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Classification and existence of positive entire k-convex radial solutions for generalized nonlinear k-Hessian system

ZHANG Li-hong[1] YANG Ze-dong[1]
WANG Guo-tao[1,*] Mohammad M. Rashidi[2]

Abstract. In this paper, we consider the following generalized nonlinear k-Hessian system

$$\begin{cases} \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right)=\varphi(|x|, z_1, z_2), & x \in \mathbb{R}^N, \\ \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right)=\psi(|x|, z_1, z_2), & x \in \mathbb{R}^N, \end{cases}$$

where $\mathcal{G}$ is a nonlinear operator and $S_k\left(\lambda\left(D^2 z\right)\right)$ stands for the k-Hessian operator. We first are interested in the classification of positive entire k-convex radial solutions for the k-Hessian system if $\varphi(|x|, z_1, z_2)=b(|x|)\varphi(z_1, z_2)$ and $\psi(|x|, z_1, z_2)=h(|x|)\psi(z_1)$. Moreover, with the help of the monotone iterative method, some new existence results on the positive entire k-convex radial solutions of the k-Hessian system with the special non-linearities ψ, φ are given, which improve and extend many previous works.

§1 Introduction

In this paper, the main purpose is to consider the positive k-convex radial solutions of the following generalized k-Hessian system involving a nonlinear operator

$$\begin{cases} \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right)=\varphi(|x|, z_1, z_2), & x \in \mathbb{R}^N, \\ \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right)=\psi(|x|, z_1, z_2), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $\psi, \varphi \in C(\mathbb{R}^N \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, $\mathcal{G}$ is a nonlinear operator satisfying

$$\Upsilon=\left\{\mathcal{G} \in C^2([0,+\infty),[0,+\infty)) \mid \textit{there exists a constant } p>0 \textit{ such that } \textit{for any } 0<l<1,\ \mathcal{G}(lt) \leq l^p \mathcal{G}(t)\right\},$$

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*Corresponding author.

which is defined in [1]. $S_k\left(\lambda\left(D^2 z\right)\right)$ stands for the k-Hessian operator as follows

$$S_k\left(\lambda\left(D^2 z\right)\right)=\sum_{1\le j_1<\cdots<j_k\le N}\lambda_{j_1}\cdot\lambda_{j_2}\cdots\lambda_{j_k},\quad k=1,2,...,N,$$

where $D^2 z$ is the Hessian matrix of z, $\lambda_1,\lambda_2,...,\lambda_N$ are the eigenvalues of $D^2 z$ and $\lambda\left(D^2 z\right)$ $=(\lambda_1,\lambda_2,\cdots,\lambda_N)$ is the vector of eigenvalues of $D^2 z$. Here define

$$\Gamma_k:=\left\{\lambda\in\mathbb{R}^N: S_l\left(\lambda\left(D^2 z\right)\right)>0, 1\le l\le k\right\}.$$

A function $z\in C^2(\mathbb{R}^N)$ is k-convex in $\mathbb{R}^N$ if $\lambda\left(D^2 z(x)\right)\in\Gamma_k$ for all $x\in\mathbb{R}^N$.

It is not hard to see that $S_k\left(\lambda\left(D^2 z\right)\right)$ is a discrete collection of partial differential operators including the Laplace operator, the Monge-Ampère operator and many well known other operators. For $k=1$, $S_1\left(\lambda\left(D^2 z\right)\right)$ is the so-called Laplace operator $\triangle z$ and $k=N$, $S_N\left(\lambda\left(D^2 z\right)\right)$ is the so-called Monge-Ampère operator $\det D^2 z$, i.e.

Laplace operator	Monge-Ampère operator
$S_1\left(\lambda\left(D^2 z(x)\right)\right)=\sum_{i=1}^N\lambda_i=\Delta z,$	$S_N\left(\lambda\left(D^2 z(x)\right)\right)=\prod_{i=1}^N\lambda_i=\det D^2 z.$

In recent years, Laplace problem and Monge-Ampère problem are hot topics. Existence, uniqueness and asymptotic behavior of solutions to the above problem have been investigated extensively by many authors in different contexts [2-8,39,40]. In 2009, Ghanmi, Maagli, Radulescu and Zeddini [2] investigated the existence and the nonexistence of blow up or bounded solutions to the following system of nonlinear elliptic equations

$$\begin{cases}\triangle z_1=b(|x|)\psi(z_2), & x\in\mathbb{R}^N,\\ \triangle z_2=h(|x|)\varphi(z_1), & x\in\mathbb{R}^N.\end{cases}$$

In 2019, Covei [3] showed the existence of positive radially symmetric solutions to the following problem

$$\triangle z=b(|x|)\psi(z)+h(|x|)\varphi(z),\quad x\in\mathbb{R}^N. \tag{2}$$

For the problem (2), when $\psi(z)=z^\alpha$, $\varphi(z)=z^\beta$ and $0<\alpha\le\beta$, Lair [4] gave sufficient conditions for the nonexistence (existence) of nonnegative entire blow up solutions.

It is well known that k-Hessian equations are fully nonlinear PDEs for $k\ge 2$ [9,10]. There are many important applications in fluid mechanic, geometric problem and other applied subjects. For instance, the k-Hessian problem can describe Weingarten curvature or reflector shape design [11], and it can also describe some phenomena of non-equilibrium phase transitions and statistical physics [12,13]. In recent years, many authors have shown increasing interest in the subject of the k-Hessian problem. A good number of investigative results on the k-Hessian problem have been given by applying different methods, such as the variational method [14,15,16], the monotone iterative method [17,18,19], the method of moving planes [20], the method of sub- and super-solutions [21,22,23], the fixed point theorem [24,25].

Here, we give a definition on the classification of solutions.

Definition 1.1. *[26] A solution $(z_1,z_2)\in C^2([0,\infty))\times C^2([0,\infty))$ of system (1) is called an entire bounded solution if condition (3) holds; it is called an entire blow up solution if condition (4) holds; it is called a semifinite entire blow up solution when (5) or (6) holds.*

Finite Case: Both components (z_1, z_2) *are bounded, namely*

$$\lim_{|x|\to\infty} z_1(|x|) < \infty \quad and \quad \lim_{|x|\to\infty} z_2(|x|) < \infty. \tag{3}$$

Infinite Case: Both components (z_1, z_2) *are blow up, namely*

$$\lim_{|x|\to\infty} z_1(|x|) = \infty \quad and \quad \lim_{|x|\to\infty} z_2(|x|) = \infty. \tag{4}$$

Semifinite Case: One of the components is bounded, whereas the other is blow up, namely

$$\lim_{|x|\to\infty} z_1(|x|) < \infty \quad and \quad \lim_{|x|\to\infty} z_2(|x|) = \infty \tag{5}$$

or

$$\lim_{|x|\to\infty} z_1(|x|) = \infty \quad and \quad \lim_{|x|\to\infty} z_2(|x|) < \infty. \tag{6}$$

In 2015, Zhang and Zhou [27] studied the following k-Hessian system

$$\begin{cases} S_k\left(\lambda\left(D^2 z_1\right)\right) = b(|x|)\psi(z_2), & x \in \mathbb{R}^N, \\ S_k\left(\lambda\left(D^2 z_2\right)\right) = h(|x|)\varphi(z_1), & x \in \mathbb{R}^N. \end{cases} \tag{7}$$

They only obtained the existence of positive entire k-convex radial solution for the Finite Case and Infinite Case by applying Arzela-Ascoli theorem and the monotone iterative method.

In 2018, Covei [28] considered the Laplacian system

$$\begin{cases} \triangle z_1 = b(|x|)\psi(z_1, z_2), & x \in \mathbb{R}^N, \\ \triangle z_2 = h(|x|)\varphi(z_1), & x \in \mathbb{R}^N. \end{cases}$$

He analyzed the existence of positive entire radial solution under the Finite Case, Infinite Case and Semifinite Case by applying the monotone iterative method.

