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Z-control on dynamics of pollution-allergy model

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Abstract. Allergy due to toxic pollutants present in the environment has become a major public health threat affecting the lives of more than one billion people worldwide. Its prevalence and impact are on the rise due to urbanization and growing chemical industries, on the other hand emerging toxic allergens heavily impact the budgets of public health systems. In the current work, the dynamical model of allergy due to pollution is constructed and studied. Moreover, the basic reproduction number is formulated to calculate the threshold value of allergic diseases. The model is stabilized locally and globally, besides the conditions that emerged from stability theory are used to study the effects of various parameters. Z-type control mechanism has been used to optimize the stability of the model. Numerical simulations validate, that chaotic oscillated compartments are stabilized under the effect of Z-control which shows that allergic diseases due to the polluted environment can be controlled.

§1 Introduction

The new idea of "allergy" was developed based on antigen resulted change in reactivity which indicates harmful changes in certain situations [13]. Risk factors of allergy are based on biological or chemical imbalances in food, fluids, medications, insect stings, toxins interacting with proteins, genetics, hygiene hypothesis, and different environmental factors. Of these, allergic diseases are influenced by environmental pollutants arisen due to climatic disturbances, and hence climate change due to pollution is a matter of concern [9, 11, 2]. Following the lifestyle depending on various chemical-based industries results in industrial development which increases the exposure rate to allergic diseases due to pollution, which leads to a reduction in life expectancy [5]. Moreover, due to rapid urbanization, various pollutants exposed humans to ambient air pollution that determines the exacerbation of morbidity and mortality in respiratory disease [6, 14]. During inhalation, environmental pollutants enter the body orally or percutaneously and

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create allergic disruptions in the immune system. Various documentation and observations in the last three decades show that there has been a remarkable increase observed in the cases of allergic diseases in industrialized countries [10, 17]. Global premature mortality rates due to various respiratory diseases and lung cancer are remarkably increasing due to the interrupted level in particulate matter (PM2.5) and ozone [12]. As per WHO statistics, one amongst every ten people is suffering from several allergic diseases [19]. Globally more than 1 billion people are under health threat which is highly impacting public health budgets. Mathematical analysis-based modeling has been used to enhance our understanding of the spread of infectious diseases. Gross et al. have examined the extended model based upon the Th1-Th2 paradigm and hypothesized that immunotherapy mainly acts on the T cell level with help of a dynamical phenomenon [7]. While Kim et al. develops a mathematical model of asthma development and using modeling, they have represented the complex network of interactions between cells and molecules [20]. Whereas, Eggo et al. introduced a dynamic model of common cold transmission, estimated epidemiological characteristics of common cold viruses, and rigorously assessed the relative contributions of proposed infectious and non- infectious drivers of asthma exacerbations [18]. In the present study, a pollution-allergy model is constructed in the second section to study the influence of pollution associated allergic diseases on social lifespan. Furthermore, the intens spread of allergy is been measured through the basic reproduction number. In the third section, local and global stability of the system is proved. In the fourth section, the Z-type control mechanism is introduced and applied to the pollution-allergy model, further analysis is completed by simulating the model and discussing conclusions.

§2 Formulation of mathematical model

The model consists of three compartmental class, class of people who are direct or indirect comes in contact with pollution but not showing any symptoms of allergy (P), class of people who are get allergic due to pollution (A) and class of people who needs medication (M). In the model a, b and c are saturation constants. C_1 and C_2 are conversion coefficients of medication on allergic and polluted class, respectively. The other parameters are interpreted as given in table 1.

Table 1. Model parameters and their interpretation.

Parameters	**Description**
B	Rate at which individuals enters in to polluted class
β_1	Inter-class competition coefficients
β_2	Force of infection
β_3	Medication rate in class of polluted population
β_4	Medication rate in class of allergic population
μ_P	Escape rate of individuals who is not get affected by pollution
μ_A	Escape rate of not caching allergy
μ_M	Escape rate of not caching allergy

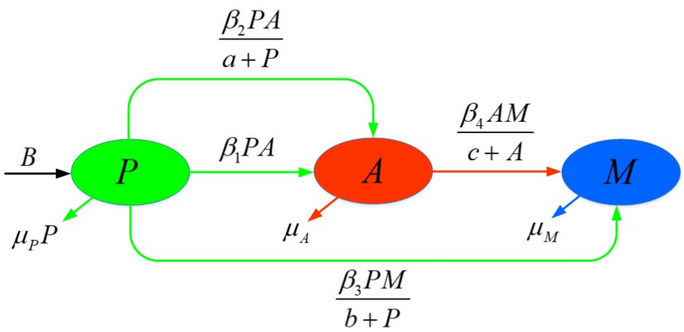


Figure 1. Flow diagram of pollution-allergy model.

The pollution-allergy model is written in the form of dynamical system of non-linear differential equations as given below:

$$\begin{aligned} \frac{dP}{dt} &= BP - \beta_1 PA - \frac{\beta_2 PA}{a+P} - \frac{\beta_3 PM}{b+P} - \mu_P P^2 \\ \frac{dA}{dt} &= \beta_1 PA + \frac{\beta_2 PA}{a+P} - \frac{\beta_4 AM}{c+A} - \mu_A A \\ \frac{dM}{dt} &= \frac{c_1 \beta_4 AM}{c+A} + \frac{c_2 \beta_3 PM}{b+P} - \mu_M M \end{aligned} \tag{1}$$

2.1 Equilibrium points and basic reproduction number

By solving the system (1) of non-linear differential equation, we get five equilibrium points:

(i) $E_0 = \{\frac{B}{\mu_P}, 0, 0\}$

(ii) $E_1 = \left\{0, -\frac{\mu_M c}{C_1 \beta_4 + \mu_M}, \frac{\mu_A c C_1}{-C_1 \beta_4 + \mu_M}\right\}$

(iii) $E_2 = \left\{-\frac{\mu_M b}{-C_2 \beta_3 + \mu_M}, 0, -\frac{b C_2 (\mu_M \mu_P b - B C_2 \beta_3 + \mu_M B)}{(-C_2 \beta_3 + \mu_M)^2}\right\}$

(iv) $E_3 = \{r_1, \frac{v}{\beta_1}, 0\}$ where, $r_1 = Root\,of\ \left\{\beta_1 x^2 + (a\beta_1 - \mu_A + \beta_2)x - \mu_A a\right\}$ and

$$v = \frac{\mu_M \mu_P a + \mu_M \mu_P r_1 - \mu_P \beta_2 r_1 - B a \beta_1 - B \beta_1 r_1}{(a\beta_1 + \beta_1 r_1 + \beta_2)}$$

