C_0^\infty(\mathbb{R}) : \sup_{0\le x\le a} |\xi(x)| \le 1\big\}$ where $a > 0$, there exists demi-distribution $f \in C_0^\infty(\mathbb{R})^{(\gamma,U)}$ such that *supp* $f = \{x_0\}$ is compact but the condition (1.1) fails to hold for f (see Exam. 2.2). However, Th. 4.2 shows that if $f \in C_0^\infty(\Omega)^{[\gamma,U]}$ has compact support then not only (1.1) holds for f but the more strong (4.1)$'$ holds for f. Thus, the most important properties of demi-distributions heavily depend on the splitting degree of demi-distributions.

Theorem 4.3. *Let* $M \ge 1$*,* $\gamma(t) = Mt$ *for* $t \in \mathbb{C}$*. If* $f \in C^\infty(\Omega)^{[\gamma,V]}$ *and* $f_0(\xi) = f(\xi)$ *for* $\xi \in C_0^\infty(\Omega)$*, then* $f_0 \in C_0^\infty(\Omega)^{[\gamma,U]}$ *where* $U = V \cap C_0^\infty(\Omega) \in \mathcal{N}(C_0^\infty(\Omega))$*, and there exist compact* $L \subset \Omega$*,* $C > 0$ *and* $k \in \mathbb{N} \cup \{0\}$ *such that*

$$\big|f_0(\xi)\big| \le C \sum_{|\alpha|\le k} \sup_L \big|\partial^\alpha \xi\big|, \quad \forall\, \xi \in C_0^\infty(\Omega)$$

so supp f_0 *is compact,* supp $f_0 =$ supp f *and* f_0 *is of order* $\le k$*. Moreover,* $f = \widetilde{f_0}$*, the canonical extension of* f_0*.*

Proof. By Th. 3.7 and Th. 4.1, we only need to show $\widetilde{f_0} = f$*.*

By Th. 4.1, there exist compact $L \subset \Omega$*,* $C > 0$ *and* $k \in \mathbb{N} \cup \{0\}$ *such that*

$$\big|f(\xi)\big| \le C \sum_{|\alpha|\le k} \sup_L \big|\partial^\alpha \xi\big|, \quad \forall\, \xi \in C^\infty(\Omega).$$

If $x \in \Omega\backslash$supp f *then there is an open* $N_x \subset \Omega\backslash$supp f *such that* $f(\xi) = 0$ *for all* $\xi \in C^\infty(N_x)$ *and so* $f_0(\xi) = f(\xi) = 0$*,* $\forall\, \xi \in C_0^\infty(N_x) \subset C^\infty(N_x)$*, that is,* $x \notin$ supp f_0*. Thus* supp $f_0 \subset$ supp $f \subset L$*.*

Pick a $\mathcal{X} \in C_0^\infty(\Omega)$ such that $\mathcal{X} = 1$ in a neighborhood of L. Then by Th. 3.4 and Def. 3.2 we have that $\widetilde{f}_0(\xi) = f_0(\mathcal{X}\xi) = f(\mathcal{X}\xi),\ \forall\, \xi \in C^\infty(\Omega)$.

Let $\xi \in C^\infty(\Omega)$ and pick a $p \in \mathbb{N}$ such that $\frac{1}{p}(1-\mathcal{X})\xi \in V$. Since $1-\mathcal{X} = 0$ in a neighborhood of L, $\partial^\alpha[\frac{1}{p}(1-\mathcal{X})\xi](x) = 0$ for all $x \in L$ and all multi-index α. Then

$$\big|f\big[\frac{1}{p}(1-\mathcal{X})\xi\big]\big| \le C \sum_{|\alpha|\le k} \sup_L \big|\partial^\alpha\big[\frac{1}{p}(1-\mathcal{X})\xi\big]\big| = 0, \text{ i.e., } f\big[\frac{1}{p}(1-\mathcal{X})\xi\big] = 0,$$

$$\begin{aligned} f(\xi) &= f\big[\mathcal{X}\xi + (1-\mathcal{X})\xi\big] = f\big[\mathcal{X}\xi + (p-1)\frac{1}{p}(1-\mathcal{X})\xi + \frac{1}{p}(1-\mathcal{X})\xi\big] \\ &= f\big[\mathcal{X}\xi + (p-1)\frac{1}{p}(1-\mathcal{X})\xi\big] + sf\big[\frac{1}{p}(1-\mathcal{X})\xi\big] = f\big[\mathcal{X}\xi + (p-1)\frac{1}{p}(1-\mathcal{X})\xi\big] \\ &\quad \cdots \quad \cdots \\ &= f(\mathcal{X}\xi) = f_0(\mathcal{X}\xi) = \widetilde{f}_0(\xi). \end{aligned}$$

Thus $f = \widetilde{f}_0$ and supp f_0 = supp $\widetilde{f}_0$ = supp f by Th. 3.6. $\square$

Corollary 4.1. *Let $M \ge 1$, $\gamma(t) = Mt,\ \forall\, t \in \mathbb{C}$. Then*

$$\bigcup_{V\in\mathcal{N}(C^\infty(\Omega))} C^\infty(\Omega)^{[\gamma,V]} = \bigcup_{U\in\mathcal{N}(C_0^\infty(\Omega))} \big\{\widetilde{f} : f \in C_0^\infty(\Omega)^{[\gamma,U]},\ \text{supp}\, f \text{ is compact}\big\}.$$

Now we can improve Th. 2.4 as follows.