In 2020, Zhang, Liu, Wu and Cui [29] considered the k-Hessian equation

$$\mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z\right)\right) = \varphi(|x|, z), \quad x \in \mathbb{R}^N,$$

they gave a sufficient and necessary condition of existence of radial solutions under the Infinite Case by applying the monotone iterative method.

On the other hand, if $\varphi(|x|, z_1, z_2) = b(|x|)\varphi(z_1, z_2)$ and $\psi(|x|, z_1, z_2) = h(|x|)\psi(z_1, z_2)$, then the k-Hessian system (1) turns to the following form

$$\begin{cases} \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right) = b(|x|)\varphi(z_1, z_2), & x \in \mathbb{R}^N, \\ \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right) = h(|x|)\psi(z_1, z_2), & x \in \mathbb{R}^N. \end{cases} \tag{8}$$

In 2020, by applying the iterative technique, in paper [30], we presented sufficient conditions for the existence of the entire radial solutions under the finite case and infinite case and gave the estimation of positive entire radial solutions for the problem (8).

Inspired by the above works, we firstly consider, in addition to those cases [27,29,30], the existence of entire positive k-convex radial solutions under the semifinite cases (5) and (6) for the k-Hessian system (1) if $\varphi(|x|, z_1, z_2) = b(|x|)\varphi(z_1, z_2)$ and $\psi(|x|, z_1, z_2) = h(|x|)\psi(z_1)$. Secondly, we show the existence of entire positive k-convex bounded solutions and blow up radial solutions and also give the estimation of entire positive k-convex radial solutions for the k-Hessian system (1), where $\varphi(|x|, z_1, z_2) = b_1(|x|)\psi(z_1) + h_1(|x|)\varphi(z_2)$ and $\psi(|x|, z_1, z_2) = b_2(|x|)\psi(z_2) + h_2(|x|)\varphi(z_1)$. Here we extend the study in [3,28] from semilinear problem to fully nonlinear problem. The approach employed to deal with the k-Hessian system (1) is the monotone iterative method, which is a powerful tool for studying nonlinear problem, see [3,11,17,18,19], [27-37] and the references therein. The main results of this paper improve and complement the works [4,7,8,18,27,29,30] on the problem (1) with more special nonlinearities ψ, φ and weights b, h, b_i, h_i $(i=1,2)$.

§2 Existence results for Hessian system with two weights

In this section, we will discuss the following k-Hessian problem for $\varphi(|x|, z_1, z_2) = b(|x|)\varphi(z_1, z_2)$ and $\psi(|x|, z_1, z_2) = h(|x|)\psi(z_1)$ in the system (1)

$$\begin{cases} \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right) = b(|x|)\varphi(z_1, z_2), & x \in \mathbb{R}^N, \\ \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right) = h(|x|)\psi(z_1), & x \in \mathbb{R}^N, \end{cases} \tag{9}$$

where the weights $b, h \in C([0,+\infty),(0,+\infty))$ are spherically symmetric.

Here we assume that the functions ψ and φ satisfy the following conditions.

(S_1) : $\varphi \in C([0,\infty)\times[0,\infty),(0,\infty))$ is increasing in each variable and $\varphi(t_1, t_2) > 0$ for all $t_1, t_2 > 0$;

(S_2) : $\psi \in C([0,\infty),(0,\infty))$ is increasing and $\psi(t) > 0$ for all $t > 0$;

(S_3) : There exist positive constants c, δ and γ, the function $f \in C([0,\infty)\times[0,\infty),(0,\infty))$ and the function $g \in C([0,+\infty),(0,+\infty))$ such that

$$\varphi(t_1, t_2 s_2) \le c f(t_1, t_2) g(s_2), \quad \forall t_1 \ge 0, \quad \forall t_2 \ge M \quad \textit{and} \quad \forall s_2 \ge 1,$$

where f is increasing in each variable and

$$M = \begin{cases} \delta, & \delta > (\psi(\gamma)+1)^{\frac{1}{p+1}}, \\ (\psi(\gamma)+1)^{\frac{1}{p+1}}, & \delta \le (\psi(\gamma)+1)^{\frac{1}{p+1}}. \end{cases}$$

Here, we introduce a lemma about the properties of the operator $\mathcal{R}(t)$.

Lemma 2.1. *[1] Let $\mathcal{R}(t) = t\mathcal{G}(t)$, where $\mathcal{G} \in \Upsilon$. Thus, it has the following properties.*

(1) : *$\mathcal{R}(t)$ has an inverse mapping $\mathcal{R}^{-1}(t)$, which is nonnegative increasing;*

(2) : *for $0 < \beta < 1$, we have*

$$\mathcal{R}^{-1}(\beta t) \ge \beta^{\frac{1}{p+1}} \mathcal{R}^{-1}(t);$$

(3) : *for $\beta \ge 1$, we have*

$$\mathcal{R}^{-1}(\beta t) \le \beta^{\frac{1}{p+1}} \mathcal{R}^{-1}(t).$$

For $r \ge 0$, denote

$$F(r) = \int_\gamma^r \frac{dt}{f\left(t, M(\psi(t)+1)^{\frac{1}{p+1}}\right)+1}, \quad r \ge \gamma > 0, \quad F(\infty) := \lim_{r\to\infty} F(r),$$

$$B(r) = \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}(b(s))\right]^k ds\right)^{\frac{1}{k}} dt,$$

$$H(r) = \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}(h(s))\right]^k ds\right)^{\frac{1}{k}} dt,$$

$$\overline{P}(r)=\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)g(1+H(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt,$$

$$\overline{Q}(r)=\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi\left(F^{-1}\left((c+1)^{\frac{k}{p+1}}\overline{P}(s)\right)\right)\Big)\right]^k ds\right)^{\frac{1}{k}}dt,$$

$$\underline{P}(r)=\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)\varphi(\gamma,\delta+\left(\min\left\{\frac{1}{3},\psi(\gamma)\right\}\right)^{\frac{1}{p+1}}H(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt,$$

$$\underline{Q}(r)=\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi(\gamma+\left(\min\left\{\frac{1}{3},\varphi(\gamma,\delta)\right\}\right)^{\frac{1}{p+1}}B(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt,$$

and

$$\overline{P}(\infty):=\lim_{r\to\infty}\overline{P}(r),\quad \underline{P}(\infty):=\lim_{r\to\infty}\underline{P}(r),\quad \overline{Q}(\infty):=\lim_{r\to\infty}\overline{Q}(r),\quad \underline{Q}(\infty):=\lim_{r\to\infty}\underline{Q}(r).$$

Our assumptions are as follows:

$(P_1): F(\infty)=\infty$;

$(P_2): \overline{P}(\infty)<\infty$, $\overline{Q}(\infty)<\infty$;

$(P_3): \underline{P}(\infty)=\infty$, $\underline{Q}(\infty)=\infty$;

$(P_4): \overline{P}(\infty)<\infty$, $\underline{Q}(\infty)=\infty$;

$(P_5): \underline{P}(\infty)=\infty$, $\overline{\overline{Q}}(\infty)<\infty$;

$(P_6): \overline{P}(\infty)<F(\infty)<\infty$, $\overline{Q}(\infty)<\infty$;

$(P_7): \overline{P}(\infty)<F(\infty)<\infty$, $\underline{Q}(\infty)=\infty$.

Before we give a detailed description of our main results, we state the following lemmas, which are important for our proofs.