(v) $E^* = \left(r_2, -\frac{c(-C_2 \beta_3 r_2 + \mu_M b + \mu_M r_2)}{-C_1 b \beta_4 - C_1 \beta_4 r_2 - C_2 \beta_3 r_2 + \mu_M b + \mu_M r_2}, \frac{n^*}{m^*}\right)$ where,

$$
\begin{aligned}
m^* = &(a-b)\left((C_1-C_2)\left(\mu_A C_1 a\beta_3\beta_4(C_1\beta_4+C_2\beta_3)-2\mu_M ab\beta_1\beta_3\beta_4\right.\right.\\
&+\mu_M b\beta_2\beta_4\left(C_1\beta_4+C_2\beta_3\right)+\mu_A C_1^3 a\beta_3\beta_4^2+\left(C_1-2C_2\right)\mu_M^2 b\left(a\beta_1\beta_3-\beta_2\beta_4\right)\big)\\
&+C_1 ab\beta_1\beta_3\beta_4^2\left(C_1^2+C_2^2\right)+\mu_A\mu_M a\beta_3\left(2C_2^2+\mu_M C_1\right)+\mu_M^3 b\beta_2\big)\\
&+\left(C_1-C_2\right)\left(\mu_A C_2 a^2\beta_3^2\left(-\mu_M+C_2\beta_3\right)+\mu_M ab\beta_3\left(-C_2 a\beta_1\beta_3-2C_1\beta_2\beta_4\right)\right.\\
&\left.+\mu_M^2 ab\beta_2\beta_3+\mu_A C_1 C_2 a^2\beta_3^2\beta_4\right)+(a-b)^2\left(\mu_M C_1 b\beta_1\beta_4^2\left(C_1-C_2\right)\right.\\
&\left.+\mu_A\mu_M C_1\beta_4\left(C_1\beta_4-2\mu_M\right)+\mu_M^2 b\beta_1\beta_4\left(-2C_1+C_2\right)+\mu_M^3\left(b\beta_1+\mu_A\right)\right)\\
&+\left(C_1\beta_4+C_2\beta_3\right)\left(ab\beta_2\beta_3\beta_4(C_1^2+C_2^2)-2C_1C_2 ab\beta_2\beta_3\beta_4\right)\\
&+(a+b)\left(\mu_M\beta_3(2\mu_A C_1^2 a\beta_4+\mu_M C_2 b\beta_2)\right)+C_1^2 C_2 ab\beta_1\beta_3\beta_4\left(a\beta_3-2a\beta_4+2b\beta_4\right)\\
&+C_2^2 a^2 b\beta_1\beta_3^2\beta_4\left(-2C_1+C_2\right)+\mu_A\mu_M C_1C_2\beta_3\beta_4\left(4a^2-5ab+b^2\right)\\
&+\mu_M C_1C_2 ab\beta_2\beta_3\left(\beta_3+\beta_4\right)+\mu_M C_2^2 ab\beta_2
\end{aligned}
$$

$$
\begin{aligned}
n^* = &(b+r_2)\{(a-b)\left((C_1-C_2)\,C_1^2 ab\beta_1\beta_4^2\left(-\mu_P r_2+B\right)+\mu_A\mu_P C_1^2 a\beta_4 r_2\left(-C_1\beta_4-2C_1\beta_3\right)\right.\\
&+\left(C_1\beta_4+C_2\beta_3\right)\left(\mu_A C_1 a(2\mu_M\mu_P r_2+BC_1\beta_4)\right)+\mu_A\mu_M BC_1C_2 a\beta_3\\
&+\left(b\beta_1+\mu_A\right)\left(\mu_M^2 C_1 a(-\mu_P r_2+B)-2\mu_M BC_1^2 a\beta_4\right)\big)\\
&+\left(C_1-C_2\right)\left((a\beta_1+\beta_2)(C_1C_2 ab\beta_3\beta_4(-\mu_P r_2+B))\right.\\
&+(\mu_M-C_1\beta_4)(C_1C_2 ab\beta_3\beta_4(-\mu_P r_2+B)))(C_2\beta_3+C_1\beta_4)(C_1 a(\mu_A BC_2 a\beta_3\\
&+\mu_A\mu_M b\beta_2 r_2))+\mu_M^2 C_1 ab\beta_2(B+\mu_P r_2)-\mu_M BC_1C_2 a^2 b\beta_1\beta_2\\
&+\mu_M BC_1C_2 a\beta_3\left(-b\beta_2-\mu_A\right)\}\\
&+\mu_M^2\mu_P C_2 b^2\beta_1 r_2+\mu_M^2 BC_2 b(-b\beta_1-\mu_A)+(2a-b)\left(\mu_A\mu_M C_2^2 b\beta_3\left(-\mu_P r_2+B\right)\right.\\
&+\left(C_1\beta_4+C_2\beta_3\right)\left(\mu_P r_2-B\right)\left(C_2 b(\mu_A C_2^2 a\beta_3-\mu_M b\beta_2)\right)\big)\\
&+\mu_M C_2^2 ab\beta_1\beta_3\left(c(b\beta_1+2\mu_A)+b(-\mu_P r_2+B)\right)\mu_M^2 C_2 b^2\beta_2(-B+\mu_P r_2)\\
&-\mu_A\mu_M C_2^2 b^2\beta_1\beta_3 c+\mu_A C_2^2 ab\beta_1\beta_3^2 c\left(C_1-C_2\right)\big)\}+\mu_P C_1C_2 a^2\beta_3 r_2(-\mu_A C_2\beta_3+\mu_M b\beta_1)\\
&+(a-b)\left\{\mu_A C_1C_2 ab\beta_1\beta_3\beta_4\left(C_1-C_2\right)+\mu_A^2 C_1C_2 a\beta_3 c\left(C_1\beta_4+C_2\beta_3\right)\right.\\
&+\mu_A\mu_M\mu_P C_1C_2 b\beta_4 r_2\left(a-r_2\right)+2\mu_A\mu_M^2 C_1 bc\left(a\beta_1+\beta_2\right)\\
&+\left(C_2-2C_1\right)\left(\mu_A\mu_M b\beta_2\beta_3 c(C_1+C_2)\right)-\mu_A\mu_M^2 C_2 b\beta_2 c\\
&\left.+2\mu_M C_1 ab\beta_1\beta_4\left(-\mu_P C_2 br_2-\mu_A C_1 c\right)\right\}\\
&+\left(C_1-C_2\right)\left\{\left(C_2\beta_3-\mu_M\right)b\beta_2 c\left(\mu_A C_2 a\beta_3-\mu_M b\beta_2\right)\right.\\
&\left.+C_1 bc\left(\mu_A C_2 a-\mu_M b\beta_2^2\beta_4\right)\right\}+\left(2a-b+r_2\right)\left(C_1-C_2\right)\left(\mu_M b^2\beta_1\beta_2 c(C_1\beta_4+\mu_M)\right)\\
&+(a-b)^2\left(\mu_A\mu_M C_1C_2 b\beta_1\beta_4 c+\mu_A^2\mu_M C_1 c(-C_1\beta_4+\mu_M)\right)\\
&+\left(a+r_2\right)\left\{(a-b)\left((C_2-C_1)\left(\mu_M b^2\beta_1^2\beta_4 c+\mu_A C_2^2 ab\beta_1\beta_3^2 c\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
r_2 = &\mathit{Root\,of}\,\{\left(-\mu_P C_1\beta_4-\mu_P C_2\beta_3+\mu_M\mu_P\right)y^3+\left(\mu_P(a-b)\left(\mu_M-C_1\beta_4\right)\right.\\
&\left.+\beta_1\beta_2 c(C_2-C_1)-\mu_P C_2 a\beta_3-\mu_M\beta_1+B(C_1\beta_4+C_2\beta_3-\mu_M)\right)y^2\\
&+\left(\mu_P ab(\mu_M-C_1\beta_4)+\beta_3 c(C_2-C_1)(a\beta_1+\beta_2)+(a+b)(BC_1\beta_4-\mu_M\beta_1 c-\mu_M B)\right.\\
&\left.+\mu_A C_1\beta_3 c+(BC_2 a\beta_3-\mu_M\beta_2 c)\right)y+C_1 a(\mu_A\beta_3 c+Bb\beta_4)-\mu_M bc(a\beta_1+\beta_2)-\mu_M Bab\}
\end{aligned}
$$

