Appl. Math. J. Chinese Univ.
2021, 36(4): 599-610

Image of polynomials under generalized Szász operators

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Abstract. The motivation behind this paper is a sequence S_n of generalized Szász operators using multiple Appell polynmials. The purpose of the present paper is to find the image of the polynomials under these operators. We find that as $n \to \infty$, $S_n(t^m; x)$ approaches to x^m for every $m \in \mathbb{N}$. Finally, We prove the image of a polynomial of degree m under these operators is another polynomial of degree m by using the linearity of these operators.

§1 Introduction

In 1950 Szász defined the following well known operators

$$P_m(f;x) = e^{-mx} \sum_{k=0}^{m} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right) \tag{1.1}$$

where $x \in [0, \infty)$ and $f \in C[0, \infty)$. Then in 1969, Jakimovski and Leviatan [6] generalized these operators using Appell polynomials [3] as the following:

$$S_m^*(f;y) = \frac{e^{-my}}{h(1)} \sum_{k=0}^{\infty} g_k(my) f\left(\frac{k}{m}\right), \tag{1.2}$$

where $g_k(ny)$ are the Appell polynomials having generating function of the form

$$h(u)e^{uy} = \sum_{k=0}^{\infty} g_k(y) u^k,$$

where $h(z) = \sum_{k=0}^{\infty} a_n z^n, a_0 \neq 0$, is an analytic function in the disk $|z| < R, R > 1, h(1) \neq 0$ and the explicit form of $g_k(y)$ is given by

$$g_k(y) = \sum_{\nu=0}^{k} a_\nu \frac{y^{k-\nu}}{(k-\nu)!} \quad k = 0, 1, 2, ...$$

If $h(z) = 1$, we have $g_k(y) = \frac{y^k}{k!}$ and hence we have from (1.2), Szász-Mirakyan operators. In [12] S. Verma generalized Szász operators using the multiple Appell polynomials.

Received: 2021-03-29. Revised: 2021-05-02.
MR Subject Classification: 41A36.
Keywords: Szász operator, multiple Appell polynomials, image of polynomials.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-021-4547-2.
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A multiple polynomial system [8] $\{S_{n_1,n_2}(x)\}$ is called multiple Appell if it has a generating function of the form

$$H(t_1,t_2)e^{x(t_1+t_2)}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(x)}{n_1!n_2!}t_1^{n_1}t_2^{n_2}, \tag{1.3}$$

where $H(t_1,t_2)$ has a series expansion

$$H(t_1,t_2)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{h_{n_1,n_2}}{n_1!n_2!}t_1^{n_1}t_2^{n_2} \tag{1.4}$$

with $H(0,0)=h_{0,0}\neq 0$ and $\dfrac{h_{n_1,n_2}}{H_{1,1}}\geq 0$ for all $n_1,n_2\in\mathbb{N}$. Also, (1.3) and (1.4) converge for $|t_1|\leq R_1,\ |t_2|\leq R_2\ (R_1,R_2)\geq 1$. Here s_{n_1,n_2} is a multiple polynomial system and for every $n_1+n_2\geq 1$, it satisfy the following relationship:

$$s'_{n_1,n_2}(x)=n_1s_{n_1-1,n_2}(x)+n_2s_{n_1,s_2-1}(x).$$

The generalization by S.Verma is as the following:

$$S_n(f;x)= \ =\ \frac{e^{-nx}}{H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}f\left(\frac{n_1+n_2}{n}\right). \tag{1.5}$$

Several researchers have worked on Appell polynomials [1, 2, 5, 7, 9–11]. In [4] author prove that the image of the polynomials under the operators (1.1) is another polynomial of degree m. Motivated by this result in this paper we also prove that image of a polynomial of degree m under the generalization of these operators i.e. under the operators (1.5) is another polynomial of degree m.

§2 Main Results

Lemma 1. *The following series*

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1(n_1-1)...(n_1-(m-1))t_1^{n_1-m}t_2^{n_2}$$

is product of function $e^{\frac{nx}{2}(t_1+t_2)}$ *and a polynomial of degree* m *in* x *for every* $m\in\mathbb{N}$ *with coefficient of* x^m *is* $\left(\dfrac{n^m}{2^m}H(t_1,t_2)\right)$ *and coefficients of* x^p *is function of* t_1 *and* t_2 *multiplied by* $\frac{n^p}{2^p}$ *and this function is nothing but sum of the partial derivatives of* $H(t_1,t_2)$ *multiplied by some constant with maximum order of the partial derivative is* m *for each* $p\in\{0,1,2,....m-1\}$.