Corollary 4.2. *Let $M \ge 1$, $\gamma(t) = Mt$ for $t \in \mathbb{C}$. Let $f \in \mathscr{K}_{\gamma,U}(C_0^\infty(\Omega), \mathbb{C})$ for which supp f is compact. Then f is continuous if and only if the condition (1.1) holds for f.*

For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Lemma 4.1. *Let K and F be nonempty subsets of Ω. If K is compact and F is closed in Ω and $K \cap F = \emptyset$, then there exist $x_0 \in K$ and $y_0 \in \overline{F}$ such that $\inf_{x\in K, y\in F} |x-y| = |x_0-y_0| > 0$.*

Proof. Let $d = \inf_{x\in K,y\in F} |x-y|$. There exist sequences $\{x_v\} \subset K$ and $\{y_v\} \subset F$ such that $d = \lim_v |x_v - y_v|$. Since K is compact and $\{x_v - y_v\}$ is bounded, we may assume that $x_v \to x_0 \in K$ and $x_v - y_v \to b \in \mathbb{R}^n$. Then $y_v = y_v - x_v + x_v \to x_0 - b = y_0$ and $|x_v - y_v| \to |x_0 - y_0|$, $d = |x_0 - y_0|$. If $y_0 \in F$ then $y_0 \notin K$ so $y_0 \ne x_0$ and $d = |x_0 - y_0| > 0$. If $y_0 \notin F$ then $y_0 \notin \Omega$ so $y_0 \ne x_0$ and $d = |x_0 - y_0| > 0$. $\square$

We have a fact which is different from Lemma 2.2 as follows.

Theorem 4.4. *Let $M \ge 1$, $\gamma(t) = Mt$ for $t \in \mathbb{C}$. If $f \in C^\infty(\Omega)^{[\gamma,V]}$ and $\xi \in C^\infty(\Omega)$ such that supp $f \cap$ supp $\xi = \emptyset$, then $f(\xi) = 0$.*

Proof. By Th. 4.1, supp f is compact. Then $\inf\big\{|x-y| : x \in \text{supp}\, f,\ y \in \text{supp}\,\xi\big\} = d > 0$ by Lemma 4.1. Let $f_0(\xi) = f(\xi)$ for $\xi \in C_0^\infty(\Omega)$ and $U = V \cap C_0^\infty(\Omega)$. By Th. 4.3, $f_0 \in C_0^\infty(\Omega)^{[\gamma,U]}$ and the canonical extension $\widetilde{f}_0 = f$, supp f_0 = supp f. Since $d > 0$, for $\varepsilon \in (0, d/3)$ there is a $\mathcal{X} \in C_0^\infty(\Omega)$ such that $0 \le \mathcal{X} \le 1$ and $\mathcal{X} = 1$ in $G = \big\{y \in \Omega : |y-x| < \varepsilon$ for some $x \in$ supp $f_0\big\}$,

$$\mathcal{X} = 0 \text{ outside } B_{3\varepsilon} = \big\{y \in \Omega : |y-x| \le 3\varepsilon \text{ for some } x \in \text{supp}\, f_0\big\}.$$

Then $\operatorname{supp}\mathcal{X} \subset B_{3\varepsilon}$, $\operatorname{supp} f_0 \cap \operatorname{supp}(\mathcal{X}\xi) \subset \operatorname{supp} f \cap \operatorname{supp}\xi = \emptyset$ *and* $\operatorname{supp}[(1-\mathcal{X})\xi] \cap \operatorname{supp} f_0 \subset (\Omega\backslash G) \cap \operatorname{supp} f_0 = \emptyset$. *Hence* $\xi = \mathcal{X}\xi + (1-\mathcal{X})\xi$ *is a* f_0*-decomposition of* ξ *and* $\operatorname{supp}(\mathcal{X}\xi) \cap \operatorname{supp} f_0 = \emptyset$. *Then* $f(\xi) = \widetilde{f}_0(\xi) = f_0(\mathcal{X}\xi) = 0$ *by Lemma 2.2.* □

Theorem 4.5. *Let* $M \geq 1$, $\gamma(t) = Mt$ *for* $t \in \mathbb{C}$. *If* $f \in C^\infty(\Omega)^{[\gamma,V]}$ *is of order* $\leq k$ *and* $\xi \in C^\infty(\Omega)$ *such that* $\partial^\alpha\xi(x) = 0$ *when* $|\alpha| \leq k$ *and* $x \in \operatorname{supp} f$, *then* $f(\xi) = 0$.

Proof. *By Th. 4.1,* $\operatorname{supp} f$ *is compact so for sufficiently small* $\varepsilon > 0$ *the set* $B_\varepsilon = \big\{y \in \mathbb{R}^n : |y-x| \leq \varepsilon$ *for some* $x \in \operatorname{supp} f\big\}$ *is compact and contained in* Ω. *There is a* $\mathcal{X}_\varepsilon \in C_0^\infty(\Omega)$ *with* $0 \leq \mathcal{X}_\varepsilon \leq 1$ *such that* $\mathcal{X}_\varepsilon = 1$ *in a neighborhood of* $\operatorname{supp} f$ *and* $\mathcal{X}_\varepsilon = 0$ *outside* B_ε *[4, p.46]. Moreover,* $|\partial^\alpha\mathcal{X}_\varepsilon| \leq C_\alpha\varepsilon^{-|\alpha|}$ *where* C_α *is independent of* ε *[4, p.5] so there is a* $C > 0$ *such that* $|\partial^\alpha\mathcal{X}_\varepsilon| \leq C\varepsilon^{-|\alpha|}$ *for all* $|\alpha| \leq k$ *and all* $\varepsilon \in (0, \varepsilon_0]$, *where* $\varepsilon_0 > 0$ *and* $B_{\varepsilon_0} \subset \Omega$.