Here, E_0 and E^* are disease-free equilibrium point and endemic equilibrium point respectively. The basic reproduction number R_0, is defined as expected number of allergy cases produced by single allergic individual. R_0 is the dominant eigenvalue of next generation matrix $G = \left(FV^{-1}\right)$ [15, 16]. Where, $F = \left(\frac{\partial f_i(E_0)}{\partial x_j}\right)$ and $V = \left(\frac{\partial v_i(E_0)}{\partial x_j}\right)$.
The F_j are the new allergy, while the V_i transfers of allergy from one compartment to another. Let $X = (P(t), A(t), M(t))$, then system (1) can be written as
$X' = f(X) - v(X)$ where,

$$f(X) = \begin{bmatrix} \beta_1 PA + \frac{\beta_2 PA}{a+P} \\ \frac{c_1\beta_4 AM}{c+A} + \frac{c_2\beta_3 PM}{b+P} \\ 0 \end{bmatrix} \text{ and } v(X) = \begin{bmatrix} \frac{\beta_4 AM}{c+A} + \mu_A A \\ \mu_M M \\ -BP + \beta_1 PA + \frac{\beta_2 PA}{a+P} + \frac{\beta_3 PM}{b+P} + \mu_P P^2 \end{bmatrix}$$

Hence, the next generation matrix is:

$$G = FV^{-1} = \begin{bmatrix} \frac{\beta_1 B}{\mu_P} + \frac{\beta_2 B}{a\mu_P + B} & 0 & 0 \\ 0 & \frac{C_2\beta_3 B}{(b\mu_P + B)\mu_M} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence, calculated reproduction number is

$$R_0 = \frac{B\left(\mu_A\mu_P C_2\beta_3\left(\mu_P a + B\right) + \left(\mu_P b + B\right)\left(\mu_M\mu_P\left(a\beta_1 + \beta_2\right) + \mu_M B\beta_1\right)\right)}{\mu_P\left(\mu_P a + B\right)\mu_A\left(\mu_P b + B\right)\mu_M} \qquad (2)$$

§3 Stability analysis of pollution-allergy model

In this section, local and global stability of the pollution-allergy model are analyzed.

3.1 Local stability

If system is reserved around, the equilibrium, then system have tendency to move towards the equilibrium point under some condition then the equilibrium point is called locally stable. In this section, local stability of equilibrium points of the system (1) is analysed.

Theorem 1: The infection free equilibrium point E_0 is locally asymptotically stable if $BC_2\beta_3 < \mu_M\left(\mu_P b + B\right)$ and $\mu_P B\beta_2 < \left(\mu_P a + B\right)\left(\mu_A\mu_P - B\beta_1\right)$.

Proof: The Jacobian matrix $J(E_0)$ for system (1) associated with point E_0 is given by:

$$J(E_0) = \begin{bmatrix} -B & -\frac{\beta_1 B}{\mu_P} - \frac{\beta_2 B}{a\mu_P + B} & -\frac{\beta_3 B}{(b\mu_P + B)} \\ 0 & \frac{\beta_1 B}{\mu_P} + \frac{\beta_2 B}{a\mu_P + B} - \mu_A & 0 \\ 0 & 0 & \frac{C_2\beta_3 B}{(b\mu_P + B)} - \mu_M \end{bmatrix}$$

The eigenvalues of matrix $J(E_0)$ are negative if
$\mu_P B\beta_2 < \left(\mu_P a + B\right)\left(\mu_A\mu_P - B\beta_1\right)$ and $BC_2\beta_3 < \mu_M\left(\mu_P b + B\right)$.

Theorem 2: The equilibrium point E_1 is locally asymptotically stable if $C_1\beta_4 < \mu_M$ and $\frac{\mu_A C_1\beta_3}{b} + \frac{BC_1\beta_4}{c} > \mu_M\left(\beta_1 + \frac{B}{c} + \frac{\beta_2}{a}\right)$.

Proof: The Jacobian matrix $J(E_1)$ for system (1) associated with point E_1 is given by:

$$\begin{bmatrix} B + \frac{1}{(-C_1\beta_4 + \mu_M)}\left(\frac{\beta_2\mu_M c}{a} - \frac{\beta_3\mu_A c C_1}{b} + \frac{\beta_1\mu_M c}{1}\right) & 0 & 0 \\ \frac{-\mu_M c}{(-C_1\beta_4 + \mu_M)}\left(\beta_1 + \frac{\beta_2}{a}\right) & -\frac{\mu_M\mu_A}{C_1\beta_4} - 2\mu_A & -\frac{\mu_M}{C_1} \\ \frac{C_2\beta_3\mu_A c C_1}{(-C_1\beta_4 + \mu_M)b} & -\mu_A C_1 + \frac{\mu_M\mu_A}{\beta_4} & -2\mu_M \end{bmatrix}$$

Eigenvalues of matrix $J(E_1)$ are:

$$\lambda_1^1 = -\frac{1}{2}\frac{\mu_M\mu_A - \sqrt{4\mu_M\mu_A C_1^2\beta_4^2 - 4\mu_M^2\mu_A C_1\beta_4 + \mu_M^2\mu_A^2}}{C_1\beta_4}$$
$$\lambda_2^1 = -\frac{1}{2}\frac{\mu_M\mu_A + \sqrt{4\mu_M\mu_A C_1^2\beta_4^2 - 4\mu_M^2\mu_A C_1\beta_4 + \mu_M^2\mu_A^2}}{C_1\beta_4}$$
$$\lambda_3^1 = -\frac{\mu_A C_1\beta_3 ac - \mu_M\beta_1 abc + BC_1 ab\beta_4 - \mu_M Bab - \mu_M b\beta_2 c}{(-C_1\beta_4 + \mu_M)\, ab}$$

$\lambda_1^1 < 0$ and $\lambda_2^1 < 0$ when $C_1\beta_4 < \mu_M$
and $\lambda_3^1 < 0$ when $\frac{\mu_A C_1\beta_3}{b} + \frac{BC_1\beta_4}{c} > \mu_M\left(\beta_1 + \frac{B}{c} + \frac{\beta_2}{a}\right)$.
Hence, the equilibrium point E_1 is locally asymptotically stable under these conditions.
Theorem 3: The equilibrium point E_1 is locally asymptotically stable if following three conditions are satisfied:

i. $\mu_M < C_2\beta_3$ and $\mu_M\mu_P b + \mu_M B < BC_2\beta_3 < \mu_M\mu_P b$

ii. $((a-b)\mu_M > C_2 a\beta_3)$

iii. $a > b$, $\mu_M b\beta_2 > \mu_A C_2 a\beta_3$ and $\mu_M\beta_1 c > C_2(\beta_1\beta_3 c + \mu_P b\beta_4)$