Proof. From 1.3, we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}t_1^{n_1}t_2^{n_2}=H(t_1,t_2)e^{\frac{nx}{2}(t_1+t_2)}.$$

Differentiating both sides with respect to t_1, it follows that

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1 \frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!} t_1^{n_1-1}t_2^{n_2} = \Big(H_{t_1}(t_1,t_2) + \frac{nx}{2}H(t_1,t_2)\Big)e^{\frac{nx}{2}(t_1+t_2)}. \tag{2.1}$$

Hence, the result is true for $m = 1$.

Let the result is true for $m = r$ where $r \in \mathbb{N}$,

$$i.e. \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1(n_1-1)(n_1-2)...(n_1-(r-1))\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}t_1^{n_1-r}t_2^{n_2}$$

$$= e^{\frac{nx}{2}(t_1+t_2)}\Big(\frac{n^r x^r}{2^r}H(t_1,t_2) + \frac{n^{r-1}x^{r-1}}{2^{r-1}}f_{r,1}(t_1,t_2) + ... + \frac{nx}{2}f_{r,r-1}(t_1,t_2)$$

$$+ f_{r,r}(t_1,t_2)\Big),$$

where $f_{r,r-i}(t_1,t_2)$ denotes the required function of t_1 and t_2.

Differentiating both sides with respect to t_1, we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1(n_1-1)(n_1-2)...(n_1-(r-1))(n_1-r)\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}t_1^{n_1-(r+1)}t_2^{n_2}$$

$$= e^{\frac{nx}{2}(t_1+t_2)}\Big[\Big(\frac{n^r x^r}{2^r}H_{t_1}(t_1,t_2) + \frac{n^{r-1}x^{r-1}}{2^{r-1}}f'_{r,1}(t_1,t_2) + ... + \frac{nx}{2}f'_{r,r-1}(t_1,t_2)$$

$$+ f'_{r,r}(t_1,t_2)\Big)$$

$$+ \Big(\frac{n^r x^r}{2^r}H(t_1,t_2) + \frac{n^{r-1}x^{r-1}}{2^{r-1}}f_{r,1}(t_1,t_2) + ... + \frac{nx}{2}f_{r,r-1}(t_1,t_2) + f_{r,r}(t_1,t_2)\Big)\frac{nx}{2}\Big],$$

$$= e^{\frac{nx}{2}(t_1+t_2)}\Big(\frac{n^{r+1}x^{r+1}}{2^{r+1}}H(t_1,t_2) + \frac{n^r x^r}{2^r}(H_{t_1}(t_1,t_2) + f_{r,1}(t_1,t_2)) + ...$$

$$+ \frac{nx}{2}(f'_{r,r-1}(t_1,t_2) + f_{r,r}(t_1,t_2)) + f'_{r,r}(t_1,t_2)\Big),$$

$$= e^{\frac{nx}{2}(t_1+t_2)}\Big(\frac{n^{r+1}x^{r+1}}{2^{r+1}}H(t_1,t_2) + \frac{n^r x^r}{2^r}f_{r+1,1}(t_1,t_2) + ... + \frac{nx}{2}f_{r+1,r}(t_1,t_2)$$

$$+ f_{r+1,r+1}(t_1,t_2)\Big),$$

where $f'_{r,r-i}(t_1,t_2)$ denotes the derivative of f with respect to t_1, for all $i = 0, 1, ...r-1$.

Therefore, the result is true for $m = r$. Hence by the Principle of Mathematical Induction the lemma holds. □

Proposition 1. *The following series*

$$\frac{e^{-nx}}{H(1,1)n^m}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m$$

is a polynomial of degree m in x with coefficient of x^m, $\frac{1}{2^m}$ for every $m \in \mathbb{N}$, while coefficient of x^p is a function of t_1 and t_2 having it's value at (1,1) multiplied by $\frac{1}{n^{m-p}2^pH(1,1)}$ and this function is nothing but sum of the partial derivatives of $H(t_1, t_2)$ multiplied by some constant with maximum order of the partial derivative is m for each $p \in \{0, 1, 2, ...m-1\}$.

Proof. Putting $t_1 = t_2 = 1$ in the equation (2.1) and multiplying both sides by $\frac{e^{-nx}}{nH(1,1)}$, we get

$$\frac{e^{-nx}}{H(1,1)n}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1 = \frac{x}{2} + \frac{H_{t_1}(1,1)}{nH(1,1)}.$$