For $\varepsilon \in (0, \varepsilon_0]$ *pick a* $p \in \mathbb{N}$ *for which* $\frac{1}{p}(1-\mathcal{X}_\varepsilon)\xi \in V$. *Since* $\operatorname{supp}[\frac{1}{p}(1-\mathcal{X}_\varepsilon)\xi] \cap \operatorname{supp} f = \emptyset$, $f[\frac{1}{p}(1-\mathcal{X}_\varepsilon)\xi] = 0$ *by Th. 4.4. Then*

$$f(\xi) = f\big[\xi\mathcal{X}_\varepsilon + p\frac{1}{p}(1-\mathcal{X}_\varepsilon)\xi\big] = f\big[\xi\mathcal{X}_\varepsilon + (p-1)\frac{1}{p}(1-\mathcal{X}_\varepsilon)\xi\big] = \cdots = f(\xi\mathcal{X}_\varepsilon).$$

Since f *is of order* $\leq k$, *there is an* $A > 0$ *such that* $|f(\eta)| \leq A\sum_{|\alpha|\leq k}\sup_{B_\varepsilon}|\partial^\alpha\eta|$, $\forall\,\eta \in C^\infty(B_\varepsilon)$. *If* $0 < \varepsilon' < \varepsilon$ *and* $\eta \in C^\infty(B_{\varepsilon'})$, *then* $\eta \in C^\infty(B_\varepsilon)$ *and* $\sup_{B'_\varepsilon}|\partial^\alpha\eta| = \sup_{B_\varepsilon}|\partial^\alpha\eta|$ *for all* α *so* $|f(\eta)| \leq A\sum_{|\alpha|\leq k}\sup_{B_\varepsilon}|\partial^\alpha\eta| = A\sum_{|\alpha|\leq k}\sup_{B'_\varepsilon}|\partial^\alpha\eta|$. *So the constant* A *is available for all* $\varepsilon' \in (0, \varepsilon]$. *Since* $\xi\mathcal{X}_\varepsilon \in C_0^\infty(B_\varepsilon) = C^\infty(B_\varepsilon)$, *it follows from the Leibniz formula that*

$$\begin{aligned}\big|f(\xi)\big| = \big|f(\xi\mathcal{X}_\varepsilon)\big| &\leq A\sum_{|\alpha|\leq k}\sup_{B_\varepsilon}\big|\partial^\alpha(\xi\mathcal{X}_\varepsilon)\big| \leq A_1\sum_{|\alpha|+|\beta|\leq k}\sup_{B_\varepsilon}\big|\partial^\alpha\xi\big|\big|\partial^\beta\mathcal{X}_\varepsilon\big| \\ &\leq A_1C\sum_{|\alpha|\leq k}\varepsilon^{|\alpha|-k}\sup_{B_\varepsilon}\big|\partial^\alpha\xi\big|,\end{aligned}$$

where both A_1 *and* C *are independent of* ε. *Observing* $\partial^\alpha\xi(x) = 0$ *for all* $x \in \operatorname{supp} f$ *and* $|\alpha| \leq k$, *we have* $\lim_{\varepsilon\to0}\varepsilon^{|\alpha|-k}\sup_{B_\varepsilon}|\partial^\alpha\xi| = 0$ *for all* $|\alpha| \leq k$ *[4, p.46]. Thus* $f(\xi) = 0$. □

Notice that in the notation $(\bullet - a)^\alpha$ the symbol $\bullet$ denotes the variable, that is, $(\bullet - a)^\alpha$ is a function such that $[(\bullet - a)^\alpha](x) = (x-a)^\alpha = (x_1-a_1)^{\alpha_1}\cdots(x_n-a_n)_n^\alpha$ [4, p.47].

Corollary 4.3. *Let* $M \geq 1$, $\gamma(t) = Mt$ *for* $t \in \mathbb{C}$. *If* $f \in C^\infty(\Omega)^{[\gamma,V]}$ *is of order* k *and* $\operatorname{supp} f = \{y\}$, *a singleton, then we have* $f(\xi) = f\big[\sum_{|\alpha|\leq k}\partial^\alpha\xi(y)(\bullet - y)^\alpha/(\alpha!)\big]$, $\forall\,\xi \in C^\infty(\Omega)$, *where for* $0 = (0, \cdots, 0)$ *the term* $\alpha^0\xi(y)(\bullet - y)^0/(0!) = \xi(y)$ *is the function* $\eta \in C^\infty(\Omega)$ *for which* $\eta(x) = \xi(y)$, $\forall\,x \in \Omega$.

Example 4.1. *Let* $y \in \Omega$ *and* $f(\xi) = \sin|\xi(y)|$ *for* $\xi \in C^\infty(\Omega)$. *Clearly,* $V = \big\{\xi \in C^\infty(\Omega) : |\xi(y)| \leq 1\big\} \in \mathcal{N}(C^\infty(\Omega))$. *If* $\xi \in C^\infty(\Omega)$, $\eta \in V$ *and* $|t| \leq 1$, *then* $f(\xi+t\eta) = \sin|\xi(y)+t\eta(y)| = \sin(|\xi(y)| + s|\eta(y)|) = \sin|\xi(y)| + \theta\sin|\eta(y)| = f(\xi) + \theta f(\eta)$ *where* $|\theta| \leq \frac{\pi}{2}|s| \leq \frac{\pi}{2}|t|$. *Letting* $\gamma(t) = \frac{\pi}{2}t$ *for* $t \in \mathbb{C}$, $f \in C^\infty(\Omega)^{[\gamma,V]}$ *and* $\operatorname{supp} f = \{y\}$. *For every compact* $K \subset \Omega$ *and* $\xi \in C^\infty(K) = C_0^\infty(K)$ *we have that* $|f(\xi)| = |\sin|\xi(y)|| = 0 \leq \sup_K|\partial^0\xi|$ *when* $y \notin K$, $|f(\xi)| = |\sin|\xi(y)|| \leq |\xi(y)| \leq \sup_K|\xi| = \sup_K|\partial^0\xi|$ *when* $y \in K$. *Thus,* f *is of order* 0.