Proof: Let $J(E_2) = [a_{ij}]$ be the Jacobian matrix for system (1) associated with point E_2. Where,
$a_{11} = B + \frac{2\mu_P\mu_M b}{-C_2\beta_3+\mu_M} + \frac{\mu_P\mu_M b - BC_2\beta_3 + \mu_M B}{-C_2\beta_3+\mu_M} + \frac{\mu_M(\mu_P\mu_M b - BC_2\beta_3 + \mu_M B)}{C_2\beta_3(-C_2\beta_3+\mu_M)}$,
$a_{12} = \frac{\beta_1\mu_M b}{-C_2\beta_3+\mu_M} + \frac{\beta_2\mu_M b}{\mu_M a - C_2\beta_3 a - \mu_M b}$, $a_{13} = -\frac{\mu_M}{C_2}$, $a_{21} = 0$,
$a_{22} = -\frac{\beta_1\mu_M b}{-C_2\beta_3+\mu_M} - \frac{\beta_2\mu_M b}{\mu_M a - C_2\beta_3 a - \mu_M b} + \frac{\beta_4 bC_2(\mu_P\mu_M b - BC_2\beta_3 + \mu_M B)}{(-C_2\beta_3+\mu_M)^2 c} - \mu_A$, $a_{23} = 0$,
$a_{31} = \frac{C_2(\mu_P\mu_M b - BC_2\beta_3 + \mu_M B)}{(-C_2\beta_3+\mu_M)} - \frac{\mu_M(\mu_P\mu_M b - BC_2\beta_3 + \mu_M B)}{\beta_3(-C_2\beta_3+\mu_M)}$,
$a_{32} = -\frac{\beta_4 bC_1C_2(\mu_P\mu_M b - BC_2\beta_3 + \mu_M B)}{(-C_2\beta_3+\mu_M)^2 c}$ and $a_{33} = 0$.
The eigenvalues of matrix $J(E_2)$ are:

$$\lambda_1^2 = \frac{\alpha_1 + \sqrt{\alpha_2}}{2C_2\beta_3(-C_2\beta_3 + \mu_M)},\ \lambda_2^2 = \frac{\alpha_1 - \sqrt{\alpha_2}}{2C_2\beta_3(-C_2\beta_3 + \mu_M)} \text{ and } \lambda_3^2 = \frac{n}{m}.$$

Where, $\alpha_1 = \mu_M C_2\beta_3(\mu_P b - B) + \mu_M^2(\mu_P b + B)$,

$$\begin{aligned}\alpha_2 =& (\mu_P b + 4C_2\beta_3)\left(\mu_M^2\mu_P b(C_2^2\beta_3^2 + \mu_M^2)\right)(\mu_P b - 4C_2\beta_3)\left(2\mu_M^2 C_2\beta_3(\mu_M\mu_P b - BC_2\beta_3)\right)\\ &+ 4\mu_M BC_2^3\beta_3^2(\mu_M + C_2\beta_3) - 2\mu_M^3 BC_2\beta_3(B + 6C_2\beta_3) + 2\mu_M^4 B(\mu_P b + 2C_2\beta_3)\\ &+ \mu_M^2 B^2\left(C_2^2\beta_3^2 + \mu_M^2\right),\end{aligned}$$

$$\begin{aligned}-n =& \mu_M(a-b)\left(\mu_A c(\mu_M - C_2\beta_3)^2 + C_2 b\beta_3(-\mu_M\beta_1 c + BC_2\beta_4)\right.\\ &\left.+ \mu_M b(-\mu_P C_2 b\beta_4 + \mu_M\beta_1 c - BC_2\beta_4)\right) + \left(\mu_M^2 - C_2\beta_3\right)^2\\ &(\mu_M b\beta_2 c - \mu_A C_2 a\beta_3 c) + C_2 ab\beta_3(BC_2\beta_4 - \mu_M\beta_1 c)\\ &+ \mu_M\mu_P C_2^2 ab^2\beta_4\beta_3 \text{ and}\\ m =& (\mu_M - C_2\beta_3)^2((a-b)\mu_M - C_2 a\beta_3).\end{aligned}$$

Let $\mu_M < C_2\beta_3$. Note that, real part of eigenvalues λ_1^2 and λ_2^2 are negative when $\alpha > 0$ and $\alpha_1^2 > \alpha_2$ when which implies $\mu_M\mu_P b + \mu_M B < BC_2\beta_3 < \mu_M\mu_P b$. Eigenvalue λ_3^2 is negative when $n,\, m > 0$. $m > 0$ if $((a-b)\mu_M > C_2 a\beta_3)$ whereas $n > 0$ if $a > b$, $\mu_M b\beta_2 > \mu_A C_2 a\beta_3$ and $\mu_M \beta_1 c > C_2\left(\beta_1\beta_3 c + \mu_P b\beta_4\right)$. Hence, the equilibrium point E_2 is locally asymptotically stable under all these conditions.

Theorem 4: The equilibrium point E_3 is locally asymptotically stable if the model satisfies following conditions:

i. $\beta_1 c > v$ and $(b+r_1)\mu_M > C_2\beta_3 r_1$

ii. $\mu_P > \beta_1$, $\mu_A > B$ and $\mu_P r_1 > \max\left\{\beta_2 + v,\ \left(1+\frac{\beta_2}{\beta_1}\right)v\right\}$

iii. $a\beta_1 > 8\mu_A > \beta_1 r_1$, $\min\left\{Ba^2\beta_1 r_1,\ B\beta_1 r_1^3\right\} > 2\mu_A a^2 v$,
$\beta_1 r_1^4 > \mu_A a^3$, $B > 2\mu_P r_1$ and $\mu_A > B > 2\mu_P r_1$

Proof: $J(E_3) = [b_{ij}]$ is the Jacobian matrix for system (1) associated with point E_3. Where,
$b_{11} = -2\mu_P r_1 + B + \frac{\beta_2 a v}{\beta_1 (a+r_1)^2} + v$, $b_{12} = -\frac{\beta_2 r_1}{a+r_1} - \beta_1 r_1$, $b_{13} = -\frac{\beta_3 r_1}{b+r_1}$,
$b_{21} = -v - \frac{\beta_2 v a}{\beta_1 (a+r_1)^2}$, $b_{22} = \beta_1 r_1 + \frac{\beta_2 r_1}{(a+r_1)} - \mu_A$, $b_{23} = \frac{\beta_4 v}{(\beta_1 c - v)}$, $b_{31} = 0$, $b_{32} = 0$,
$b_{33} = -\frac{C_1\beta_4 v}{(\beta_1 c - v)} + \frac{C_2\beta_3 r_1}{b+r_1} - \mu_M$
The eigenvalues of matrix $J(E_3)$ are:

$$\lambda_1^3 = -\frac{(\beta_1 c - v)\left((b+r_1)\mu_M - C_2\beta_3 r_1\right) + (b+r_1)\, C_1\beta_4 v}{(b+r_1)\,(\beta_1 c - v)}$$

$$\lambda_2^3 = -\frac{1}{2\beta_1 (a+r_1)^2}\left(\alpha_3 - \sqrt{\alpha_4}\right) \text{ and } \lambda_3^3 = -\frac{1}{2\beta_1 (a+r_1)^2}\left(\alpha_3 + \sqrt{\alpha_4}\right)$$

