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Fuzzy Zorn's lemma with applications

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Abstract. We introduced the fuzzy axioms of choice, fuzzy Zorn's lemma and fuzzy well-ordering principle, which are the fuzzy versions of the axioms of choice, Zorn's lemma and well-ordering principle, and discussed the relations among them. As an application of fuzzy Zorn's lemma, we got the following results: (1) Every proper fuzzy ideal of a ring was contained in a maximal fuzzy ideal. (2) Every nonzero ring contained a fuzzy maximal ideal. (3) Introduced the notion of fuzzy nilpotent elements in a ring R, and proved that the intersection of all fuzzy prime ideals in a commutative ring R is the union of all fuzzy nilpotent elements in R. (4) Proposed the fuzzy version of Tychonoff Theorem and by use of fuzzy Zorn's lemma, we proved the fuzzy Tychonoff Theorem.

§1 Introduction

Zorn's lemma is a useful result to appear in proofs of some non-constructive esistence theorems throughout many mathematical branches. In 1933 Artin and Chevalley first referred to the principle as Zorns lemma. Especially, the equivalences of axioms of choice, Zorn's lemma, well-ordering principle and comparability principle were discussed ([4,8,7]).

The theory of fuzzy sets which was introduced by Zadeh ([16]) was applied to the branches of pure and applied mathematics. The study of fuzzy relations was started by Zadeh ([17]) in 1971. In [17], the author introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy orderings. Fuzzy orderings have broad utility. They can be applied, for example, when expressing our preferences with a set of alternatives. Since then many notions and results from the theory of ordered sets have been extended to the fuzzy ordered sets. In [15], Venugopalan introduced a definition of fuzzy ordered set (foset) (P, μ) and presented an example on the set of positive integers. He extended this concept to obtain a fuzzy lattice in which he defined a (fuzzy) relation as a generalization of equivalence. The

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notion of a multichain in a fuzzy ordered set is defined in [1]. In [13], *Šešelja* and Tepavčević presented a survey on representations of ordered structures by fuzzy sets. An order relation and a ranking method for type-2 fuzzy values are proposed in [9]. On the other hand, few years after the inception of the notion of fuzzy set, Rosenfeld started the pioneer work in the domain of fuzzification of the algebraic objects, with his work on fuzzy groups ([12]). This work is a contribution to the theory founded on the ideas of those authors and their followers. Das ([6]) characterized fuzzy subgroups by their level subgroups. In [10], Lin applied the concept of fuzzy sets to the theory of rings and introduced and examined the notion of a fuzzy ideal of a ring.

Zorn's lemma has many applications in mathematics. By using of Zorn's lemma, W. J. Lin proved that every proper ideal of a ring is contained in a maximal ideal ([10]). In 2009, D. D. Anderson etc. applied Zorn's lemma to study some existence theorems in modules, groups, and integral domains ([2]). In 2015, as a direct application of Zorn's lemma, Haruo Tsukada proved Tychonoff Theorem ([14]).

In the set theory, ones have been proved that axioms of choice, Zorn's lemma, well-ordering principle and comparability of cardinalities are all equivalent. By use of a special fuzzy order, Chapin ([5]) studied the basic logical axioms of fuzzy set theory and also introduced the fuzzy axiom of choice. Ismat Beg ([3]) proved fuzzy Zorn's lemma by using fuzzy axiom of choice due to Chapin. But the authors did not prove that such fuzzy axiom of choice and fuzzy Zorn's lemma are equivalent.

In this paper, by the use of Zadeh's fuzzy order, we introduce the fuzzy axioms of choice, fuzzy Zorn's lemma and fuzzy well-ordering principle, which are the fuzzy versions of the axioms of choice, Zorn's lemma and well-ordering principle, and discuss the relations among them. As an application of fuzzy Zorn's lemma, we prove that every proper fuzzy ideal of a ring is contained in a maximal fuzzy ideal. Moreover we give the fuzzy version of Tychonoff Theorem. By use of the fuzzy Zorn's lemma, we prove that fuzzy Tychonoff Theorem.

§2 Preliminary

In this section, we recollect some definitions and results which will be used in the following.