Putting $t_1 = t_2 = 1$ and $m = 2$ in the Lemma 1 and multiplying it by $\dfrac{e^{-nx}}{n^2H(1,1)}$, we obtain

$$\frac{e^{-nx}}{H(1,1)n^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1(n_1-1) = \left(\frac{x^2}{2^2} + \frac{x}{2n}\frac{f_{2,1}(1,1)}{H(1,1)} + \frac{1}{n^2}\frac{f_{2,2}(1,1)}{H(1,1)}\right)$$

$$\begin{aligned}
&\Rightarrow \frac{e^{-nx}}{H(1,1)n^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^2,\\
&= \frac{e^{-nx}}{H(1,1)n^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1 + \left(\frac{x^2}{2^2} + \frac{x}{2n}\frac{f_{2,1}(1,1)}{H(1,1)} + \frac{1}{n^2}\frac{f_{2,2}(1,1)}{H(1,1)}\right),\\
&= \frac{x}{2n} + \frac{H_{t_1}(1,1)}{n^2H(1,1)} + \left(\frac{x^2}{2^2} + \frac{x}{2n}\frac{f_{2,1}((1,1)}{H(1,1)} + \frac{1}{n^2}\frac{f_{2,2}(1,1)}{H(1,1)}\right),\\
&= \frac{x^2}{2^2} + \frac{x}{2n}\left(1 + \frac{f_{2,1}(1,1)}{H(1,1)}\right) + \frac{1}{n^2}\left(\frac{H_{t_1}(1,1)}{H(1,1)} + \frac{f_{2,2}(1,1)}{H(1,1)}\right),\\
&= \frac{x^2}{2^2} + \frac{x}{2n}(g_{2,1}(1,1)) + \frac{1}{n^2}g_{2,2}(1,1),
\end{aligned}$$

where $g_{2,1}(t_1,t_2) = \left(1 + \dfrac{f_{2,1}(t_1,t_2)}{H(1,1)}\right)$ and $g_{2,2}(t_1,t_2) = \left(\dfrac{H_{t_1}(t_1,t_2)}{H(1,1)} + \dfrac{f_{2,2}(t_1,t_2)}{H(1,1)}\right)$ Therefore the result is true for $m = 1$ and $m = 2$.

Now, we prove that if the result is true for every $m \leq r-1$ where $r \in \mathbb{N}$, then the result is true for $m = r$, then by the Principle of Strong Mathematical Induction the result will be true for every $m \in \mathbb{N}$.
So, let the result is true for every $m \leq r-1$ where $m, r \in \mathbb{N}$, i.e.

$$\begin{aligned}
\frac{e^{-nx}}{H(1,1)n^m}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m &= \frac{x^m}{2^m} + \frac{x^{m-1}}{2^{m-1}n}g_{m,1}(1,1) + ... + \frac{x}{2n^{m-1}}g_{m,m-1}(1,1)\\
&+ \frac{1}{n^m}g_{m,m}(1,1),
\end{aligned}$$

here $g_{m,m-i}(t_1,t_2)$ is the required function of t_1, t_2 , for each $i \in \{1, 2, ...m-1\}$. Then, putting $t_1 = t_2 = 1$, in the Lemma 1, we have

$$\frac{e^{-nx}}{H(1,1)n^r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1(n_1-1)...(n_1-(r-1))$$

$$= \left(\frac{x^r}{2^r} + \frac{x^{r-1}}{n2^{r-1}}\frac{f_{r,1}(1,1)}{H(1,1)} + ... + \frac{x}{2n^{r-1}}\frac{f_{r,r-1}(1,1)}{H(1,1)} + \frac{1}{n^r}\frac{f_{r,r}(1,1)}{H(1,1)}\right).$$

Therefore,

$$\frac{e^{-nx}}{H(1,1)n^r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^r$$
$$= c_1\frac{e^{-nx}}{H(1,1)n^r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^{r-1} + ... + (-1)^r c_{r-1}\frac{e^{-nx}}{H(1,1)n^r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1$$
$$+ \left(\frac{x^r}{2^r} + \frac{x^{r-1}}{n2^{r-1}}\frac{f_{r,1}(1,1)}{H(1,1)} + ... + \frac{x}{2n^{r-1}}\frac{f_{r,r-1}(1,1)}{H(1,1)} + \frac{1}{n^r}\frac{f_{r,r}(1,1)}{H(1,1)}\right),$$

where $c_1 = 1 + 2 + ... + (r-1)$, $c_2 = 1.2 + 1.3 + ... + 1.(r-1) + 2.3 + 2.4 + ...$ $+ 2.(r-1) + ... + (r-2)(r-1), ..., c_{r-1} = 1.2.3...(r-1)$ are positive constants.