Corollary 4.4. *Let $M \geq 1$, $\gamma(t) = Mt$ for $t \in \mathbb{C}$. If $f \in C^\infty(\Omega)^{[\gamma,V]}$ is of order 0 and supp $f = \{y\}$, then $f(\xi) = f\big(\xi(y)\big)$, $\forall\, \xi \in C^\infty(\Omega)$, where $\xi(y)$ is a function in $C^\infty(\Omega)$ such that $\xi(y)(x) = \xi(y)$, $\forall\, x \in \Omega$.*

In [2] we gave a very clear-cut characterization of demi-linear functions in $\mathscr{K}_{M,\varepsilon}(\mathbb{R},\mathbb{R})$ [2, Th. 1.1]. We have a similar description for demi-linear functions in $\mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$ as follows.

Lemma 4.2. *Let $g : \mathbb{C} \to \mathbb{C}$ be a function such that $g(0) = 0$ and $g'(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$. Let $\varepsilon > 0$. Then $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$ for some $M \geq 1$ if and only if*

(1) g is continuous,

(2) $g(z) \neq 0$ for $0 < |z| \leq \varepsilon$,

(3) $\inf_{0<|u|\leq\varepsilon} \big|\frac{g(u)}{u}\big| > 0$,

(4) $\sup_{z,u\in\mathbb{C},\, 0<|u|\leq\varepsilon} \big|\frac{g(z+u)-g(z)}{u}\big| < +\infty$.

Proof. Suppose that $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$ where $M \geq 1$. If $z_k \to z$ in $\mathbb{C}$, then for sufficiently large $k \in \mathbb{N}$ we have that $|\frac{z_k-z}{\varepsilon}| < 1$ and $g(z_k) = g(z + \frac{z_k-z}{\varepsilon}\varepsilon) = g(z) + s_k g(\varepsilon)$ where $|s_k| \leq M|\frac{z_k-z}{\varepsilon}| \to 0$ so $g(z_k) \to g(z)$, that is, g is continuous. Assume that $g(u) = 0$ for some $0 < |u| \leq \varepsilon$ and $z \in \mathbb{C}$, $z \neq 0$. Then $|\frac{z}{ku}| < 1$ for some $k \in \mathbb{N}$ and $g(z) = g(k\frac{z}{ku}u) = g[(k-1)\frac{z}{ku}u + \frac{z}{ku}u] = g[(k-1)\frac{z}{ku}u] + s_1 g(u) = g[(k-1)\frac{z}{ku}u] = \cdots = g(\frac{z}{ku}u) = s_k g(u) = 0$. Thus $g = 0$ but $g'(z_0) \neq 0$. This contradiction shows that (2) holds for g.

If $\inf_{0<|u|\leq\varepsilon} \big|\frac{g(u)}{u}\big| = 0$, then $\frac{g(u_k)}{u_k} \to 0$ for some $\{u_k\} \subset \big\{z \in \mathbb{C} : |z| \leq \varepsilon\big\} \setminus \{0\}$. May assume that $u_k \to u_0$. If $u_0 \neq 0$, then $|\frac{g(u_k)}{u_k}| \to |\frac{g(u_0)}{u_0}| > 0$ by (1) and (2), a contradiction. So $u_0 = 0$, $u_k \to 0$. Then $g(z_0 + u_k) = g(z_0) + s_k g(u_k)$ where $|s_k| \leq M|1| = M$, $0 \neq g'(z_0) = \lim_k \frac{g(z_0+u_k)-g(z_0)}{u_k} = \lim_k \frac{s_k g(u_k)}{u_k} = 0$. This contradiction shows that (3) holds for g.

Let $z, u \in \mathbb{C}$, $0 < |u| \leq \varepsilon$. Since $g(u) = g\big(\frac{u}{\varepsilon}\varepsilon\big) = sg(\varepsilon)$ where $|s| \leq M\big|\frac{u}{\varepsilon}\big| = \frac{M}{\varepsilon}|u|$ and $g(z+u) = g(z) + s_1 g(u)$ where $|s_1| \leq M|1| = M$, $|g(u)| \leq \frac{M}{\varepsilon}|g(\varepsilon)u|$ and $\Big|\frac{g(z+u)-g(z)}{u}\Big| = \Big|\frac{s_1 g(u)}{u}\Big| \leq \frac{M^2}{\varepsilon}|g(\varepsilon)|$. Thus, (4) holds for g.

Conversely, assume that (1), (2), (3) and (4) hold for g. Since $g(0) = 0$ and $\inf_{0<|u|\leq\varepsilon} \big|\frac{g(u)}{u}\big| = \inf_{0<|u|\leq\varepsilon} \big|\frac{g(0+u)-g(0)}{u}\big| \leq \sup_{0<|u|\leq\varepsilon} \big|\frac{g(0+u)-g(0)}{u}\big|$,

$$M = \Big[\sup_{z,u\in\mathbb{C},\, 0<|u|\leq\varepsilon} \Big|\frac{g(z+u)-g(z)}{u}\Big|\Big] \Big/ \inf_{0<|u|\leq\varepsilon} \Big|\frac{g(u)}{u}\Big| \geq 1.$$

Let $z, u, t \in \mathbb{C}$, $0 < |u| \leq \varepsilon$, $0 < |t| \leq 1$. Then $g(u) \neq 0$ by (2) and $g(z + tu) = g(z) + g(z+tu) - g(z) = g(z) + \Big[\frac{g(z+tu)-g(z)}{tu}\frac{u}{g(u)}t\Big]g(u)$, where

$$\Big|\frac{g(z+tu)-g(z)}{tu}\frac{u}{g(u)}t\Big| = \Big|\frac{g(z+tu)-g(z)}{tu}\Big/\frac{g(u)}{u}\Big||t| \leq M|t|.$$

Thus, $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$. $\square$

We now have the following representation theorem.