Lemma 2.2. *[38] Assume that $y(r)\in C^2[0,R)$ with $y'(0)=0$. Then we have the function $z(|x|)=y(r)\in C^2(B_R)$,*

$$\lambda\left(D^2z\right)=\begin{cases}\left(y''(r),\frac{y'(r)}{r},\ldots,\frac{y'(r)}{r}\right), & 0<r<R,\\ (y''(0),y''(0),\ldots,y''(0)), & r=0,\end{cases}\tag{10}$$

and

$$S_k\left(\lambda\left(D^2z\right)\right)=\begin{cases}C_{N-1}^{k-1}y''(r)\left(\frac{y'(r)}{r}\right)^{k-1}+C_{N-1}^k\left(\frac{y'(r)}{r}\right)^k, & 0<r<R,\\ C_N^k\left(y''(0)\right)^k, & r=0,\end{cases}\tag{11}$$

where $r=|x|<R$, $B_R=\left\{x\in\mathbb{R}^N:|x|<R\right\}$ and $C_N^k=\frac{N!}{k!(N-k)!}$.

Lemma 2.3. *[30] $(z_1(|x|),z_2(|x|))=(y_1(r),y_2(r))$ is a radial solution of the k-Hessian system (9) if and only if $(y_1(r),y_2(r))$ is a solution of the following ordinary differential system*

$$\begin{cases}\left\{\frac{r^{N-k}}{k}\left[(y_1(r))'\right]^k\right\}'=\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)\varphi(y_1(r),y_2(r))\Big)\right]^k, & r\geq 0,\\ \left\{\frac{r^{N-k}}{k}\left[(y_2(r))'\right]^k\right\}'=\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(r)\psi(y_1(r))\Big)\right]^k, & r\geq 0,\\ y_1(0)=\gamma,\quad y_2(0)=\delta,\quad y_1'(0)=0,\quad y_2'(0)=0.\end{cases}\tag{12}$$

Our main results for the k-Hessian system (9) are as follows.

Theorem 2.1. *Assume (S_1), (S_2), (S_3) and (P_1) hold. Then the k-Hessian system (9) has infinitely many positive entire k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$. Moreover,*

(1) if (P_2) holds, then $\lim\limits_{|x|\to\infty} z_1(|x|) < \infty$ and $\lim\limits_{r\to\infty} z_2(|x|) < \infty$.

(2) If (P_3) holds, then $\lim\limits_{|x|\to\infty} z_1(|x|) = \infty$ and $\lim\limits_{r\to\infty} z_2(|x|) = \infty$.

(3) If (P_4) holds, then $\lim\limits_{|x|\to\infty} z_1(|x|) < \infty$ and $\lim\limits_{r\to\infty} z_2(|x|) = \infty$.

(4) If (P_5) holds, then $\lim\limits_{|x|\to\infty} z_1(|x|) = \infty$ and $\lim\limits_{r\to\infty} z_2(|x|) < \infty$.

Theorem 2.2. *Assume (S_1), (S_2) and (S_3) hold. Moreover,*

(1) if (P_6) holds, then the k-Hessian system (9) has infinitely many positive entire k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$ satisfying

$$\gamma + \underline{P}(r) \le z_1(|x|) \le F^{-1}\left((c+1)^{\frac{k}{p+1}}\overline{P}(r)\right), \quad \forall r \ge 0$$

and

$$\delta + \underline{Q}(r) \le z_2(|x|) \le \delta + \overline{Q}(r), \quad \forall r \ge 0.$$

(2) If (P_7) holds, then $\lim\limits_{|x|\to\infty} z_1(|x|) < \infty$ and $\lim\limits_{|x|\to\infty} z_2(|x|) = \infty$.

Remark 2.1. *By the simple calculation, we know that the similar results as Theorem 2.1 and 2.2 can be obtained for the k-Hessian system*

$$\begin{cases} S_k^{\frac{1}{k}}(\lambda(D^2 z_1)) = b(|x|)\varphi(z_1, z_2), & x \in \mathbb{R}^N, \\ S_k^{\frac{1}{k}}(\lambda(D^2 z_2)) = h(|x|)\psi(z_1), & x \in \mathbb{R}^N, \end{cases}$$

under the condition as follows

$$F(r) = \int_\gamma^r \frac{dt}{f\left(t, M\psi(t)\right)}, \quad r \ge \gamma > 0, \quad F(\infty) := \lim_{r\to\infty} F(r).$$

So we consider the more interesting system (9).

§3 Proofs of Theorem 2.1 and Theorem 2.2

In this section, we give proofs of Theorem 2.1 and Theorem 2.2.

3.1 Proof of Theorem 2.1

It is well known [30] that the radial solution of the system (9) is the solution of the integral system

$$\begin{cases} y_1(r) = \gamma + \displaystyle\int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b(s)\varphi(y_1(s), y_2(s))\Big)\right]^k ds\right)^{\frac{1}{k}} dt, \quad r \ge 0, \\ y_2(r) = \delta + \displaystyle\int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(h(s)\psi(y_1(s))\Big)\right]^k ds\right)^{\frac{1}{k}} dt, \quad r \ge 0. \end{cases} \tag{13}$$

Let $\left\{y_1^{(m)}(r)\right\}_{m\geqslant 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\geqslant 0}$ on $[0,\infty)$ be two sequences of functions given by

$$\begin{cases} y_1^{(0)}(r)=y_1(0)=\gamma, y_2^{(0)}(r)=y_2(0)=\delta, \quad r\ge 0, \quad \gamma>0 \quad and \quad \delta>0, \\ y_1^{(m)}(r)=\gamma+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)\varphi(y_1^{(m-1)}(s),y_2^{(m-1)}(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt, \quad r\ge 0, \\ y_2^{(m)}(r)=\delta+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi(y_1^{(m)}(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt, \quad r\ge 0. \end{cases} \tag{14}$$

By the same way in [30], we can conclude that $\left\{y_1^{(m)}(r)\right\}_{m\ge 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\ge 0}$ are nondecreasing on $[0,\infty)$. Furthermore, we are going to prove that $\left\{y_1^{(m)}(r)\right\}_{m\ge 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\ge 0}$ are bounded on $[0,\infty)$. It follows from (14) that $(y_1^{(m)})'(r)\ge 0$ and $(y_2^{(m)})'(r)\ge 0$. By the monotonicity of $\left\{y_1^{(m)}(r)\right\}_{m\ge 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\ge 0}$, (S_1), (S_2), (S_3) and Lemma 2.1, one can see that

$$\begin{aligned}
&\left\{\frac{r^{N-k}}{k}\left[\left(y_1^{(m)}(r)\right)'\right]^k\right\}' \\
=&\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)\varphi(y_1^{(m-1)}(r),y_2^{(m-1)}(r))\Big)\right]^k \\
\le&\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)\varphi(y_1^{(m)}(r),y_2^{(m)}(r))\Big)\right]^k \\
=&\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)\varphi(y_1^{(m)}(r),\delta+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi(y_1^{(m)}(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt)\Big)\right]^k \\
\le&\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)\varphi(y_1^{(m)}(r),\delta+(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}}H(r))\Big)\right]^k \\
=&\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)\varphi(y_1^{(m)}(r),(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}}(\frac{\delta}{(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}}}+H(r)))\Big)\right]^k \\
\le&\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)\varphi(y_1^{(m)}(r),(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}}(\frac{\delta}{(\psi(\gamma)+1)^{\frac{1}{p+1}}}+H(r))\Big)\right]^k \\
\le&\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)\varphi(y_1^{(m)}(r),M(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}}(1+H(r))\Big)\right]^k \\
\le&\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)cf(y_1^{(m)}(r),M(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}})g(1+H(r))\Big)\right]^k \\
\le&(c+1)^{\frac{k}{p+1}}(f(y_1^{(m)}(r),M(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}})+1)^{\frac{k}{p+1}}\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)g(1+H(r))\Big)\right]^k \\
\le&(c+1)^{\frac{k}{p+1}}(f(y_1^{(m)}(r),M(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}})+1)\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(r)g(1+H(r))\Big)\right]^k,
\end{aligned}$$

which leads to

$$\begin{aligned}\left(y_1^{(m)}(r)\right)' \le&(c+1)^{\frac{k}{p+1}}(f(y_1^{(m)}(r), M(\psi(y_1^{(m)}(r))+1)^{\frac{1}{p+1}})+1)\\ &\times\left(\frac{k}{r^{N-k}}\int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)g(1+H(s))\Big)\right]^k ds\right)^{\frac{1}{k}}.\end{aligned} \tag{15}$$