So, by using our assumption, we have

$$\frac{e^{-nx}}{H(1,1)n^r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\dfrac{nx}{2})}{n_1!n_2!}n_1^r$$
$$= \frac{c_1}{n}\left(\frac{x^{r-1}}{2^{r-1}} + ... + \frac{g_{r-1,r-1}(1,1)}{n^{r-1}}\right) + ... + (-1)^{r-1}\frac{c_{r-1}}{n^{r-1}}\left(\frac{x}{2} + \frac{g_{1,1}(1,1)}{n}\right)$$
$$+ \left(\frac{x^r}{2^r} + \frac{x^{r-1}}{n2^{r-1}}\frac{f_{r,1}(1,1)}{H(1,1)} + ... + \frac{x}{2n^{r-1}}\frac{f_{r,r-1}(1,1)}{H(1,1)} + \frac{1}{n^r}\frac{f_{r,r}(1,1)}{H(1,1)}\right),$$
$$= \frac{x^r}{2^r} + \frac{x^{r-1}}{n2^{r-1}}\left(\frac{f_{r,1}(1,1)}{H(1,1)} + c_1\right) + ...$$
$$+ \frac{1}{n^r}\left(\frac{f_{r,r}(1,1)}{H(1,1)} + c_1 g_{r-1,r-1}(1,1) - c_2 g_{r-2,r-2} + ... + (-1)^{r-1}c_{r-1}g_{1,1}(1,1)\right),$$
$$= \frac{x^r}{2^r} + \frac{x^{r-1}}{n2^{r-1}}g_{r,1}(1,1) + ... + g_{r,r}(1,1).$$ □

Proposition 2. *The following series*

$$\frac{e^{-nx}}{H(1,1)n^m}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\dfrac{nx}{2})}{n_1!n_2!}n_2^m$$

is a polynomial of degree m in x with coefficient of x^m, $\frac{1}{2^m}$ for every $m \in \mathbb{N}$, while coefficient of x^p is a function of t_1 and t_2 having it's value at (1,1) multiplied by $\frac{1}{n^{m-p}2^p H(1,1)}$ and this function is nothing but sum of the partial derivatives of $H(t_1,t_2)$ multiplied by some constant with maximum order of the partial derivative is m for each $p \in \{0,1,2,....m-1\}$.

Proof. Since

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\dfrac{nx}{2})}{n_1!n_2!}t_1^{n_1}t_2^{n_2} = H(t_1,t_2)e^{\dfrac{nx}{2}(t_1+t_2)}.$$

Therefore, by symmetricity of t_1 and t_2 and Proposition 1, the result is true. □

Lemma 2. *The following equality is true for every* $m \in \mathbb{N}$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^{m-1}n_2t_1^{n_1-(m-1)}t_2^{n_2-1}=\Big(\sum_{i=0}^{m}\frac{1}{t_1^{m-i}}h_{m,m-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)e^{\frac{nx(t_1+t_2)}{2}},$$

where $h_{m,m-i}(t_1,t_2)$ *is a function of* t_1 *and* t_2 *in terms of the sum of the partial derivatives of* $H(t_1,t_2)$ *multiplied by some power of* t_1 *and some constant for all* $i = 0,1,2,...m$ *and* $h_{m,0}(t_1,t_2) = H(t_1,t_2)$ *for every* $m \in \mathbb{N}$ *(max. power of* t_1 *is* m *and maximum order of partial derivatives is* m*).*

Proof. From equation (2.1) and by symmetricity of t_1 and t_2, we can see that the result is true for $m = 1$.
Differentiating (2.1) with respect to t_2, we get

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1n_2t_1^{n_1-1}t_2^{n_2-1}$$

$$=\Big(\frac{n^2x^2}{2^2}H(t_1,t_2)+\frac{nx}{2}\Big(H_{t_2}(t_1,t_2)+H_{t_1}(t_1,t_2)\Big)+H_{t_1t_2}(t_1,t_2)\Big)e^{\frac{nx(t_1+t_2)}{2}},$$

$$\Big(\frac{n^2x^2}{2^2}H(t_1,t_2)+\frac{nx}{2t_1}\Big(t_1H_{t_2}(t_1,t_2)+t_1H_{t_1}(t_1,t_2)\Big)+\frac{1}{t_1^2}t_1^2H_{t_1t_2}(t_1,t_2)\Big)e^{\frac{nx(t_1+t_2)}{2}},$$

$$=\Big(\frac{n^2x^2}{2^2}h_{2,0}(t_1,t_2)+\frac{nx}{2t_1}h_{2,1}(t_1,t_2)+\frac{1}{t_1^2}h_{2,2}(t_1,t_2)\Big)e^{\frac{nx(t_1+t_2)}{2}}.$$

Therefore the result is true for $m = 2$.
Let the result is true for $m = p$ where $p \in \mathbb{N}$ i.e.