Theorem 4.6. *Let $M \geq 1$, $\gamma(t) = Mt$ for $t \in \mathbb{C}$. If $f \in C^\infty(\Omega)^{[\gamma,V]}$ is of order 0 and supp $f = \{y\}$, then there exist $\varepsilon > 0$ and $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$ such that*

$$f(\xi) = g\big(\xi(y)\big), \quad \forall\, \xi \in C^\infty(\Omega). \tag{4.2}$$

Conversely, every $\varepsilon > 0$ and $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$ give a $V \in \mathcal{N}(C^\infty(\Omega))$ and a $f \in C^\infty(\Omega)^{[\gamma,V]}$ through (4.2) such that f is of order 0 and supp $f = \{y\}$.

Proof. Suppose that $f \in C^\infty(\Omega)^{[\gamma,V]}$ is of order 0 and supp $f = \{y\}$. For $z \in \mathbb{C}$ let $\zeta_z(x) = z$ for all $x \in \Omega$. Then $\zeta_z \in C^\infty(\Omega)$ and $\lim_{z\to 0}\zeta_z = 0$ in $C^\infty(\Omega)$. Hence there is an $\varepsilon > 0$ such that $\zeta_z \in V$ when $|z| \le \varepsilon$. Define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = f(\zeta_z)$, $\forall\, z \in \mathbb{C}$. Then $g(0) = f(\zeta_0) = f(0) = 0$. For $z, u, t \in \mathbb{C}$ with $|u| \le \varepsilon$ and $|t| \le 1$, $\zeta_u \in V$ and

$$g(z + tu) = f(\zeta_{z+tu}) = f(\zeta_z + \zeta_{tu}) = f(\zeta_z + t\zeta_u) = f(\zeta_z) + sf(\zeta_u) = g(z) + sg(u),$$

where $|s| \le |\gamma(t)| = M|t|$. Thus $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$. By Cor. 4.4 we have $f(\xi) = f(\xi(y)) = f(\zeta_{\xi(y)}) = g(\xi(y))$, $\forall\, \xi \in C^\infty(\Omega)$.

Conversely, let $\varepsilon > 0$, $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$ and $y \in \Omega$. Since the Dirac measure $\delta_y : C^\infty(\Omega) \to \mathbb{C}$, $\delta_y(\xi) = \xi(y)$ is continuous and $\delta_y(0) = 0$, $V = \{\xi \in C^\infty(\Omega) : |\xi(y)| \le \varepsilon\} = \delta_y^{-1}([-\varepsilon,\varepsilon]) \in \mathcal{N}(C^\infty(\Omega))$. Then define $f : C^\infty(\Omega) \to \mathbb{C}$ by $f(\xi) = g(\xi(y))$, $\xi \in C^\infty(\Omega)$. For $\xi \in C^\infty(\Omega)$, $\eta \in V$ and $|t| \le 1$, $|\eta(y)| \le \varepsilon$ and $f(\xi + t\eta) = g((\xi + t\eta)(y)) = g(\xi(y) + t\eta(y)) = g(\xi(y)) + sg(\eta(y)) = f(\xi) + sf(\eta)$ where $|s| \le |\gamma(t)| = M|t|$. Thus $f \in \mathscr{K}_{\gamma,V}(C^\infty(\Omega),\mathbb{C})$ and f is continuous because $\xi_\lambda \to \xi$ in $C^\infty(\Omega)$ implies $\xi_\lambda(y) \to \xi(y)$ and $f(\xi_\lambda) = g(\xi_\lambda(y)) \to g(\xi(y)) = f(\xi)$ by Lemma 4.2, that is, $f \in C^\infty(\Omega)^{[\gamma,V]}$.

Let $y_0 \in \Omega$, $y_0 \ne y$. Pick a $\theta > 0$ such that $K = \{x \in \mathbb{R}^n : |x - y_0| \le \theta\} \subset \Omega\backslash\{y\}$. Then for every $\xi \in C^\infty(K)$ we have $\xi(y) = 0$ and $f(\xi) = g(\xi(y)) = g(0) = 0$. This shows that $y_0 \notin$ supp f, supp $f \subset \{y\}$. If G is an open set in $\mathbb{R}^n$ such that $y \in G \subset \Omega$, then there is a $\mathcal{X} \in C_0^\infty(G)$ such that $0 < |\mathcal{X}(y)| \le \varepsilon$. By Lemma 4.2, $f(\mathcal{X}) = g(\mathcal{X}(y)) \ne 0$ so $y \in$ supp f and supp $f = \{y\}$.