Where,

$$\begin{aligned}\alpha_3 =& (\mu_P - \beta_1)\left(a^2\beta_1 r_1 + 2a\beta_1 r_1^2 + \beta_1 r_1^3\right) + (\mu_A - B)\left(a^2\beta_1 + 2a\beta_1 r_1 + \beta_1 r_1^2\right)\\ &+ a\left(\mu_P a\beta_1 r_1 - av(\beta_1+\beta_2)\right) + \beta_1 r_1 a\left(\mu_P r_1 - v\right) + \left(\mu_P r_1 - \beta_2 - v\right)\left(\beta_1 r_1 a + \beta_1 r_1^2\right)\\ \alpha_4 =& (a+r_1)^4\left((4\mu_P\beta_1^2 r_1^2 - 2\mu_A\beta_1^2 r_1)(\mu_P + \beta_1) + \beta_1^4 r_1^2(\beta_1^2 v - 8\mu_P\beta_1^2 r_1)(B+v)\right.\\ &\left.-2\mu_P\mu_A\beta_1^2 r_1 + (\mu_A + B)(\mu_A\beta_1^2 + B\beta_1^2) + B\beta_1^2 v\right) + (a+r_1)^3\left(2\beta_1^2\beta_2 r_1^2(\mu_P + \beta_1)\right.\\ &\left.- 2\beta_1^2\beta_2 r_1(\mu_A + B) + 2\beta_1^2\beta_2 r_1(\mu_P r_1 + v)\right) + (a+r_1)^2\left(\beta_1^2\beta_2(r_1^2 + 2Bav)\right.\\ &\left.+2a\beta_1\beta_2 v(\mu_A + v - 2\mu_P r_1)\right) + \beta_2^2 v(2a\beta_1 r_1 + av)(a+r_1)\end{aligned}$$

The equilibrium point E_3 is locally asymptotically stable when all these eigenvalues have negative real part. Clearly, $\lambda_1^3 < 0$ if $\beta_1 c > v$ and $(b+r_1)\mu_M > C_2\beta_3 r_1$.
$\alpha_3 > 0$ if $\mu_P > \beta_1$, $\mu_A > B$ and $\mu_P r_1 > \max\left\{\beta_2 + v,\ \left(1+\frac{\beta_2}{\beta_1}\right)v\right\}$.
For $\alpha_3 > 0$ and $\alpha_4 < 0$, real part of eigenvalues λ_2^3 and λ_3^3 are negative; if $\alpha_3 > 0$ and $\alpha_4 > 0$ then these two e eigenvalue are negative only when $\alpha_3^2 - \alpha_4 > 0$.
$\alpha_3^2 - \alpha_4 > 0$ for the conditions: $a\beta_1 > 8\mu_A > \beta_1 r_1$, $\min\left\{Ba^2\beta_1 r_1,\ B\beta_1 r_1^3\right\} > 2\mu_A a^2 v$, $\beta_1 r_1^4 > \mu_A a^3$, $B > 2\mu_P r_1$ and $\mu_A > B > 2\mu_P r_1$.

Theorem 5: The endemic equilibrium point E^* is locally asymptotically stable if following three conditions are satisfied:

i. $B < 2\mu_P P^* + \beta_1 A^*$

ii. $\beta_1 P^* + \frac{\beta_2 P^*}{a+P^*} < \frac{\beta_4 M^* c}{(c+A^*)^2} + \mu_A$

iii. $\frac{C_1\beta_4 A^*}{c+A^*} + \frac{C_2\beta_3 P^*}{b+P^*} < \mu_M$

Proof: $J(E^*) = [y_{ij}]$ is a the Jacobian matrix for system (1) associated with point E^*. Where, $y_{11} = -2\mu_P P^* + B - \frac{\beta_2 A^* a}{(a+P^*)^2} - \frac{\beta_3 M^* b}{(b+P^*)^2} - \beta_1 A^*$, $y_{12} = -\frac{\beta_2 P^*}{a+P^*} - \beta_1 P^*$
$y_{13} = -\frac{\beta_3 P^*}{b+P^*}$, $y_{21} = \beta_1 A^* + \frac{\beta_2 A^* a}{(a+P^*)^2}$, $y_{22} = \beta_1 P^* + \frac{\beta_2 P^*}{a+P^*} - \frac{\beta_4 M^* c}{(c+A^*)^2} - \mu_A$, $y_{23} = -\frac{\beta_4 A^*}{c+A^*}$,
$y_{31} = \frac{C_2\beta_3 M^* b}{(b+P^*)^2}$, $y_{32} = \frac{C_1\beta_4 M^* c}{(c+A^*)^2}$,
$y_{33} = \frac{C_1\beta_4 A^*}{c+A^*} + \frac{C_2\beta_3 P^*}{b+P^*} - \mu_M$.
Characteristic polynomial of matrix $J(E^*)$ is $Ch_{E^*}(x)$, and it defines as:

$$\begin{aligned} Ch_{E^*}(x) = x^3 &- (y_1 + y_5 + y_9)x^2 + (y_1y_5 + y_1y_9 - y_2y_4 - y_3y_7 + y_5y_9 \\ &- y_6y_8)x - y_1y_5y_9 + y_1y_6y_8 + y_2y_4y_9 - y_2y_6y_7 - y_3y_4y_8 \\ &+ y_3y_5y_7 \end{aligned}$$

With the help of Routh-Harwitz criteria, we can say that real part of all the roots of characteristic polynomial $Ch_{E^*}(x)$ are negative if the model satisfies following three conditions:

$$(-y_1 - y_5 - y_9) > 0, -y_1y_5y_9 + y_1y_6y_8 + y_2y_4y_9 - y_2y_6y_7 - y_3y_4y_8 + y_3y_5y_7 > 0 \text{ and}$$

$$\begin{aligned} (-y_1 - y_5 - y_9)\left(y_1(y_5 + y_9) - y_2y_4 - y_3y_7 + y_5y_9 - y_6y_8\right) > \\ \left(y_1(y_6y_8 - y_5y_9) + y_2(y_4y_9 - y_6y_7) + y_3(y_5y_7 - y_4y_8)\right). \end{aligned}$$

Which implies, y_1, y_5, $y_9 < 0$ and it generate following three conditions,
$B < 2\mu_P P^* + \beta_1 A^* \Rightarrow y_1 < 0$, $\beta_1 P^* + \frac{\beta_2 P^*}{a+P^*} < \frac{\beta_4 M^* c}{(c+A^*)^2} + \mu_A \Rightarrow y_5 < 0$ and $\frac{C_1\beta_4 A^*}{c+A^*} + \frac{C_2\beta_3 P^*}{b+P^*} < \mu_M \Rightarrow y_9 < 0$. Hence, equilibrium point E^* is locally asymptotically stable under all these conditions.