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^{p-1}n_2t_1^{n_1-(p-1)}t_2^{n_2-1}=\Big(\sum_{i=0}^{p}\frac{1}{t_1^{p-i}}h_{p,p-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)e^{\frac{nx(t_1+t_2)}{2}}.$$

Differentiating both sides with respect to t_1, we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}(n_1-(p-1))n_1^{p-1}n_2t_1^{n_1-(p+1-1)}t_2^{n_2-1}$$

$$=\Big(\sum_{i=0}^{p}\frac{1}{t_1^{p-i}}h'_{p,p-i}(t_1,t_2)\frac{n^ix^i}{2^i}+\sum_{i=0}^{p}\frac{1}{t_1^{p+1-i}}(i-p)h_{p,p-i}(t_1,t_2)\frac{n^ix^i}{2^i}$$

$$+\Big(\sum_{i=0}^{p}\frac{1}{t_1^{p-i}}h_{p,p-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)\frac{nx}{2}\Big)e^{\frac{nx(t_1+t_2)}{2}},$$

where $h'_{p,p-i}(t_1,t_2)$ denotes the derivative of $h_{p,p-i}(t_1,t_2)$ with respect to t_1 for every $i \in \{0,1,2,...p\}$.
Therefore,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^{p}n_2t_1^{n_1-(p+1-1)}t_2^{n_2-1}$$

$$= (p-1)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^{p-1}n_2\frac{t_1^{n_1-(p-1)}}{t_1}t_2^{n_2-1}$$

$$+\Big(\sum_{i=0}^{p}\frac{1}{t_1^{p+1-i}}(t_1h'_{p,p-i}(t_1,t_2)+(i-p)h_{p,p-i}(t_1,t_2))\frac{n^ix^i}{2^i}$$

$$+\sum_{i=1}^{p+1}\frac{1}{t_1^{p+1-i}}h_{p,p+1-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)e^{\dfrac{nx(t_1+t_2)}{2}},$$

$$= \Big((p-1)\sum_{i=0}^{p}\frac{1}{t_1^{p+1-i}}h_{p,p-i}(t_1,t_2)\frac{n^ix^i}{2^i}$$

$$+\sum_{i=1}^{p}\frac{1}{t_1^{p+1-i}}\Big(t_1h'_{p,p-i}(t_1,t_2)+(i-p)h_{p,p-i}(t_1,t_2)+h_{p,p+1-i}(t_1,t_2)\Big)\frac{n^ix^i}{2^i}$$

$$+\frac{1}{t_1^{p+1}}\Big(t_1h'_{p,p}(t_1,t_2)+(-p)h_{p,p}(t_1,t_2)\Big)+h_{p,0}(t_1,t_2)\frac{n^{p+1}x^{p+1}}{2^{p+1}}\Big)e^{\dfrac{nx(t_1+t_2)}{2}},$$

$$=\sum_{i=1}^{p}\frac{1}{t_1^{p+1-i}}\Big((p-1)h_{p,p-i}(t_1,t_2)+t_1h'_{p,p-i}(t_1,t_2)+(i-p)h_{p,p-i}(t_1,t_2)$$

$$+h_{p,p+1-i}(t_1,t_2)\Big)\frac{n^ix^i}{2^i}$$

$$+\frac{1}{t_1^{p+1}}\Big(t_1h'_{p,p}(t_1,t_2)+(-p)h_{p,p}(t_1,t_2)+(p-1)h_{p,p}(t_1,t_2)\Big)$$

$$+h_{p,0}(t_1,t_2)\frac{n^{p+1}x^{p+1}}{2^{p+1}}\Big)e^{\frac{nx(t_1+t_2)}{2}},$$

$$=\Big(\sum_{i=1}^{p}\frac{1}{t_1^{p+1-i}}h_{p+1,p+1-i}(t_1,t_2)+\frac{1}{t_1^{p+1}}\Big(h_{p+1,p+1}(t_1,t_2)\Big)$$

$$+h_{p+1,0}(t_1,t_2)\frac{n^{p+1}x^{p+1}}{2^{p+1}}\Big)e^{\frac{nx(t_1+t_2)}{2}},$$

where $h_{p+1,p+1-i}(t_1,t_2)=(p-1)h_{p,p-i}(t_1,t_2)+t_1h'_{p,p-i}(t_1,t_2)+(i-p)h_{p,p-i}(t_1,t_2)$ $+h_{p,p+1-i}(t_1,t_2)$ for each $i\in\{1,2,...,p\}$, $h_{p+1,p+1}(t_1,t_2)=t_1h'_{p,p}(t_1,t_2)$ $+(-p)h_{p,p}(t_1,t_2)+(p-1)h_{p,p}(t_1,t_2)$ and $h_{p+1,0}(t_1,t_2)=h_{p,0}(t_1,t_2)$.