Let $K \subset \Omega$ be compact. If $y \in K$ and $\xi \in C^\infty(K)$ then there is a $p \in \mathbb{N}$ such that $|\frac{\xi(y)}{p\varepsilon}| < 1$ and

$$\begin{aligned}
|f(\xi)| &= |g(\xi(y))| = |g(p\frac{\xi(y)}{p\varepsilon}\varepsilon)| = |g[(p-1)\frac{\xi(y)}{p\varepsilon}\varepsilon + \frac{\xi(y)}{p\varepsilon}\varepsilon]| = |g[(p-1)\frac{\xi(y)}{p\varepsilon}\varepsilon] + s_1 g(\varepsilon)| \\
&\quad \cdots \quad \cdots \\
&= |g(\frac{\xi(y)}{p\varepsilon}\varepsilon) + s_{p-1}g(\varepsilon) + \cdots + s_1 g(\varepsilon)| \\
&= |s_p g(\varepsilon) + s_{p-1}g(\varepsilon) + \cdots + s_1 g(\varepsilon)| = |\sum_{v=1}^{p} s_v||g(\varepsilon)|,
\end{aligned}$$

where each $|s_v| \le |\gamma(\frac{\xi(y)}{p\varepsilon})| = M\frac{|\xi(y)|}{p\varepsilon}$ so $|\sum_{v=1}^p s_v| \le \sum_{v=1}^p |s_v| \le pM\frac{|\xi(y)|}{p\varepsilon} = \frac{M}{\varepsilon}|\xi(y)|$. Then

$$|f(\xi)| \le \frac{M}{\varepsilon}|g(\varepsilon)||\xi(y)| \le \frac{M}{\varepsilon}|g(\varepsilon)|\sup_K|\xi| = \frac{M}{\varepsilon}|g(\varepsilon)|\sup_K|\partial^0\xi|.$$

If $y \notin K$, $\xi \in C^\infty(K)$ then $\xi(y) = 0$ and $|f(\xi)| = |g(\xi(y))| = |g(0)| = 0 \le \frac{M}{\varepsilon}|g(\varepsilon)|\sup_K|\partial^0\xi|$.

Thus, f is of order 0. Moreover, the constant $\frac{M}{\varepsilon}|g(\varepsilon)|$ is available for all compact $K \subset \Omega$. □

Corollary 4.5. *Let $M \ge 1$, $\gamma(t) = Mt$ for $t \in \mathbb{C}$. If $f \in C^\infty(\Omega)^{[\gamma,V]}$ is of order 0 and supp $f = \{y\}$, then there exist $\varepsilon > 0$ and $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C},\mathbb{C})$ such that $f(\xi) = a_\xi M\frac{g(\varepsilon)}{\varepsilon}\xi(y)$, $\forall\, \xi \in C^\infty(\Omega)$, where $|a_\xi| \le 1$. Hence, $A = M|\frac{g(\varepsilon)}{\varepsilon}| > 0$ and $|f(\xi)| \le A|\xi(y)|$, $\forall\, \xi \in C^\infty(\Omega)$.*

Corollary 4.6. *Let $M \geq 1$, $\gamma(t) = Mt$ for $t \in \mathbb{C}$. If $f \in C^\infty(\Omega)^{[\gamma,V]}$ is of order 0 and $\operatorname{supp} f = \{y\}$, then f is Lipschitz, that is, there is a $A > 0$ such that*

$$\big|f(\xi) - f(\eta)\big| \leq A\big|\xi(y) - \eta(y)\big|, \quad \forall\, \xi, \eta \in C^\infty(\Omega), \text{ and } \big|f(\xi)\big| \leq A\big|\xi(y)\big|, \quad \forall\, \xi \in C^\infty(\Omega).$$

For $z = (z_1, \cdots, z_m) \in \mathbb{C}^m$ let $|z| = \sqrt{|z_1|^2 + \cdots + |z_m|^2}$ and $\mathscr{K}_{M,\varepsilon}(\mathbb{C}^m, \mathbb{C}) = \big\{g \in \mathbb{C}^{\mathbb{C}^m} : g(0) = 0$; for $z, u \in \mathbb{C}^m$ with $|u| \leq \varepsilon$ and $t \in \mathbb{C}$ with $|t| \leq 1$, $g(z + tu) = g(z) + sg(u)$ where $|s| \leq M|t|\big\}$. For $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C}^m, \mathbb{C})$ and $1 \leq j \leq m$ define $g_j : \mathbb{C} \to \mathbb{C}$ by $g_j(w) = g((0, \cdots, 0, \overset{(j)}{w}, 0, \cdots, 0))$, $\forall\, w \in \mathbb{C}$, then $g_j \in \mathscr{K}_{M,\varepsilon}(\mathbb{C}, \mathbb{C})$.

If $k \in \mathbb{N}$ then $\big\{\text{multi-index } \alpha : |\alpha| \leq k\big\}$ is a finite set $\big\{\alpha_1, \alpha_2, \cdots, \alpha_{m_k}\big\}$ which is lexicographically ordered such that $\alpha_1 = (0, \cdots, 0)$, $\alpha_2 = (0, \cdots, 0, 1)$, $\cdots$, $\alpha_{m_k} = (k, 0, \cdots, 0)$, and we can write $(z_{\alpha_1}, z_{\alpha_2}, \cdots, z_{\alpha_{m_k}}) = (z_\alpha)_{|\alpha| \leq k}$ in $\mathbb{C}^{m_k}$.

Theorem 4.7. *Let $M \geq 1$, $\gamma(t) = Mt$ for $t \in \mathbb{C}$. If $f \in C^\infty(\Omega)^{[\gamma,V]}$ is of order k and $\operatorname{supp} f = \{y\}$, then there exist $\varepsilon > 0$ and $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C}^{m_k}, \mathbb{C})$ such that*

$$f(\xi) = g\Big(\big(\partial^\alpha \xi(y)\big)_{|\alpha| \leq k}\Big), \quad \forall\, \xi \in C^\infty(\Omega).$$