3.2 Global stability

Global stability of endemic equilibrium point is verified using some basic results of graph-theoretic method described in Harary [8] and West DB [4]. We need to understand some results and definitions from graph theory to prove global stability using graph-theoretic method.
A directed graph $G(V, E)$ consists of a set of vertices V and a set of arcs E where arc is defined as an order pair (i, j) from its initial vertex i to terminal vertex j, and if $i = j$ then arc is called self-loop defined on vertex i or j. The in-degree of a vertex i is the number of arcs in the graph whose terminal vertex is i and it is denoted by $d^-(i)$, and out-degree $d^+(i)$ of vertex i is the number of arcs whose initial vertex is i. A spanning set K of graph G having same vertex set is called sub-digraph. A directed graph G is called weighted if each arc is assigned a positive weight. The weight $\mathcal{W}(K)$ of sub-graph K is product of weight on all its arcs.
A rooted tree T is a connected sub-digraph of G, in which the in-degree of one vertex is zero, whereas each of the remaining vertices has in-degree one. A spanning tree is a sub-graph of a graph, which includes all the vertices of with minimum number of arcs. A directed path in a directed graph is a sequence of arcs, which connect a sequence of vertices where all the arcs should be directed in the same direction. If (G, A) is a weighted directed graph with n vertices then the weight matrix is denoted by $A = (a_{ij})_{n\times n}$, where $a_{ij} > 0$ is weight of arc (i, j) if it exist and zero otherwise. A directed graph G is strongly connected if, there exist a directed

path from any pair of distinct vertices. A weighted directed graph $(G,\, A)$ is strongly connected if and only if the weight matrix is irreducible [1]. The Laplacian matrix $L = (l_{ij})$ of G is defined as:

$$l_{ij} = \begin{cases} -a_{ij} & i \neq j \\ \sum_{i \neq k} a_{ik} & i = k \end{cases} \tag{3}$$

Theorem 6: (Kirchhoff's matrix tree theorem) For $n \leq 2$, assume that c_i is the cofactor of $l_i i$ in L. Then $c_i = \sum_{T \in T_i} \mathcal{W}(T)$, $i = 1,\, 2,\, ..., n$, where T_i is the set of all spanning tree T of graph $(G,\, A)$ that are rooted at vertex i. Moreover, if graph $(G,\, A)$ is strongly connected, then $c_i > 0$ for $1 \leq i \leq n$.

Lemma 1: Let c_i be as given in Kirchhoff's matrix tree theorem. If $a_{ij} > 0$ and $d^+(j) = 1$ for some $1 \leq i, j \leq n$, then $c_i a_{ij} = \sum\limits_{k=1}^{n} c_j a_{jk}$. **Lemma 2:** Let c_i be as given in the in Kirchhoff?s matrix tree theorem. If $a_{ij} > 0$ and $d^-(j) = 1$ for some $1 \leq i, j \leq n$, then $c_i a_{ij} = \sum\limits_{k=1}^{n} c_k a_{ki}$.

Theorem 7: Suppose that the following assumptions are satisfied:

i. There exist functions $V_i : U \to \mathbb{R}$, $G_{ij} : U \to \mathbb{R}$ and constants $a_{ij} \geq 0$ such that for every $1 \leq i \leq n$, $V'_i \leq \sum\limits_{j=1}^{n} G_{ij}(x)$ for $x \in U$.

ii. For $M = [a_{ij}]$, each directed cycle C of $(G,\, M)$ has $\sum\limits_{(s,\, r) \in \varepsilon(C)} G_{rs}(x) \leq 0$ for $x \in U$, where $\varepsilon(C)$ denotes the arc set of the directed cycle C.

Then the function $V(x) = \sum\limits_{i=1}^{n} c_i V_i(x)$, with constants $c_i \geq 0$ as given in the lemma 1, satisfies $V^{'} \leq 0$, Hence V is a Lyapunov function for the system.

Theorem 8: The endemic equilibrium point E^* of system (1) is globally asymptotically stable.

Proof: First we construct the Lyapunov function $V(t)$ using graph-theoretic method [23]. For that let we take,

$$V_1 = P - P^* - P^* ln\frac{P}{P^*},\; V_2 = A - A^* - A^* ln\frac{A}{A^*} \text{ and } V_3 = M - M^* - M^* ln\frac{M}{M^*}$$

Differentiating V_1 with respect to t, we get,

$$\begin{aligned} V'_1 &= \left(1 - \frac{P^*}{P}\right)\left(BP - \beta_1 PA - \frac{\beta_2 PA}{a+P} - \frac{\beta_3 PM}{b+P} - \mu_P P^2\right) \\ &= \left(1 - \frac{P^*}{P}\right)\Bigg(\beta_1 P^* A^* - \beta_1 PA + \frac{\beta_2 P^* A^*}{a+P^*} - \frac{\beta_2 PA}{a+P} + \frac{\beta_3 P^* M^*}{b+P^*} - \frac{\beta_3 PM}{b+P} \\ &\quad + \mu_P {P^*}^2 - \mu_P P^2\Bigg) \\ &= \left(1 - \frac{P^*}{P}\right)\left(\left(\beta_1 + \frac{\beta_2}{a+P^*}\right)(P^* A^* - PA) + \frac{\beta_3}{b+P^*}(P^* M^* - PM)\right) \\ &\quad - \frac{\mu_P}{P}(P^* - P)^2 (P^* + P) \end{aligned}$$

$$\leq (\beta_1+\beta_2)\,P^*A^*\left(1-\frac{P^*}{P}\right)\left(1-\frac{PA}{P^*A^*}\right)+P^*M^*\beta_3\left(1-\frac{P^*}{P}\right)\left(1-\frac{PM}{P^*M^*}\right)$$
$$= a_{12}G_{12}+a_{13}G_{13}$$

Similarly,

$$V'_2=\left(1-\frac{A^*}{A}\right)\left(\beta_1 PA+\frac{\beta_2 PA}{a+P}-\frac{\beta_4 AM}{c+A}-\mu_A A\right)$$
$$\leq (\beta_1+\beta_2)\,P^*A^*\left(1-\frac{A^*}{A}\right)\left(1-\frac{PA}{P^*A^*}\right)=a_{21}G_{21}$$
$$V'_3=\left(1-\frac{M^*}{M}\right)\left(\frac{c_1\beta_4 AM}{c+A}+\frac{c_2\beta_3 PM}{b+P}-\mu_M M\right)$$
$$\leq C_1\beta_4A^*M^*\left(1-\frac{M^*}{M}\right)\left(1-\frac{AM}{A^*M^*}\right)+C_2\beta_3P^*M^*\left(1-\frac{M^*}{M}\right)\left(1-\frac{PM}{P^*M^*}\right)$$
$$= a_{32}G_{32}+a_{31}G_{31}$$

Hence, the weighted graph is been constructed as below: The above weighted directed graph

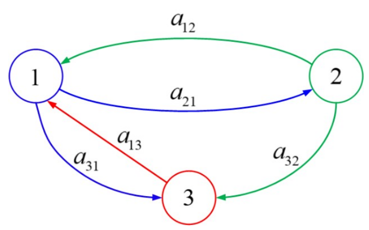


Figure 2. weighted graph

G has three cycles $G_{21}+G_{12}=0$, $G_{21}+G_{32}+G_{13}=0$ and $G_{13}+G_{31}=0$. As describe in theorem 7, $V(x)=\sum_{i=1}^{n} c_iV_i(x)$ is a Lyapunov function where relations between constants c_i's are generated using lemma 1 and lemma 2 as $c_1a_{13}=c_3a_{31}+c_3a_{32}$ and $c_2a_{21}=c_1a_{12}+c_3a_{32}$. Let $c_3=n$, which implies

$$c_1=\frac{n}{a_{13}}\,(a_{31}+a_{32})\ \text{ and } c_2=\frac{n}{a_{21}}\left(\frac{a_{12}}{a_{13}}(a_{31}+a_{32})+a_{32}\right).$$

Calculated Lyapunov function is

$$V(t)=\frac{n}{a_{13}}\,(a_{31}+a_{32})\,V_1+\frac{n}{a_{21}}\left(\frac{a_{12}}{a_{13}}(a_{31}+a_{32})+a_{32}\right)V_2+nV_3$$

Note that, $\{E^*\}$ is the only invariant set and $V^{'}=0$ which implies point E^* is globally asymptotically stable.