Integrating the above inequality (15) from 0 to r, one gets

$$\int_0^r \frac{\left(y_1^{(m)}(t)\right)'}{f\left(y_1^{(m)}(t), M(\psi(y_1^{(m)}(t))+1)^{\frac{1}{p+1}}\right)+1}dt \le (c+1)^{\frac{k}{p+1}}\overline{P}(r),$$

i.e.

$$\int_\gamma^{y_1^{(m)}(r)} \frac{1}{f\left(\tau, M(\psi(\tau)+1)^{\frac{1}{p+1}}\right)+1}d\tau \le (c+1)^{\frac{k}{p+1}}\overline{P}(r).$$

Consequently,

$$F\left(y_1^{(m)}(r)\right) \le (c+1)^{\frac{k}{p+1}}\overline{P}(r), \quad \forall r \ge 0. \tag{16}$$

In view of the fact that F is a bijection with the inverse function F^{-1} strictly increasing on $[0,\infty)$, we have by (16)

$$y_1^{(m)}(r) \le F^{-1}\left((c+1)^{\frac{k}{p+1}}\overline{P}(r)\right), \quad \forall r \ge 0. \tag{17}$$

It follows from (17) that

$$\begin{aligned} y_2^{(m)}(r) &= \delta + \int_0^r \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi(y_1^{(m)}(s))\Big)\right]^k ds\right)^{\frac{1}{k}} dt\\ &\le \delta + \int_0^r \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi\left(F^{-1}\left((c+1)^{\frac{k}{p+1}}\overline{P}(r)\right)\right)\Big)\right]^k ds\right)^{\frac{1}{k}} dt\\ &= \delta + \overline{Q}(r).\end{aligned} \tag{18}$$

By (17) and (18), one gets for arbitrary $c_0 > 0$

$$y_1^{(m)}(r) \le F^{-1}\left((c+1)^{\frac{k}{p+1}}\overline{P}(r)\right) \le F^{-1}\left((c+1)^{\frac{k}{p+1}}\overline{P}(c_0)\right) = C_1, \quad \forall r \in [0, c_0] \tag{19}$$

and

$$y_2^{(m)}(r) \le \delta + \overline{Q}(r) \le \delta + \overline{Q}(c_0) = C_2, \quad \forall r \in [0, c_0], \tag{20}$$

which mean that the sequences $\left\{y_1^{(m)}(r)\right\}_{m\ge 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\ge 0}$ are bounded on $[0, c_0]$ for arbitrary $c_0 > 0$. By (14), one gets for arbitrary $c_0 > 0$

$$\begin{aligned}\left(y_1^{(m)}(r)\right)' &= \left(\frac{k}{r^{N-k}}\int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)\varphi(y_1^{(m-1)}(s), y_2^{(m-1)}(s))\Big)\right]^k ds\right)^{\frac{1}{k}}\\ &\le \left(\frac{k}{r^{N-k}}\int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)\varphi(y_1^{(m)}(s), y_2^{(m)}(s))\Big)\right]^k ds\right)^{\frac{1}{k}}\end{aligned} \tag{21}$$

$$
\begin{aligned}
&\leq \left(\frac{kr^{N-1}}{r^{N-k}} \parallel b(r)+1 \parallel_{\infty}^{\frac{k}{p+1}} \int_0^r \frac{1}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(\varphi(y_1^{(m)}(s), y_2^{(m)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} \\
&\leq \left(\frac{k}{C_{N-1}^{k-1}} r^{k-1} \parallel b(r)+1 \parallel_{\infty}^{\frac{k}{p+1}} \left[\mathcal{R}^{-1}\Big(\varphi(y_1^{(m)}(r), y_2^{(m)}(r))\Big)\right]^k \int_0^r ds \right)^{\frac{1}{k}} \\
&\leq \left(\frac{k}{C_{N-1}^{k-1}} c_0^k \parallel b(r)+1 \parallel_{\infty}^{\frac{k}{p+1}} \left[\mathcal{R}^{-1}\Big(\varphi(C_1, C_2)\Big)\right]^k \right)^{\frac{1}{k}} \\
&\leq \left(\frac{k}{C_{N-1}^{k-1}} \right)^{\frac{1}{k}} c_0 \parallel b(r)+1 \parallel_{\infty}^{\frac{1}{p+1}} \left[\mathcal{R}^{-1}\Big(\varphi(C_1, C_2)\Big)\right], \quad \forall r \in [0, c_0]
\end{aligned}
$$

and

$$
\begin{aligned}
\Big(y_2^{(m)}(r)\Big)' &= \left(\frac{k}{r^{N-k}} \int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(h(s)\psi(y_1^{(m)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} \\
&\leq \left(\frac{k}{C_{N-1}^{k-1}} \right)^{\frac{1}{k}} c_0 \parallel h(r)+1 \parallel_{\infty}^{\frac{1}{p+1}} \left[\mathcal{R}^{-1}\Big(\psi(C_1)\Big)\right], \quad \forall r \in [0, c_0]\,.
\end{aligned}
\tag{22}
$$

In view of (21) and (22), the sequences $\left\{ \Big(y_1^{(m)}(r)\Big)' \right\}_{m\geq 0}$ and $\left\{ \Big(y_2^{(m)}(r)\Big)' \right\}_{m\geq 0}$ are bounded on $[0, c_0]$ for arbitrary $c_0 > 0$. It follows from Arzela-Ascoli theorem that $\left\{ y_1^{(m)}(r) \right\}_{m\geq 0}$ and $\left\{ y_2^{(m)}(r) \right\}_{m\geq 0}$ have subsequences converging uniformly to y_1 and y_2 on $[0, c_0]$. Due to the arbitrariness of $\gamma, \delta, c_0 > 0$, one concludes that the system (12) has infinitely many positive entire k-convex solutions(y_1, y_2). Therefore, with the help of Lemma 2.3, one can easily draw a conclusion that the k-Hessian system (9) has infinitely many positive entire k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$. Next, the proof of $z_i \in C^2[0,\infty)(i = 1, 2)$ is similar with [30] and is omitted.

Since $F(\infty) = \infty$, we can know that

$$
F^{-1}(\infty) = \infty.
$$

(1) It follows from $\overline{P}(\infty) < \infty$, $\overline{Q}(\infty) < \infty$, (17) and (18) that

$$
y_1(r) \leq F^{-1}\left((c+1)^{\frac{k}{p+1}} \overline{P}(\infty) \right) < \infty, \quad \forall r \geq 0
$$

and

$$
y_2(r) \leq \delta + \overline{Q}(\infty) < \infty, \quad \forall r \geq 0,
$$

which mean that $\lim_{r\to\infty} y_1(r) < \infty$ and $\lim_{r\to\infty} y_2(r) < \infty$. Thus, the positive radial solutions $(y_1, y_2) \in C^2[0,\infty) \times C^2[0,\infty)$ are bounded. Therefore, the k-Hessian system (9) has infinitely many positive entire bounded radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$.