Therefore, the result is true for $m=p+1$. Hence, the result is true for every $m\in\mathbb{N}$. □

Lemma 3. *The following series*

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2(n_2-1)...(n_2-(r-1))t_1^{n_1-m}t_2^{n_2-r}$$

is a product of function $e^{\dfrac{nx}{2}(t_1+t_2)}$ *and a polynomial of degree* $m+r$ *in* x *for each* m,r $\in\mathbb{N}$ *with coefficient of* x^{m+r} *is* $\left(\dfrac{n^{m+r}}{2^{m+r}}H(t_1,t_2)\right)$ *and coefficients of* x^p *is* $\dfrac{n^pq_{m+r,m+r-p}(t_1,t_2)}{t_1^{m+r-p}2^p}$, *a function of* t_1 *and* t_2 *which is nothing but the sum of the partial derivatives of* $H(t_1,t_2)$ *multiplied by some constant and some power of* t_1 *for each* $p\in\{0,1,2,....m+r-1\}$*(max power* *of t_1 is $m+r$ and maximum order of the partial derivative is $m+r$).*

Proof. The result is true for $r=1$ and any $m\in\mathbb{N}$ by Lemma 2.

Let the result is true for $r=s$ and and any $m\in\mathbb{N}$,(max. order of partial derivatives is $s+m$), then

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2(n_2-1)...(n_2-(s-1))t_1^{n_1-m}t_2^{n_2-s}$$
$$=\Big(\sum_{i=0}^{m+s}\frac{1}{t_1^{m+s-i}}q_{m+s,m+s-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)e^{\frac{nx}{2}(t_1+t_2)}.$$

Here $q_{m+s,0}(t_1,t_2)=H(t_1,t_2)$.

Differentiating both sides with respect to t_2, we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2(n_2-1)...(n_2-(s-1))(n_2-s)t_1^{n_1-m}t_2^{n_2-(s+1)}$$
$$=\Big(\sum_{i=1}^{m+s+1}\frac{1}{t_1^{m+s+1-i}}q_{m+s,m+s+1-i}(t_1,t_2)\frac{n^ix^i}{2^i}$$
$$+\sum_{i=0}^{m+s}\frac{1}{t_1^{m+s-i}}q'_{m+s,m+s-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)e^{\frac{nx}{2}(t_1+t_2)},$$
$$=\sum_{i=1}^{m+s}\frac{1}{t_1^{m+s+1-i}}(t_1q'_{m+s,m+s-i}(t_1,t_2)+q_{m+s,m+s+1-i})(t_1,t_2)\frac{n^ix^i}{2^i}$$
$$+\frac{1}{t_1^{m+s+1}}t_1q'_{m+s,m+s}(t_1,t_2)+q_{m+s,0}(t_1,t_2)\frac{n^{m+s+1}x^{m+s+1}}{2^{m+s+1}}\Big)e^{\frac{nx}{2}(t_1+t_2)},$$
$$=\Big(\sum_{i=0}^{m+s+1}\frac{1}{t_1^{m+s+1-i}}q_{m+s+1,m+s+1-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)e^{\frac{nx}{2}(t_1+t_2)},$$

where $q'_{m+s,m+s-i}(t_1,t_2)$ denotes the derivative of $q_{m+s,m+s-i}(t_1,t_2)$ with respect to t_2

and $q_{m+s+1,m+s+1-i}(t_1,t_2)=(t_1q'_{m+s,m+s-i}(t_1,t_2)+q_{m+s,m+s+1-i}(t_1,t_2)$ for each $i\in\{1,...m+s\}$,

$q_{m+s+1,m+s+1}(t_1,t_2)=t_1q'_{m+s,m+s}(t_1,t_2)$ *and*$q_{m+s+1,0}(t_1,t_2)=q_{m+s,0}(t_1,t_2)$. □

Therefore the result is true for $r=s+1$. Hence by principle of mathematical induction the result is true for every $m\in\mathbb{N}$.

Proposition 3. *The following series*

$$\frac{e^{-nx}}{n^{m+s}H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2^s$$

is a polynomial of degree $m+s$ in x, where coefficient of x^{m+s} is $\frac{1}{2^{m+s}}$ and coefficients of x^p is function of t_1 and t_2 having its value at $(1,1)$ multiplied by $\frac{1}{2^pn^{m+s-p}}$ and this function is nothing *but sum of the partial derivatives of $H(t_1,t_2)$ multiplied by some constant with maximum order of the partial derivative is $m+s$ for each $p \in \{0,1,2,....m+s-1\}$.*

Proof. The result is true for $s=1$ and for any $m \in \mathbb{N}$ by Lemma 3.
Putting $r=2$ in Lemma 3, we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2^2 t_1^{n_1-m}t_2^{n_2-2}$$
$$=\Big(\sum_{i=0}^{m+2}\frac{1}{t_1^{m+2-i}}q_{m+2,m+2-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)e^{\frac{nx}{2}(t_1+t_2)}+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2 t_1^{n_1-m}\frac{t_2^{n_2-1}}{t_2},$$
$$=\Big(\sum_{i=0}^{m+1}\frac{1}{t_2t_1^{m+1-i}}q_{m+1,m+1-i}(t_1,t_2)\frac{n^i}{x^i}$$
$$+\Big(\sum_{i=0}^{m+2}\frac{1}{t_1^{m+2-i}}q_{m+2,m+2-i}(t_1,t_2)\frac{n^ix^i}{2^i}\Big)\Big)e^{\frac{nx}{2}(t_1+t_2)}.$$