§4 Z-Control

The spread of allergic diseases is not always easily controllable as the immunity is decreasing day by day due to modern lifestyle and fast food. In this section, the Z-type control mechanism is applied to the pollution-allergy model to investigate impact of pollution on allergy spread. In an indirect Z-control mechanism, the most affected variable can be restriced with the help of other variables, the controller can be applied to the latter one. This mechanism helps to stabilize the systems of differential equations using the error-based dynamic method [3][21][22]. In this control theory, by adding the control variable on class of medicated individuals, the pollution-allergy model (1) is modified as below:

$$\begin{aligned}
\frac{dP}{dt} &= BP - \beta_1 PA - \frac{\beta_2 PA}{a+P} - \frac{\beta_3 PM}{b+P} - \mu_P P^2 \\
\frac{dA}{dt} &= \beta_1 PA + \frac{\beta_2 PA}{a+P} - \frac{\beta_4 AM}{c+A} - \mu_A A \\
\frac{dM}{dt} &= \frac{c_1\beta_4 AM}{c+A} + \frac{c_2\beta_3 PM}{b+P} - \mu_M M - u(t)M
\end{aligned} \tag{4}$$

Where $u(t)$ is the indirect control variable for the medicated population acting on the disease dynamics, which can be positive or negative quantity. The purpose is to adjust the medication pressure on the allergic population to achieve a desired state by forcing the error function $e(t)$ to converge exponentially to zero, that is $e(t) = y(t) - y_d(t) \to 0$ as $t \to \infty$, where $y(t)$ is actual output and $y_d(t)$ is desire output. Such a purpose can be achieved by adopting the following design formula,

$$\dot{e}(t) = -\lambda e(t) \tag{5}$$

Equation (4) is named as design formula and $\lambda > 0 \in \mathbb{R}$ is the design parameter. λ use to measure convergence rate. To designing the control law using the Z-type dynamic method, first an error function $e(t)$ should be defined. Second, obtain an explicit expression of the control variable $u(t)$, here the design formula is used for this purpose.

Let first error function defined as $v_1 = e_1 = A(t) - A_d(t)$. Using the design formula of the Z-type dynamic method, we have $\dot{A}(t) - \dot{A}_d(t) = -\lambda A(t) - A_d(t)$. Let the second error function is defined as $e_2 = v_2 = \dot{v}_1 + \lambda v_1$, hence $v_2 = \dot{A}(t) - \dot{A}_d(t) + \lambda(A(t) - A_d(t))$.

Let suppose that error functions v_1 and v_2 decays exponentially in time, i.e., $\dot{v}_1(t) = -\lambda v_1(t)$ and $\dot{v}_2(t) = -\lambda v_2(t)$. Using design formula and error function, control variable $u(t)$ can be formulated as below:

$$\begin{aligned}
&\ddot{A}(t) - \ddot{A}_d(t) + \lambda(\dot{A}(t) - \dot{A}_d(t)) = -\lambda\left[\dot{A}(t) - \dot{A}_d(t) + \lambda(A(t) - A_d(t))\right] \\
&\Rightarrow \beta_1\dot{P}A + \beta_1 P\dot{A} + \frac{\beta_2\dot{A}P}{a+P} + \frac{\beta_2 A\dot{P}a}{(a+P)^2} - \frac{\beta_4 A\dot{M}}{c+A} - \frac{\beta_4\dot{A}M}{(c+A)^2} - \mu_A\dot{A}(t) \\
&\qquad - \ddot{A}_d(t) + 2(\dot{A}(t) - \dot{A}_d(t)) + \lambda^2(A(t) - A_d(t)) = 0 \\
&\Rightarrow \dot{M} = \frac{c+A}{\beta_4 A}\left(\beta_1\dot{P}A + \beta_1 P\dot{A} + \frac{\beta_2\dot{A}P}{a+P} + \frac{\beta_2 A\dot{P}a}{(a+P)^2} - \frac{\beta_4\dot{A}M}{(c+A)^2}\right. \\
&\qquad \left. -\mu_A\dot{A}(t) - \ddot{A}_d(t) + 2(\dot{A}(t) - \dot{A}_d(t)) + \lambda^2(A(t) - A_d(t))\right)
\end{aligned}$$

Putting $\dot{M}$ in system (1), we get formulation of Z-type controller:

$$\begin{aligned}
u(t) =& \frac{c_1\beta_4 A}{c+A} + \frac{c_2\beta_3 P}{b+P} - \mu_M - \frac{c+A}{\beta_4 AM}\left(\beta_1\dot{P}A + \beta_1 P\dot{A} + \frac{\beta_2\dot{A}P}{a+P} + \frac{\beta_2 A\dot{P}a}{(a+P)^2}\right. \\
&\left. -\frac{\beta_4\dot{A}Mc}{(c+A)^2} - \mu_A\dot{A}(t) - \ddot{A}_d(t) + 2(\dot{A}(t) - \dot{A}_d(t)) + \lambda^2(A(t) - A_d(t))\right)
\end{aligned}$$

Theorem 9: For a continuously differentiable and bounded desire state $A_d(t)$, starting from a positive initial state $[P(0), A(0), M(0)]^T$, the tracking error of e_1 given model (4) equipped with Z-type controller converges to zero exponentially.

Proof: Let $\bar{e} = [e_1, e_2]^T = [A(t) - A_d(t),\ \dot{A}(t) - \dot{A}_d(t) + \lambda(A(t) - A_d(t))]^T$ be the error vector of the modified pollution-allergy model equipped with the controller $u(t)$. Let Lyapunov function constructed as follow:

$$L = \frac{1}{2}\bar{e}^T\bar{e} = \frac{1}{2}\left(e_1^2 + e_2^2\right) \geq 0$$

Observe that $L = 0$, only when $\bar{e} = 0$ and L is positively definite. Its derivative with respect to time is given by: $L' = e_1\dot{e}_1 + e_2\dot{e}_2 = -\lambda(e_1^2 + e_2^2) \leq 0$. Since $L' < 0$ if and only if $e \neq 0$. L' is negatively definite. The Lyapunov function $L \to \infty$ if $e \to \infty$. According to Lyapunov stability theory, the error vector e converge to zero.

Error function can be rewritten as $L' = -\lambda(e_1^2 + e_2^2) = -\lambda e^T e = -2\lambda L$.

Thus, $L = L(0)\,exp(-2\lambda L),\ \forall t \geq 0$, that is, error vector e converges to zero exponentially and λ is the rate of convergence. Hence using Z-type mechanism we can drive the system to the desire state at optimum rate by making error function converge to zero. It is that medication can play crucial role in controlling spread of allergy, which is also investigated by Z-type controlling theory.

§5 Numerical simulation

It can be interpreted from the above figure 3 that the elevated levels of pollution results in allergic reactions; hence, during the initial days an incline towards allergic diseases is observed. As we improve medication, it is observed in graph that allergy decreases. Hence, from the graph it can be inferred that medication plays a crucial role in controlling spread of allergy.