Let X be a non-empty set. A map $\mu : X \to [0,1]$ is called a fuzzy subset of X. Denote the set of all fuzzy subsets of X by $F(X)$. For a fuzzy subset $\mu \in F(X)$, we define $supp\{\mu\} = \{x \in X \mid \mu(x) > 0\}$, which is called support set of μ. If $\mid supp\{\mu\} \mid \leq 1$, the fuzzy subset μ is called a fuzzy point of X. Denote the set of all fuzzy points of X by $FP(X)$. Some times, for $\lambda \in FP(X)$, we write λ by λ_x, where $\lambda(x) > 0$. Let $PF(X)$ denote the sets of all subsets of the set of fuzzy subsets on X.

For $\mu, \nu \in F(X)$, we give the following notions:

$(\mu \cap \nu)(x) = min\{\mu(x), \nu(x)\}$, for all $x \in X$,

$(\mu \cup \nu)(x) = max\{\mu(x), \nu(x)\}$, for all $x \in X$,

$\mu \subseteq \nu$ iff $\mu(x) \leq \nu(x)\}$, for all $x \in X$.

Now we present the concept of Zadeh's fuzzy partial orders.

Definition 2.1. ([16]) Let X be a crisp set. A fuzzy partial order (for short fuzzy order) μ_R on X is a fuzzy subset μ_R of $X \times X$ with the following properties:

(FO1) for all $x \in X$, $\mu_R(x,x) = 1$ (Reflexivity),

(FO2) for all $x, y \in X$, $\mu_R(x, y) > 0$ and $\mu_R(y, x) > 0$ imply $x = y$, (Antisymmetric)
(FO3) for all $x, y, z \in X$, $\mu_R(x, z) \geq \vee_{y \in X}(\mu_R(x, y) \wedge \mu_R(y, z))$. (Transitive)

Let X be a set and μ_R be a fuzzy order on X. A pair (X, μ_R) is called a fuzzy ordered set. If A is a subset of X, then we call (A, μ_R) a fuzzy ordered subset of X (for shortly we call A is a fuzzy ordered subset of X).

Definition 2.2. Let (X, μ_R) be a fuzzy ordered set and A be a fuzzy ordered subset of X.
(1) The fuzzy order μ_R is said to be total if for all $x, y \in X$ we have either $\mu_R(x, y) = 1$ or $\mu_R(y, x) = 1$.
(2) If the fuzzy order μ_R is total on A, then A is called a fuzzy chain.
(3) An element $x \in A$ is called fuzzy maximal element of A if there is no $y(\neq x)$ in A for which $\mu_R(x, y) = 1$.
(4) An $x \in X$ satisfying $\mu_R(y, x) = 1$ for all $y \in A$ is called fuzzy upper bound of A.
(5) An $x \in A$ satisfying $\mu_R(y, x) = 1$ for all $y \in A$ is called fuzzy greatest element of A.
Similarly, we can define fuzzy lower bound, fuzzy minimal and least elements of A.

Definition 2.3. Let X be a set and μ, λ be two fuzzy sets on X.
(1) If $\mu(x) \leq \lambda(x)$ for all $x \in X$, we call that μ is less than or equal to λ, denoted by $\mu \subseteq \lambda$.
(2) If λ is a fuzzy point and $supp\{\lambda\} \in supp\{\mu\}$, we call that λ belong to μ, denoted by $\lambda \in \mu$ (sometimes by $\lambda \subseteq \mu$).
(3) If $\mu(x) = 0$ for some $x \in X$, we call μ is a proper fuzzy set.

It is well-known the result about famous axioms in set theory.

Theorem 2.4. *([2,8,7]) The following statements are equivalent:*
(1)(Axiom of choice) If $\{X_i\}_{i \in I}$ is a family of nonempty sets, then $\prod_{i \in I} X_i$ is also nonempty.
(2)(Zorn's lemma) Let P be a partially ordered set. If every chain in P has an upper bound, then X has a maximal element.
(3)(Well-ordering principle) Every set can be well-ordered.
(4)(Comparability principle) Given any two sets X and Y, there exists either a bijection between X and a subset of Y, or a bijection between Y and a subset of X.