Substituting $t_1=t_2=1$ and multiplying by $\frac{e^{-nx}}{H(1,1)n^{m+2}}$, we have

$$\frac{e^{-nx}}{H(1,1)n^{m+2}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2^2$$
$$=\sum_{i=0}^{m+1}\frac{q_{m+1,m+1-i}(1,1)}{H(1,1)}\frac{x^i}{n^{m+2-i}2^i}+\sum_{i=0}^{m+2}\frac{q_{m+2,m+2-i}(1,1)}{H(1,1)}\frac{x^i}{n^{m+2-i}2^i},$$
$$=\sum_{i=0}^{m+1}\Big(\frac{q_{m+1,m+1-i}(1,1)}{H(1,1)}+\frac{q_{m+2,m+2-i}(1,1)}{H(1,1)}\Big)\frac{x^i}{n^{m+2-i}2^i}+\frac{x^{m+2}}{2^{m+2}},$$
$$=\sum_{i=0}^{m+2}v_{m+2,m+2-i}(1,1)\frac{x^i}{n^{m+2-i}2^i},$$

where $v_{m+2,m+2-i}(1,1)=\frac{q_{m+1,m+1-i}(1,1)}{H(1,1)}+\frac{q_{m+2,m+2-i}(1,1)}{H(1,1)}$ for each $i \in \{0,1,...,m+1\}$ and $v_{m+2,0}(1,1)=1$.
So, the result is true for $s=2$.
Now, let the result is true for every $s \le r$ where $r \in \mathbb{N}$ and for any '$m \in \mathbb{N}$ i.e.

$$\frac{e^{-nx}}{n^{m+s}H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2^s=\sum_{i=0}^{m+s}v_{m+s,m+s-i}(1,1)\frac{x^i}{n^{m+s-i}2^i},$$

where $v_{m+s,0}(1,1)=1$ and $v_{m+s,m+s-i}(1,1)$ is the required function for each $i \in 0,1,2,...m+s-1$.
Form Lemma 3, we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2(n_2-1)...(n_2-r))t_1^{n_1-m}t_2^{n_2-r}$$
$$=\sum_{i=0}^{m+r+1}q_{m+r+1,m+r+1-i}(1,1)\frac{x^i}{n^{m+r+1-i}2^i}.$$

Therefore,

$$\frac{e^{-nx}}{n^{m+r+1}H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2^{r+1}$$
$$= c_1\frac{e^{-nx}}{n^{m+r+1}H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2^{r} + ...$$
$$+(-1)^{r-1}c_r\frac{e^{-nx}}{n^{m+r+1}H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m n_2$$
$$+\Big(\sum_{i=0}^{m+r+1}\frac{q_{m+r+1,m+r+1-i}(1,1)}{H(1,1)}\frac{x^i}{n^{m+r+1-i}2^i}\Big),$$
$$= c_1\sum_{i=0}^{m+r}v_{m+r,m+r-i}(1,1)\frac{x^i}{n^{m+r+1-i}2^i} - c_2\sum_{i=0}^{m+r-1}v_{m+r-1,m+r-1-i}(1,1)\frac{x^i}{n^{m+r+1-i}2^i} + ...$$
$$+(-1)^{r-1}c_r\sum_{i=0}^{m}v_{m+1,m+1-i}(1,1)\frac{x^i}{n^{m+r+1-i}2^i} + \Big(\sum_{i=0}^{m+r+1}\frac{q_{m+r+1,m+r+1-i}(1,1)}{H(1,1)}\frac{x^i}{n^{m+r+1-i}2^i}\Big),$$
$$= \frac{q_{m+r+1,0}(1,1)}{H(1,1)}\frac{x^{m+r+1}}{2^{m+r+1}} + \frac{1}{n}\Big(c_1 v_{m+r,0}(1,1) + q_{m+r+1,1}(1,1)\Big)\frac{x^{m+r}}{2^{m+r}}$$
$$+\frac{1}{n^2}\Big(c_1 v_{m+r,1}(1,1) - c_2 v_{m+r-1,0}(1,1) + q_{m+r+1,2}(1,1)\Big)\frac{x^{m+r-1}}{2^{m+r-1}} + ...$$
$$+\frac{1}{n^{m+r+1}}\Big(c_1 v_{m+r,m+r}(1,1) - c_2 v_{m+r-1,m+r-1}(1,1) + ... + (-1)^{r-1}c_r v_{m+1,m+1}(1,1)$$
$$+\frac{q_{m+r+1,m+r+1}(1,1)}{H(1,1)}\Big),$$

where $c_1 = 1 + 2 + ... + (r-1)$, $c_2 = 1.2 + 1.3 + ... + 1.(r-1) + 2.3 + 2.4 + ...$
$+2.(r-1) + ... + (r-2)(r-1), ..., c_r = 1.2.3...r$ are positive constants
and $q_{m+r+1,0}(1,1) = H(1,1)$. □