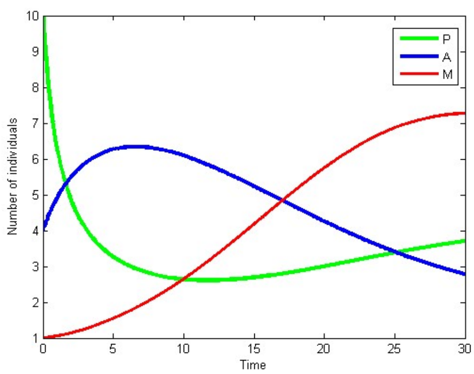
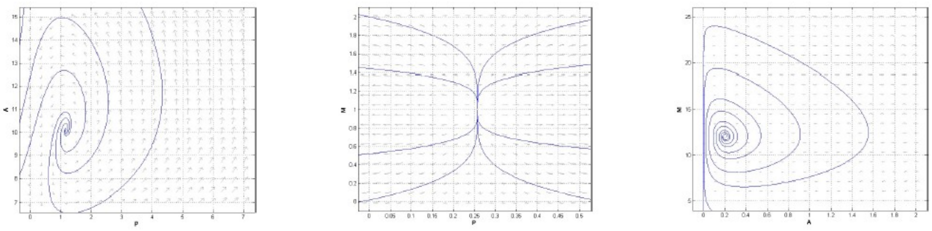


Figure 3. Variations in pollution-allergy model.

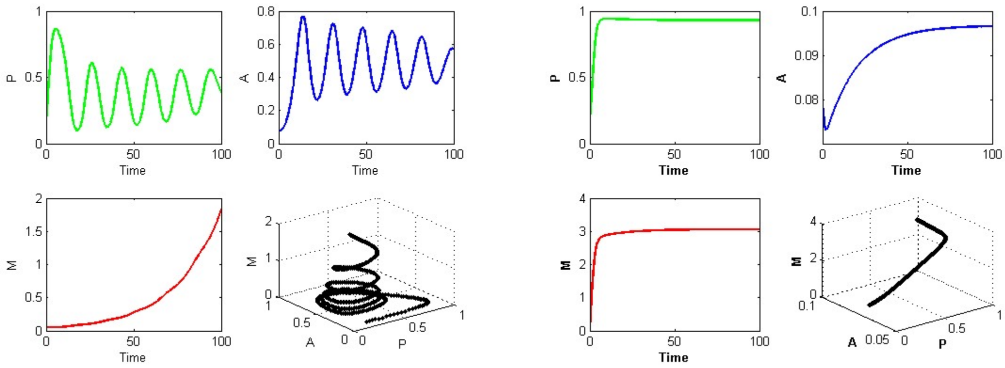
Figure 4 shows a trajectory around the endemic equilibrium point for dynamical system (1). It is observed that the initial population sizes close to the equilibrium values stay near the initial sizes, even though the populations oscillate periodically around the equilibrium points.



(a) (b) (c)

Figure 4. Solution curve and trajectory field.

Direction of trajectory towards equilibrium point shows stability of the model. Periodic oscillation in Figure (a) and (c) shows periodicity in allergic diseases due to pollution and regular need of medication in allergic class which can create chaos in society. Figure 5 (a) shows the limit



(a) (b)

Figure 5. Oscillation in compartments of the pollution-allergy model.

cycle oscillations around the endemic equilibrium point without applying Z-control mechanism. For force of infection, $\beta_2 < 0.6$ the system is stable; whereas at $\beta_2 = 0.6$, the initial oscillations can be noticed. Improvement in periodic oscillations is observed if the force of infection is increased further then 0.6, the system enters into chaotic situation. Periodic oscillation indicates repetition of allergic disease after some time, which shows the instability of the system. While after applying Z-control, it can be inferred that oscillations in the compartments cease. Figure5 (b) shows the stable controlled dynamic system after applying Z control.

Figure 6 represents the bifurcation diagram of the pollution-allergy model, which shows qualitative change in family of differential equations of system (1). A sudden qualitative change in behaviour of the dynamical system is been observed when a gradual smooth quantitative change made in the parameters. Herein the maximum and minimum values of the oscillations are plotted in blue and red colours respectively. In addition, periodicity of disease increases as force of infection increase from 0.6. It can be inferred that with a rise in the force of infection,

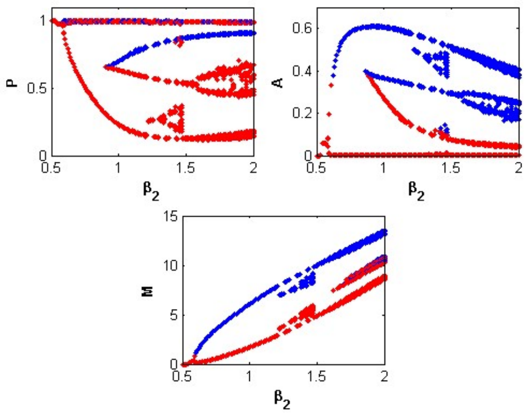


Figure 6. Bifurcation with respect to β_2.

the system tends to be more chaotic.

Figure 7 shows variation in control variable after applying Z-control on the model. Figure 8

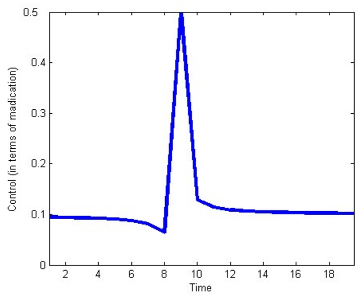


Figure 7. Control variable.

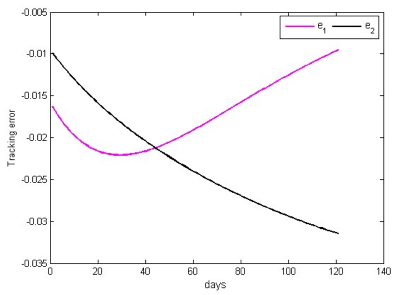


Figure 8. Error functions.

displays variation in tracking errors used in design formula. Graph of control variable shows that, after eight weeks, one needs to increase medication up to 50% to reduce the impact of allergy due to pollution. Difference between actual and desired output is minimized by applying the Z-type dynamic method. In the graph, it is observed that effect of second error function is more effective to stabilise the system.

§6 Conclusion

In the present study, the system of a dynamic model of allergic pollution is studied by mathematical modeling. Local stability of three independent equilibrium points is proved with definite conditions, whereas global stability is proved using the graph-theoretic method. It is observed that a major factor for allergy is pollution. Some factors related to the spreading of allergy are hard to control physically, those results in the chaos. Hence, it is required to apply control in a different way. To stabilize periodic oscillations in the compartments of the pollution-allergy model we put a Z-type control mechanism as a medication, which shows that allergic diseases due to a polluted environment can be controlled. It is observed that the rate of force of infection higher than 60% can create chaos. After applying the Z-control mechanism, it can be inferred that improvement in medication up to 50% helps to control the spread of allergy. This means that allergy spread can be controlled by improving the quality and dosage of medications. Numerical simulations graphically represent the visualization of threshold, evolution, and cause of expiration of allergy that also suggests that improvement in medication plays a major role to achieve a stable environment.

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