Proof. *Letting $\eta_\alpha = (\bullet - y)^\alpha/(\alpha!)$ for $|\alpha| \leq k$, we have $f(\xi) = f\Big(\sum_{|\alpha| \leq k} \partial^\alpha \xi(y)\eta_\alpha\Big)$, $\forall\, \xi \in C^\infty(\Omega)$ by Cor. 4.3. Pick a $U \in \mathcal{N}(C^\infty(\Omega))$ for which $\overbrace{U + U + \cdots + U}^{(m_k)} \subset V$. There is an $\varepsilon > 0$ such that $u\eta_\alpha \in U$ when $u \in \mathbb{C}$ with $|u| \leq \varepsilon$ and $|\alpha| \leq k$. Hence*

$$\sum_{|\alpha| \leq k} u_\alpha \eta_\alpha \in \overbrace{U + U \cdots + U}^{m_k} \subset V, \quad \forall\, (u_\alpha)_{|\alpha| \leq k} \in \mathbb{C}^{m_k} \text{ with } |(u_\alpha)_{|\alpha| \leq k}| \leq \varepsilon.$$

Define $g : \mathbb{C}^{m_k} \to \mathbb{C}$ by

$$g\Big(\big(z_\alpha\big)_{|\alpha| \leq k}\Big) = f\Big(\sum_{|\alpha| \leq k} z_\alpha \eta_\alpha\Big), \quad \forall\, (z_\alpha)_{|\alpha| \leq k} \in \mathbb{C}^{m_k}.$$

Then for $(z_\alpha)_{|\alpha| \leq k}$, $(u_\alpha)_{|\alpha| \leq k} \in \mathbb{C}^{m_k}$ with $|(u_\alpha)_{|\alpha| \leq k}| \leq \varepsilon$ and $t \in \mathbb{C}$ with $|t| \leq 1$, $\sum_{|\alpha| \leq k} u_\alpha \eta_\alpha \in V$ and

$$\begin{aligned}
g\Big(\big(z_\alpha\big)_{|\alpha| \leq k} + t\big(u_\alpha\big)_{|\alpha| \leq k}\Big) &= g\Big(\big(z_\alpha + tu_\alpha\big)_{|\alpha| \leq k}\Big) = f\Big(\sum_{|\alpha| \leq k} (z_\alpha + tu_\alpha)\eta_\alpha\Big) \\
&= f\Big(\sum_{|\alpha| \leq k} z_\alpha \eta_\alpha + t\sum_{|\alpha| \leq k} u_\alpha \eta_\alpha\Big) = f\Big(\sum_{|\alpha| \leq k} z_\alpha \eta_\alpha\Big) + sf\Big(\sum_{|\alpha| \leq k} u_\alpha \eta_\alpha\Big) \\
&= g\Big(\big(z_\alpha\big)_{|\alpha| \leq k}\Big) + sg\Big(\big(u_\alpha\big)_{|\alpha| \leq k}\Big),
\end{aligned}$$

where $|s| \leq |\gamma(t)| \leq M|t|$.

Thus $g \in \mathscr{K}_{M,\varepsilon}(\mathbb{C}^{m_k}, \mathbb{C})$ and $f(\xi) = f(\sum_{|\alpha| \leq k} \partial^\alpha \xi(y)\eta_\alpha) = g((\partial^\alpha \xi(y))_{|\alpha| \leq k})$, $\forall\, \xi \in C^\infty(\Omega)$. □

Theorem 4.8. *Let $M \geq 1$, $\gamma(t) = Mt$ for $t \in \mathbb{C}$. If $f \in C^\infty(\Omega)^{[\gamma,V]}$ is of order k and $\operatorname{supp} f = \{y\}$, then there exists $\big\{g_\alpha : \alpha \text{ is a multi-index}, |\alpha| \leq k\big\} \subset \mathscr{K}_{M,\varepsilon}(\mathbb{C}, \mathbb{C})$ such that*

$$f\big[z(\bullet - y)^\alpha/(\alpha!)\big] = g_\alpha(z), \quad \forall\, |\alpha| \leq k, \; z \in \mathbb{C},$$

$$f(\xi) = M\sum_{|\alpha| \leq k} a_{\alpha,\xi} \frac{g_\alpha(\varepsilon)}{\varepsilon} \partial^\alpha \xi(y), \quad \forall\, \xi \in C^\infty(\Omega),$$

where all $|a_{\alpha,\xi}| \le 1$.

Proof. *Letting* $\eta_\alpha = (\bullet - y)^\alpha/\alpha!$ *for* $|\alpha| \le k$, *we have* $f(\xi) = f\big(\sum_{|\alpha|\le k} \partial^\alpha \xi(y)\eta_\alpha\big)$, $\forall\, \xi \in C^\infty(\Omega)$ *by Cor. 4.3. There is an* $\varepsilon > 0$ *such that* $u\eta_\alpha \in V$ *for all* $u \in \{z \in \mathbb{C} : |z| \le \varepsilon\}$ *and* $|\alpha| \le k$. *We write* $\{\alpha : |\alpha| \le k\} = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$.

Let $\xi \in C^\infty(\Omega)$ *and pick a* $p \in \mathbb{N}$ *such that* $|\frac{\partial^\alpha \xi(y)}{p\varepsilon}| < 1$ *when* $|\alpha| \le k$. *Then*

$$f(\xi) = f\Big(\sum_{j=1}^{m} \partial^{\alpha_j}\xi(y)\eta_{\alpha_j}\Big) = f\Big(\sum_{j=1}^{m-1} \partial^{\alpha_j}\xi(y)\eta_{\alpha_j} + p\frac{\partial^{\alpha_m}\xi(y)}{p\varepsilon}\varepsilon\eta_{\alpha_m}\Big)$$
$$= f\Big(\sum_{j=1}^{m-1} \partial^{\alpha_j}\xi(y)\eta_{\alpha_j}\Big) + \big(\sum_{v=1}^{p} s_v\big) f(\varepsilon\eta_{\alpha_m}),$$

where each $|s_v| \le M|\frac{\partial^{\alpha_m}\xi(y)}{p\varepsilon}|$ *so* $|\sum_{v=1}^{p} s_v| \le pM|\frac{\partial^{\alpha_m}\xi(y)}{p\varepsilon}| = \frac{M}{\varepsilon}|\partial^{\alpha_m}\xi(y)|$.