(2) In view of $\underline{P}(\infty)=\infty$, $\underline{Q}(\infty)=\infty$,

$$\begin{aligned} y_1(r) &= \gamma+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)\varphi(y_1(s),y_2(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt \\ &\geq \gamma+\left(\min\left\{\frac{1}{3},\varphi(\gamma,\delta)\right\}\right)^{\frac{1}{p+1}}B(r) \end{aligned}$$

and

$$\begin{aligned} y_2(r) &= \delta+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi(y_1(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt \\ &\geq \delta+\left(\min\left\{\frac{1}{3},\psi(\gamma)\right\}\right)^{\frac{1}{p+1}}H(r), \end{aligned}$$

we get that

$$\begin{aligned} y_1(r) &= \gamma+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)\varphi(y_1(s),y_2(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt \\ &\geq \gamma+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}[\mathcal{R}^{-1}\Big(b(s)\varphi(y_1(s),\delta+(\min\{\frac{1}{3},\psi(\gamma)\})^{\frac{1}{p+1}}H(s))\Big)]^k ds\right)^{\frac{1}{k}}dt \\ &\geq \gamma+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b(s)\varphi(\gamma,\delta+(\min\{\frac{1}{3},\psi(\gamma)\})^{\frac{1}{p+1}}H(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt \\ &= \gamma+\underline{P}(r) \end{aligned} \tag{23}$$

and

$$\begin{aligned} y_2(r) &= \delta+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi(y_1(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt \\ &\geq \delta+\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(h(s)\psi(\gamma+(\min\{\frac{1}{3},\varphi(\gamma,\delta)\})^{\frac{1}{p+1}}B(s))\Big)\right]^k ds\right)^{\frac{1}{k}}dt \\ &= \delta+\underline{Q}(r), \end{aligned} \tag{24}$$

which imply that $\lim\limits_{r\to\infty}y_1(r)=\infty$ and $\lim\limits_{r\to\infty}y_2(r)=\infty$. Therefore, the positive entire k-convex radial solutions $(z_1,z_2)\in C^2[0,\infty)\times C^2[0,\infty)$ of (9) blow up.

(3) In the spirit of Case (1) and Case (2), we have

$$y_1(r)\leq F^{-1}\left((c+1)^{\frac{k}{p+1}}\overline{P}(\infty)\right)<\infty,\quad \forall r\geq 0$$

and

$$y_2(r)\geq \delta+\underline{Q}(r),\quad \forall r\geq 0.$$

In view of $\overline{P}(\infty)<\infty$ and $\underline{Q}(\infty)=\infty$, one gets that $\lim\limits_{r\to\infty}y_1(r)<\infty$ and $\lim\limits_{r\to\infty}y_2(r)=\infty$, which means the k-Hessian system (9) has infinitely many positive semifinite entire blow up k-convex radial solutions $(z_1,z_2)\in C^2[0,\infty)\times C^2[0,\infty)$.

(4) In this case, the idea is to mimic the proof of the Case (3). We observe that

$$y_1(r) \geq \gamma + \underline{P}(r), \quad \forall r \geq 0$$

and

$$y_2(r) \leq \delta + \overline{Q}(r), \quad \forall r \geq 0,$$

In view of $\underline{P}(\infty) = \infty$ and $\overline{Q}(\infty) < \infty$, one gets that $\lim_{r\to\infty} y_1(r) = \infty$ and $\lim_{r\to\infty} y_2(r) < \infty$. Therefore, the k-Hessian system (9) has infinitely many positive semifinite entire blow up k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$.

3.2 Proof of Theorem 2.2

(1) By a similar process to Theorem 2.1, we can obtain that $\left\{y_1^{(m)}(r)\right\}_{m\geq 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\geq 0}$ are two nondecreasing sequences. Moreover, we can also obtain (16), (18), (23) and (24). It follows from (16), (18) and (P_6) that

$$F\left(y_1^{(m)}(r)\right) \leq (c+1)^{\frac{k}{p+1}}\overline{P}(\infty) < (c+1)^{\frac{k}{p+1}}F(\infty) < \infty, \quad \forall r \geq 0$$

and

$$y_2^{(m)}(r) \leq \delta + \overline{Q}(\infty) < \infty, \quad \forall r \geq 0.$$

On the other hand, since F^{-1} is strictly increasing on $[0,\infty)$, we find out that

$$y_1^{(m)}(r) \leq F^{-1}\left((c+1)^{\frac{k}{p+1}}\overline{P}(\infty)\right) < \infty, \quad \forall r \geq 0,$$

then the nondecreasing sequences $\left\{y_1^{(m)}(r)\right\}_{m\geq 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\geq 0}$ are bounded for $\forall m \in \mathbb{N}$ and $r \in [0,\infty)$. We use this observation to conclude that $\lim_{m\to\infty} y_1^{(m)}(r) = y_1(r) < \infty$ and $\lim_{m\to\infty} y_2^{(m)}(r) = y_2(r) < \infty$, thus the limiting functions y_1 and y_2 are positive entire bounded radial solutions of the k-Hessian system (9).

(2) The proof is similar with Case (3) of Theorem 2.1, so it is omitted. This completes the proof.

§4 Existence results for Hessian system with four weights

In this section, we will discuss the following k-Hessian problem for $\varphi(|x|, z_1, z_2) = b_1(|x|)\psi(z_1) + h_1(|x|)\varphi(z_2)$ and $\psi(|x|, z_1, z_2) = b_2(|x|)\psi(z_2) + h_2(|x|)\varphi(z_1)$ in the system (1)

$$\begin{cases} \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_1\right)\right) = b_1(|x|)\psi(z_1) + h_1(|x|)\varphi(z_2), & x \in \mathbb{R}^N, \\ \mathcal{G}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right)\right) S_k^{\frac{1}{k}}\left(\lambda\left(D^2 z_2\right)\right) = b_2(|x|)\psi(z_2) + h_2(|x|)\varphi(z_1), & x \in \mathbb{R}^N, \end{cases} \tag{25}$$

where the weights $b_i, h_i \in C([0,+\infty),(0,+\infty))$ $(i = 1,2)$ are spherically symmetric.

Here we assume that the function ψ satisfies (S_2) and the function φ satisfies the following conditions.

(S_4) : there exists a positive constant $\zeta > 0$ such that $\varphi(t) < 0$ on $(0,\zeta)$ and nondecreasing on

(ζ, ∞),

$$\lim_{t\to 0^+}\varphi(t)=\varphi(\zeta)=0.$$

Denote

$$H(r)=\int_{\gamma+\delta}^{r}\frac{dt}{\varphi(t)+\psi(t)+1},\quad r\ge\gamma,\delta\ge\zeta,\quad H(\infty):=\lim_{r\to\infty}H(r),$$

$$\Lambda_1(s):=\begin{cases} b_1(s)\psi(\gamma) & \text{if}\quad \varphi(\delta)=0,\\ ((b_1(s)+h_1(s))\min\{\varphi(\gamma),\psi(\delta)\}) & \text{if}\quad \varphi(\delta)\neq 0,\end{cases}$$

$$\Lambda_2(s):=\begin{cases} b_2(s)\psi(\delta) & \text{if}\quad \varphi(\gamma)=0,\\ ((b_2(s)+h_2(s))\min\{\varphi(\delta),\psi(\gamma)\}) & \text{if}\quad \varphi(\gamma)\neq 0,\end{cases}$$

$$\underline{J}(r)=\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}(\Lambda_1(s))\right]^k ds\right)^{\frac{1}{k}}dt,\quad r\ge 0,$$

$$\underline{W}(r)=\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}(\Lambda_2(s))\right]^k ds\right)^{\frac{1}{k}}dt,\quad r\ge 0,$$

$$\overline{J}(r)=\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}(b_1(s)+h_1(s))\right]^k ds\right)^{\frac{1}{k}}dt,\quad r\ge 0,$$

$$\overline{W}(r)=\int_0^r\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}(b_2(s)+h_2(s))\right]^k ds\right)^{\frac{1}{k}}dt,\quad r\ge 0,$$

and

$$\underline{J}(\infty):=\lim_{r\to\infty}\underline{J}(r),\quad \underline{W}(\infty):=\lim_{r\to\infty}\underline{W}(r),\quad \overline{J}(\infty):=\lim_{r\to\infty}\overline{J}(r),\quad \overline{W}(\infty):=\lim_{r\to\infty}\overline{W}(r).$$

Our assumptions are as follows:

$(Q_1): H(\infty)=\infty$;
$(Q_2): H(\infty)<\infty$;
$(Q_3): \overline{J}(\infty)<\infty, \overline{W}(\infty)<\infty$;
$(Q_4): \underline{J}(\infty)=\infty, \underline{W}(\infty)=\infty$;
$(Q_5): \underline{J}(\infty)<\infty, \underline{W}(\infty)<\infty$;
$(Q_6): \overline{J}(\infty)+\overline{W}(\infty)<H(\infty)-H(2\xi)$, where $\xi>\frac{\gamma+\delta}{2}$ is a constant.