Theorem 1. *$S_n(t^m;x)$ is a polynomial of degree m in x for every $m \in \mathbb{N}$.*

Proof. $S_n(t^m;x)$

$$= \frac{e^{-nx}}{H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}\Big(\frac{n_1+n_2}{n}\Big)^m,$$
$$= \frac{e^{-nx}}{H(1,1)n^m}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}\Big(n_1^m + n_2^m + \binom{m}{1}n_1^{m-1}n_2 + \binom{m}{2}n_1^{m-2}n_2^2$$
$$+... + \tbinom{m}{m-1}n_1 n_2^{m-1}\Big),$$
$$= \frac{e^{-nx}}{H(1,1)n^m}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_1^m + \frac{e^{-nx}}{H(1,1)n^m}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}n_2^m + ...$$
$$+\tbinom{m}{m-1}\tfrac{e^{-nx}}{H(1,1)n^m}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}(n_1 n_2^{(m-1)}).$$

Now, by using propositions 1, 2 and 3, we see that

$$
\begin{aligned}
&S_n(t^m;x)\\
&= x^m\left(\tfrac{1}{2^m}+\tbinom{m}{1}\tfrac{1}{2^m}+\tbinom{m}{2}\tfrac{1}{2^m}+\ldots+\tbinom{m}{m}\tfrac{1}{2^m}\right)+\tfrac{x^{m-1}}{n}b_{m,1}(1,1)\\
&+\tfrac{x^{m-2}}{n^2}b_{m,2}(1,1)+\ldots+\tfrac{1}{n^m}b_{m,m}(1,1),\\
&= x^m\left(\tfrac{1}{2}+\tfrac{1}{2}\right)^m+\frac{x^{m-1}}{n}b_{m,1}(1,1)+\frac{x^{m-2}}{n^2}b_{m,2}(1,1)+\ldots+\frac{1}{n^m}b_{m,m}(1,1),\\
&= x^m+\frac{x^{m-1}}{n}b_{m,1}(1,1)+\frac{x^{m-2}}{n^2}b_{m,2}(1,1)+\ldots+\frac{1}{n^m}b_{m,m}(1,1),
\end{aligned}
$$

where, $b_{m,i}(1,1)$ is function of t_1 and t_2 having its value at $(1,1)$ and this function is nothing but sum of the partial derivatives of $H(t_1,t_2)$ multiplied by some constant with maximum order of the partial derivative is m for each $i \in \{0,1,2,....m-1\}$. □

Now, since $S_n(f;x)$ is linear for every $n \in \mathbb{N}$, therefore the image of a polynomial of degree m under these operators is a polynomial of degree m for every $m \in \mathbb{N}$.

§3 Conclusion

So, we prove that image of the polynomial of degree m under the operators S_n is another polynomial of degree m by using the linearity of these operators.To prove this result we have also used generating function of multiple Appell polynomials. By using this result we can prove the weighted convergence of these operators for the functions belonging to the space $R_m^* = \{f \in Rm| \lim_{x\to\infty} p_m(x)f(x) \in \mathbb{R}\}$ where $p_m(x) = 1 + x^m$ and $R_m(x)$ is the space of all continuous function defined on interval $[0,\infty)$ such that $\sup_{x\geq 0} p_m(x)|f(x)| \in \mathbb{R}$.

§4 Novelty of the research Work

In this article, we studied the generalized operators Szász operators which used multiple Appell polynomials. Motivation behind the Multiple Appell polynomials are Appell polynomials. Appell polynomials are the solutions of many differential equations. For example if we take the sequence $(A_n(x))_{n\in\mathbb{N}}$ of the Appell polynomials where $A_n(x)$ is an Appell polynomial for every $n \in \mathbb{N}$ having generating function $\frac{1}{1-h}$, then $A_n(x)$ satisfies the following equation:

$$x\frac{d^2y}{dx^2} - (x+n)\frac{dy}{dx} + ny = 0.$$

§5 Acknowledgments

The authors are thankful to the reviewers for their valuable and constructive suggestions which have substantially improved the presentation of the paper. Pooja Gupta and Ramu Dubey gratefully acknowledge, Department of Mathematics, J.C. Bose University of Science and Technology, YMCA, Faridabad-121 006, India.

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