Define $g_{\alpha_m} : \mathbb{C} \to \mathbb{C}$ *by* $g_{\alpha_m}(z) = f(z\eta_{\alpha_m})$, $\forall\, z \in \mathbb{C}$. *For* $z, u, t \in \mathbb{C}$ *with* $|u| \le \varepsilon$ *and* $|t| \le 1$, $u\eta_{\alpha_m} \in V$ *and* $g_{\alpha_m}(z + tu) = f(z\eta_{\alpha_m} + tu\eta_{\alpha_m}) = f(z\eta_{\alpha_m}) + sf(u\eta_{\alpha_m}) = g_{\alpha_m}(z) + sg_{\alpha_m}(u)$ *where* $|s| \le |\gamma(t)| = M|t|$. *Thus,* $g_{\alpha_m} \in \mathscr{K}_{M,\varepsilon}(\mathbb{C}, \mathbb{C})$ *and*

$$\Big|\big(\sum_{v=1}^{p} s_v\big) f(\varepsilon\eta_{\alpha_m})\Big| = \Big|\big(\sum_{v=1}^{p} s_v\big) g_{\alpha_m}(\varepsilon)\Big| \le M\Big|\frac{g_{\alpha_m}(\varepsilon)}{\varepsilon}\partial^{\alpha_m}\xi(y)\Big|.$$

Hence there is an $a_{\alpha_m,\xi} \in \mathbb{C}$ *such that* $|a_{\alpha_m,\xi}| \le 1$ *and*
$(\sum_{v=1}^{p} s_v) f(\varepsilon\eta_{\alpha_m}) = a_{\alpha_m,\xi} M \frac{g_{\alpha_m}(\varepsilon)}{\varepsilon}\partial^{\alpha_m}\xi(y)$.
In this way, we have

$$f(\xi) = f\Big(\sum_{j=1}^{m-1} \partial^{\alpha_j}\xi(y)\eta_{\alpha_j}\Big) + a_{\alpha_m,\xi} M \frac{g_{\alpha_m}(\varepsilon)}{\varepsilon}\partial^{\alpha_m}\xi(y)$$
$$= f\Big(\sum_{j=1}^{m-2} \partial^{\alpha_j}\xi(y)\eta_{\alpha_j}\Big) + a_{\alpha_{m-1},\xi} M \frac{g_{\alpha_{m-1}}(\varepsilon)}{\varepsilon}\partial^{\alpha_{m-1}}\xi(y) + a_{\alpha_m,\xi} M \frac{g_{\alpha_m}(\varepsilon)}{\varepsilon}\partial^{\alpha_m}\xi(y)$$
$$\cdots \quad \cdots$$
$$= \sum_{j=1}^{m} a_{\alpha_j,\xi} M \frac{g_{\alpha_j}(\varepsilon)}{\varepsilon}\partial^{\alpha_j}\xi(y) = M \sum_{|\alpha|\le k} a_{\alpha,\xi} \frac{g_\alpha(\varepsilon)}{\varepsilon}\partial^\alpha\xi(y),$$

where all $|a_{\alpha,\xi}| \le 1$. □

It is similar to Cor. 4.5 that we have

Corollary 4.7. *Let* $M \ge 1$, $\gamma(t) = Mt$ *for* $t \in \mathbb{C}$. *If* $f \in C^\infty(\Omega)^{[\gamma,V]}$ *is of order* k *and* $\operatorname{supp} f = \{y\}$, *then* f *has the following properties.*

(1) If $|\alpha| \le k$ *and* $f[z_0(\bullet - y)^\alpha/(\alpha!)] \ne 0$ *for some* $z_0 \in \mathbb{C}$, *then there exists an* $\varepsilon > 0$ *such that* $f[z_0(\bullet - y)^\alpha/(\alpha!)] \ne 0$ *when* $0 < |z| \le \varepsilon$, *that is, the equation* $f[z_0(\bullet - y)^\alpha/(\alpha!)] = 0$, $|z| \le \varepsilon$ *has the unique solution* $z = 0$. *Hence if* $|\alpha| \le k$ *and there is* $\{z_v\} \subset \mathbb{C}$ *such that each* $z_v \ne 0$, $z_v \to 0$ *and each* $f[z_v(\bullet - y)^\alpha/(\alpha!)] = 0$, *then* $f[z(\bullet - y)^\alpha/(\alpha!)] = 0$ *for all* $z \in \mathbb{C}$.

(2) If $|\alpha| \le k$, *then* $f[z(\bullet - y)^\alpha/(\alpha!)]$ *is Lipschitz, that is, there is an* $A_\alpha > 0$ *such that*

$|f[z(\bullet - y)^\alpha/(\alpha!)] - f[u(\bullet - y)^\alpha/(\alpha!)]| \le A_\alpha |z-u|, \quad \forall\, z, u \in \mathbb{C}$. *In particular, we have*

$$\Big| f\big[\partial^\alpha \xi(y)(\bullet - y)^\alpha/(\alpha!)\big] - f\big[\partial^\alpha \eta(y)(\bullet - y)^\alpha/(\alpha!)\big]\Big| \le A_\alpha \big|\partial^\alpha \xi(y) - \partial^\alpha \eta(y)\big|, \ \forall\, \xi, \eta \in C^\infty(\Omega).$$

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