Similar to the proof of Lemma 2.3, we can transform the k-Hessian system (25) to an equivalent ordinary differential equation.

Lemma 4.1. *$(z_1(|x|), z_2(|x|))=(y_1(r), y_2(r))$ is a radial solution of the k-Hessian system (25) if and only if $(y_1(r), y_2(r))$ is a solution of the following ordinary differential system*

$$\begin{cases}\left\{\frac{r^{N-k}}{k}\left[(y_1(r))'\right]^k\right\}'=\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b_1(r)\psi(y_1(r))+h_1(r)\varphi(y_2(r))\Big)\right]^k, & r\ge 0,\\ \left\{\frac{r^{N-k}}{k}\left[(y_2(r))'\right]^k\right\}'=\frac{r^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b_2(r)\psi(y_2(r))+h_2(r)\varphi(y_1(r))\Big)\right]^k, & r\ge 0,\\ y_1(0)=\gamma,\quad y_2(0)=\delta,\quad y_1'(0)=0,\quad y_2'(0)=0.\end{cases}\tag{26}$$

Our main results for the k-Hessian system (25) are as follows.

Theorem 4.1. *Assume (S_2), (S_4) and (Q_1) hold. Then the k-Hessian system (25) has infinitely many positive entire k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$. Moreover,*
(1) if (Q_3) holds, then $\lim\limits_{|x|\to\infty} z_1(|x|) < \infty$ and $\lim\limits_{|x|\to\infty} z_2(|x|) < \infty$.
(2) If (Q_4) holds, then $\lim\limits_{|x|\to\infty} z_1(|x|) = \infty$ and $\lim\limits_{|x|\to\infty} z_2(|x|) = \infty$.

Theorem 4.2. *Assume (S_2), (S_4), (Q_2), (Q_3), (Q_5) and (Q_6) hold. Then the k-Hessian system (25) has infinitely many positive entire k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$ satisfying*

$$\gamma + \underline{J}(r) \le z_1 \le H^{-1}(H(2\xi) + \overline{J}(r) + \overline{W}(r)), \quad \forall r \ge 0$$

and

$$\delta + \underline{W}(r) \le z_2 \le H^{-1}(H(2\xi) + \overline{J}(r) + \overline{W}(r)), \quad \forall r \ge 0.$$

§5 Proofs of Theorem 4.1 and Theorem 4.2

In this section, we give proofs of Theorem 4.1 and Theorem 4.2.

5.1 Proof of Theorem 4.1

Let us rewrite the system (26) to the integral form as follows

$$\begin{cases} y_1(r) = \gamma + \displaystyle\int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(r)\psi(y_1(r)) + h_1(r)\varphi(y_2(r))\Big)\right]^k ds \right)^{\frac{1}{k}} dt, \quad r \ge 0, \\ y_2(r) = \delta + \displaystyle\int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_2(r)\psi(y_2(r)) + h_2(r)\varphi(y_1(r))\Big)\right]^k ds \right)^{\frac{1}{k}} dt, \quad r \ge 0. \end{cases} \tag{27}$$

Let $\left\{y_1^{(m)}(r)\right\}_{m\ge0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\ge0}$ be the sequences of positive continuous functions on $[0,\infty)$ defined iteratively by

$$\begin{cases} y_1^{(0)}(r) = y_1(0) = \gamma, \quad y_2^{(0)}(r) = y_2(0) = \delta, \quad \gamma > 0 \quad and \quad \delta > 0, \\ y_1^{(m)}(r) = \gamma + \displaystyle\int_0^r \Big(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\mathcal{R}^{-1}(b_1(s)\psi(y_1^{(m-1)}(s)) + h_1(s)\varphi(y_2^{(m-1)}(s)))]^k ds\Big)^{\frac{1}{k}} dt, \\ y_2^{(m)}(r) = \delta + \displaystyle\int_0^r \Big(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} [\mathcal{R}^{-1}(b_2(s)\psi(y_2^{(m-1)}(s)) + h_2(s)\varphi(y_1^{(m-1)}(s)))]^k ds\Big)^{\frac{1}{k}} dt. \end{cases} \tag{28}$$

We will first prove that $\left\{y_1^{(m)}(r)\right\}_{m\ge0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\ge0}$ are nondecreasing on $[0,\infty)$. It is clear that $y_1^{(0)} \le y_1^{(1)}$ and $y_2^{(0)} \le y_2^{(1)}$. By using the mathematical induction method, we get

$$\begin{aligned}
y_1^{(1)}(r) =&\gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(s)\psi(y_1^{(0)}(s)) + h_1(s)\varphi(y_2^{(0)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
=&\gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(s)\psi(\gamma) + h_1(s)\varphi(\delta)\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\le&\gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(s)\psi(y_1^{(1)}(s)) + h_1(s)\varphi(y_2^{(1)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
=&y_1^{(2)}(r).
\end{aligned} \tag{29}$$

Similar to (29), we see that $y_2^{(1)}(r) \le y_2^{(2)}(r)$. We assume $y_1^{(m-1)}(r) \le y_1^{(m)}(r)$ *and* $y_2^{(m-1)}(r) \le y_2^{(m)}(r),\ \forall m \in \mathbb{N},\ r \in [0,\infty)$ and prove that

$$y_1^{(m)}(r) \le y_1^{(m+1)}(r) \text{ and } y_2^{(m)}(r) \le y_2^{(m+1)}(r),\ \forall m \in \mathbb{N},\ r \in [0,\infty).$$

In fact,

$$\begin{aligned}
y_1^{(m)}(r) =&\gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(s)\psi(y_1^{(m-1)}(s)) + h_1(s)\varphi(y_2^{(m-1)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\le&\gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(s)\psi(y_1^{(m)}(s)) + h_1(s)\varphi(y_2^{(m)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
=&y_1^{(m+1)}(r)
\end{aligned}$$

and

$$\begin{aligned}
y_2^{(m)}(r) =&\gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_2(s)\psi(y_2^{(m-1)}(s)) + h_2(s)\varphi(y_1^{(m-1)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\le&\gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_2(s)\psi(y_2^{(m)}(s)) + h_2(s)\varphi(y_1^{(m)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
=&y_2^{(m+1)}(r).
\end{aligned}$$

So, $\left\{y_1^{(m)}(r)\right\}_{m\ge 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\geqslant 0}$ are two nondecreasing sequences. It follows from (28) that $(y_1^{(m)})'(r) \ge 0$, and $(y_2^{(m)})'(r) \ge 0$. By the monotonicity of $\left\{y_1^{(m)}(r)\right\}_{m\ge 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\ge 0}$, (S_2), (S_4) and Lemma 2.1, we have

$$\begin{aligned}
\Big(y_1^{(m)}(r)\Big)' =& \left(\frac{k}{r^{N-k}} \int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(s)\psi(y_1^{(m-1)}(s)) + h_1(s)\varphi(y_2^{(m-1)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}} \\
\le& \left(\frac{k}{r^{N-k}} \int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(s)\psi(y_1^{(m)}(s)) + h_1(s)\varphi(y_2^{(m)}(s))\Big)\right]^k ds \right)^{\frac{1}{k}}
\end{aligned}$$

$$
\begin{aligned}
&\leq\left(\frac{k}{r^{N-k}}\int_0^r\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big((b_1(s)+h_1(s))(\psi(y_1^{(m)}(s))+\varphi(y_2^{(m)}(s)))\Big)\right]^k ds\right)^{\frac{1}{k}}\\
&\leq\left(\frac{k}{r^{N-k}}\int_0^r\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[(\psi(y_1^{(m)}(s))+\varphi(y_2^{(m)}(s))+1)\mathcal{R}^{-1}\Big(b_1(s)+h_1(s)\Big)\right]^k ds\right)^{\frac{1}{k}}\\
&\leq\Big[\Big(\varphi+\psi\Big)(y_1^{(m)}(r)+y_2^{(m)}(r))+1\Big]\\
&\quad\times\left(\frac{k}{r^{N-k}}\int_0^r\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b_1(s)+h_1(s)\Big)\right]^k ds\right)^{\frac{1}{k}}.
\end{aligned}
\tag{30}
$$

Similarly, we obtain

$$
\begin{aligned}
\Big(y_2^{(m)}(r)\Big)'\leq&\Big[\Big(\varphi+\psi\Big)(y_1^{(m)}(r)+y_2^{(m)}(r))+1\Big]\\
&\times\left(\frac{k}{r^{N-k}}\int_0^r\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b_2(s)+h_2(s)\Big)\right]^k ds\right)^{\frac{1}{k}}.
\end{aligned}
\tag{31}
$$

By (30) and (31), we know that

$$
\begin{aligned}
\Big(y_1^{(m)}(r)\Big)'+\Big(y_2^{(m)}(r)\Big)'\leq&\Big[\Big(\varphi+\psi\Big)(y_1^{(m)}(r)+y_2^{(m)}(r))+1\Big]\\
&\times\left\{\left(\frac{k}{r^{N-k}}\int_0^r\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b_1(s)+h_1(s)\Big)\right]^k ds\right)^{\frac{1}{k}}\right.\\
&\left.+\left(\frac{k}{r^{N-k}}\int_0^r\frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathcal{R}^{-1}\Big(b_2(s)+h_2(s)\Big)\right]^k ds\right)^{\frac{1}{k}}\right\},
\end{aligned}
$$

which yields

$$
\int_0^r\frac{\Big(y_1^{(m)}(t)\Big)'+\Big(y_2^{(m)}(t)\Big)'}{\varphi(y_1^{(m)}(t)+y_2^{(m)}(t))+\psi(y_1^{(m)}(t)+y_2^{(m)}(t))+1}dt\leq\overline{J}(r)+\overline{W}(r),
$$

i.e.

$$
\int_{\gamma+\delta}^{y_1^{(m)}(r)+y_2^{(m)}(r)}\frac{1}{\varphi(\tau)+\psi(\tau)+1}d\tau\leq\overline{J}(r)+\overline{W}(r).
$$

Thus

$$
H\left(y_1^{(m)}(r)+y_2^{(m)}(r)\right)\leq\overline{J}(r)+\overline{W}(r),\quad\forall r\geq 0.\tag{32}
$$

By (Q_1), one can get $H^{-1}(\infty)=\infty$. Because of the fact that H is a bijection with the inverse function H^{-1} strictly increasing on $[0,\infty)$, (32) can be rewritten to the form

$$
y_1^{(m)}(r)+y_2^{(m)}(r)\leq H^{-1}\left(\overline{J}(r)+\overline{W}(r)\right),\quad\forall r\geq 0,\tag{33}
$$

which means that the sequences $\left\{y_1^{(m)}(r)\right\}_{m\geq 0}$ and $\left\{y_2^{(m)}(r)\right\}_{m\geq 0}$ are bounded on $[0,c_0]$ for arbitrary $c_0>0$. So do $\left\{\Big(y_1^{(m)}(r)\Big)'\right\}_{m\geq 0}$ and $\left\{\Big(y_2^{(m)}(r)\Big)'\right\}_{m\geq 0}$ from (28) and (31). Identical to Theorem 2.1, we can conclude that the k-Hessian system (1) has infinitely many positive entire bounded k-convex radial solutions $(z_1,z_2)\in C^2[0,\infty)\times C^2[0,\infty)$ with the help of Lemma 4.1.

(1) we get from (Q_3) and (33) that for all $r \geq 0$

$$y_1 + y_2 \leq H^{-1}(\overline{J}(r) + \overline{W}(r)) \leq H^{-1}(\overline{J}(\infty) + \overline{W}(\infty)) < \infty,$$

which imply $\lim_{r\to\infty} y_1(r) < \infty$ and $\lim_{r\to\infty} y_2(r) < \infty$. Thus the system (26) has infinitely many positive entire bounded k-convex radial solutions $(y_1, y_2) \in C^2[0,\infty) \times C^2[0,\infty)$. By Lemma 4.1, we can see that the k-Hessian system (25) has infinitely many positive entire bounded k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$.

(2) It follows from (S_2), (S_4) and Lemma 2.1 that

$$\begin{aligned}
y_1(r) =& y_1(0) + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(r)\psi(y_1(r)) + h_1(r)\varphi(y_2(r))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\geq& \gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_1(r)\psi(\gamma) + h_1(r)\varphi(\delta)\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\geq& \gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big((b_1(r) + h_1(r))(\psi(\gamma) + \varphi(\delta))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\geq& \gamma + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(\Lambda_1(r)\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\geq& \gamma + \underline{J}(r)
\end{aligned}$$

and

$$\begin{aligned}
y_2(r) =& y_2(0) + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_2(r)\psi(y_2(r)) + h_2(r)\varphi(y_1(r))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\geq& \delta + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(b_2(r)\psi(\delta) + h_2(r)\varphi(\gamma)\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\geq& \delta + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big((b_2(r) + h_2(r))(\psi(\delta) + \varphi(\gamma))\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\geq& \delta + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathcal{R}^{-1}\Big(\Lambda_2(r)\Big)\right]^k ds \right)^{\frac{1}{k}} dt \\
\geq& \delta + \underline{W}(r).
\end{aligned}$$

Since (Q_4) holds, we note that $\lim_{r\to\infty} y_1(r) = \infty$ and $\lim_{r\to\infty} y_2(r) = \infty$. Thus the k-convex radial solutions $(y_1(r), y_2(r))$ of (26) blow up. According to Lemma 4.1 we can conclude that the k-Hessian system (25) has infinitely many positive entire blow-up k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$. This completes the proof.

5.2 Proof of Theorem 4.2

By a same process to Theorem 4.1, we can conclude that the k-Hessian system (25) has infinitely many positive entire k-convex radial solutions $(z_1, z_2) \in C^2[0,\infty) \times C^2[0,\infty)$.

It follows from (Q_6) that $H(2\xi) + \overline{J}(\infty) + \overline{W}(\infty) < H(\infty)$. We can obtain by (Q_2), (Q_3) and (33) that

$$H\left(y_1^{(m)}(r) + y_2^{(m)}(r)\right) \leq H(2\xi) + \overline{J}(r) + \overline{W}(r) < H(\infty) < \infty.$$

Since H^{-1} is strictly increasing, we have

$$y_1^{(m)}(r) + y_2^{(m)}(r) \leq H^{-1}(H(2\xi) + \overline{J}(r) + \overline{W}(r)) < \infty, \quad \forall r \geq 0. \tag{34}$$

Letting $m \to \infty$ in the (34), we get

$$y_1(r) + y_2(r) \leq H^{-1}(H(2\xi) + \overline{J}(r) + \overline{W}(r)) < \infty, \quad \forall r \geq 0.$$

Moreover, it follows from (Q_5) and the same proof of Theorem 4.1 (2) that

$$y_1(r) \geq \gamma + \underline{J}(r) \quad \textit{and} \quad y_2(r) \geq \delta + \underline{W}(r), \quad \forall r \geq 0.$$

This completes the proof.

References

[1] X Zhang, Y Wu, Y Cui. *Existence and nonexistence of blow-up solutions for a Schrödinger equation involving a nonlinear operator*, Appl Math Lett, 2018, 82: 85-91.

[2] A Ghanmi, H Maagli, V Radulescu, N Zeddini. *Large and bounded solutions for a class of nonlinear Schrödinger stationary systems*, Anal Appl, 2009, 7(4): 391-404.

[3] D P Covei. *Symmetric solutions for an elliptic partial differential equation that arises in stochastic production planning with production constraints*, Appl Math Comput, 2019, 350: 190-197.

[4] A V Lair. *Large solution of mixed sublinear/superlinear elliptic equations*, J Math Anal Appl, 2008, 346(1): 99-106.

[5] X Zhang, L Liu. *The existence and nonexistence of entire positive solutions of semilinear elliptic systems with gradient term*, J Math Anal Appl, 2010, 371(1): 300-308.

[6] Z Zhang, K Wang. *Existence and nonexistence of solutions for a class of Monge-Ampère equations*, J Differential Equations, 2009, 246(7): 2849-2875.

[7] H Li, P Zhang, Z Zhang. *A remark on the existence of entire positive solutions for a class of semilinear elliptic system*, J Math Anal Appl, 2010, 365(1): 338-341.

[8] A V Lair, A W Wood. *Existence of entire large positive solutions of semilinear elliptic systems*, J Differential Equations, 2000, 164(2): 380-394.

[9] X Wang. *A class of fully nonlinear elliptic equations and related functionals*, Indiana Univ Math J, 1994, 43(1): 25-54.

[10] X Zhang. *Analysis of nontrivial radial solutions for singular superlinear k-Hessian equations*, Appl Math Lett, 2020, 106: 106409.

[11] N Trudinger, X Wang. *The Monge-Ampère equation and its geometric applications*, Handbook of Geometric Analysis. Higher Education Press, Beijing, 2008, 1: 467-524.

[12] C Escudero. *Geometric principles of surface growth*, Phys Rev Lett, 2008, 101(19): 196102.

[13] C Escudero, E Korutcheva. *Origins of scaling relations in nonequilibrium growth*, J Phys A Math and Theor, 2012, 45(12): 125005.

[14] J F de Oliveira, J M do Ó, P Ubilla. *Existence for a k-Hessian equation involving supercritical growth*, J Differential Equations, 2019, 267(2): 1001-1024.

[15] C Escudero. *On polyharmonic regularizations of k-Hessian equations: Variational methods*, Nonlinear Anal, 2015, 125(1): 732-758.

[16] C H Lu. *A variational approach to complex Hessian equations in $\mathbb{C}^n$*, J Math Anal Appl, 2015, 431: 228-259.

[17] X Zhang, J Xu, J Jiang, Y Wu, Y Cui. *The convergence analysis and uniqueness of blow-up solutions for a Dirichlet problem of the general k-Hessian equations*, Appl Math Lett, 2020, 102: 106124.

[18] D P Covei. *A necessary and a sufficient condition for the existence of the positive radial solutions to Hessian equation and systems with weights*, Acta Math Scientia, 2017, 37(1): 47-57.

[19] D P Covei. *A remark on the existence of entire large and bounded solutions to a (k_1, k_2)-Hessian system with gradient term*, Acta Math Sinica (English Series), 2017, 33(6): 761-774.

[20] B Wang, J Bao. *Over-determined problems for k-Hessian equations in ring-shaped domains*, Nonlinear Anal, 2015, 127: 143-156.

[21] X Zhang, M Feng. *Boundary blow-up solutions to the k-Hessian equation with singular weights*, Nonlinear Anal, 2018, 167: 51-66.

[22] X Zhang, M Feng. *The existence and asymptotic behavior of boundary blow-up solutions to the k-Hessian equation*, J Differential Equations, 2019, 267(8): 4626-4672.

[23] M Feng, X Zhang. *On a k-Hessian equation with a weakly superlinear nonlinearity and singular weights*, Nonlinear Anal, 2020, 190: 111601.

[24] M Feng. *New results of coupled system of k-Hessian equations*, Appl Math Lett, 2019, 94: 196-203.

[25] P Balodis, C Escudero. *Polyharmonic k-Hessian equations in $\mathbb{R}^N$*, J Differential Equations, 2018, 265: 3363-3399.

[26] D P Covei. *The Keller-Osserman-type conditions for the study of a semilinear elliptic system*, Bound Value Probl, 2019, 2019(1): 104.

[27] Z Zhang, S Zhou. *Existence of entire positive k-convex radial solutions to Hessian equations and systems with weights*, Appl Math Lett, 2015, 50: 48-55.

[28] D P Covei. *Solutions with radial symmetry for a semilinear elliptic system with weights*, Appl Math Lett, 2018, 76: 187-194.

[29] X Zhang, L Liu, Y Wu, Y Cui. *A sufficient and necessary condition of existence of blow-up radial solutions for a k-Hessian equations with a nonlinear operator*, Nonlinear Anal Model Control, 2020, 25(1): 126-143.

[30] G Wang, Z Yang, L Zhang, D Baleanu. *Radial solutions of a nonlinear k-Hessian system involving a nonlinear operator*, Commun Nonlinear Sci Numer Simulat, 2020, 91: 105396.

[31] L Zhang, B Ahmad, G Wang. *The existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative*, Appl Math Lett, 2014, 31: 1-6.

[32] L Zhang, B Ahmad, G Wang. *Explicit iterations and extremal solutions for fractional differential equations with nonlinear integral boundary conditions*, Appl Math Comput, 2015, 268: 388-392.

[33] G Wang. *Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval*, Appl Math Lett, 2015, 47: 1-7.

[34] G Wang. *Twin iterative positive solutions of fractional q-difference Schrödinger equations*, Appl Math Lett, 2018, 76: 103-109.

[35] G Wang, X Ren, L Zhang, B Ahmad. *Explicit iteration and unique positive solution for a Caputo-Hadamard fractional turbulent flow model*, IEEE Access, 2019, 7: 109833-109839.

[36] L Zhang, N Qin, B Ahmad. *Explicit iterative solution of a Caputo-Hadamard-type fractional turbulent flow model*, Math Meth Appl Sci, 2020, DOI: 10.1002/mma.6277.

[37] G Wang, Z Yang, R P Agarwal, L Zhang. *Study on a class of Schrödinger elliptic system involving a nonlinear operator*, Nonlinear Anal Model Control, 2020, 25(5): 846-859.

[38] X Ji, J Bao. *Necessary and sufficient conditions on solvability for Hessian inequalities*, Proc Amer Math Soc, 2010, 138: 175-188.

[39] Z Zhang, H Liu. *Existence of entire radial large solutions for a class of Monge-Ampère type equations and systems*, Rocky Mountain J Math, 2020, 50(5): 1893-1899.

[40] D P Covei. *A remark on the radial solutions of a modified Schrödinger system by dual approach*, Math Commun, 2019, 24: 245-263.

[1]School of Mathematics and Computer Science, Shanxi Normal University, Linfen 041004, China.
Email: wgt2512@163.com, yangzd1229@163.com, zhanglih149@126.com

[2]Shanghai Automotive Wind Tunnel Center, Tongji University, Shanghai 201804, China.
Email: mm_rashidi@tongji.edu.cn