§3 The fuzzy axioms of choice, fuzzy Zorn's lemma and fuzzy well-ordering principle

In this section, we introduce the fuzzy axioms of choice, the fuzzy Zorn's lemma and the fuzzy well-ordering principle.

Fuzzy axioms of choice (FAC): Let X be a set. Given a family $\{\mu_{i \in I}\}$ of non-zero fuzzy sets on X, one could choice a family $\{\lambda_{i \in I}\}$ of fuzzy points, with $\lambda_i \in \mu_i$, for each $i \in I$.

Fuzzy Zorn's lemma (FZL): Let (X, μ_R) be a fuzzy ordered set. If every fuzzy chain in (X, μ_R) has a fuzzy upper bound, then X has a fuzzy maximal element.

In order to state the fuzzy well-ordering principle, we firstly give the definition of fuzzy well-ordered sets.

Definition 3.1. A fuzzy ordered set (X, μ_R) is called a fuzzy well-ordered set if it is a totally fuzzy ordered set in which every non-empty subset has a fuzzy least element.

Now we introduce the fuzzy well-ordering principle (FWOP): Every set X can be fuzzy well-ordered, i.e. for every set X, there exists a fuzzy binary relation μ_R on X which makes it a fuzzy well-ordered set.

Proposition 3.2. *If fuzzy axiom of choice holds, then axiom of choice holds, too.*

Proof. Let fuzzy axiom of choice hold. Given a family of non-empty sets $\{X_i\}_{i\in I}$, then we define $X = \cup_{i\in I} X_i$ and define a family of fuzzy sets $\{\mu_i\}_{i\in I}$ for each $i \in I$ by

$$\mu_i(x) = \begin{cases} 1, & x \in X_i \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in X$. We can see that $\{\mu_i\}_{i\in I}$ is a non-zero fuzzy set on X for each $i \in I$. By use of the fuzzy axiom of choice, we can choice a family of fuzzy points $\{\lambda_i\}_{i\in I}$, with $\lambda_i \in \mu_i$, for each $i \in I$. Hence we can choice $x_i = supp\lambda_i \in supp\mu_i = X_i$, for each $i \in I$. □

Proposition 3.3. *If Zorn's lemma holds, then fuzzy Zorn's lemma holds, too.*

Proof. Assume that Zorn's lemma holds. Let (X, μ_R) be a fuzzy ordered set, in which every fuzzy chain has a fuzzy upper bound in (X, μ_R). Define a binary relation " $\leq$ " by $x \leq y$ if $\mu_R(x, y) = 1$. Then we can check that $(X, \leq)$ is an ordered set. If $\{a_i\}_{i\in I}$ is a chain in $(X, \leq)$ with $a_i \leq a_{i+1}$ for each $i \in I$, then $\mu_R(a_i, a_{i+1}) = 1$ for each $i \in I$ and thus it is a fuzzy chain. By hypothesis, $\{a_i\}_{i\in I}$ has a fuzzy upper bound in (X, μ_R), denote it by a_0. Then we have $R(a_i, a_0) = 1$. By the definition of $\leq$, we have $a_i \leq a_0$, which means that x_0 is a upper bound for $\{a_i\}_{i\in I}$ in $(X, \leq)$. Since Zorn's lemma holds, we have that $(X, \leq)$ has a maximal element, denote it by x_0. We claim that x_0 is a fuzzy maximal element in (X, μ_R). Otherwise, if x_0 is not a fuzzy maximal element in (X, μ_R), there exists $x_1 \in X$, such that $x_1 \neq x_0$ and $R(x_0, x_1) = 1$. By the definition of $\leq$, we have $x_0 \leq x_1$ but $x_1 \neq x_0$, contradicting to that x_0 is a fuzzy maximal element in (X, μ_R). □

Proposition 3.4. *If the fuzzy Zorn's lemma holds, then the fuzzy axiom of choice holds, too.*

Proof. Assume that fuzzy Zorn's lemma holds. Let $\{\mu_i\}_{i\in I}$ be a family of non-zero fuzzy sets on a non-empty X. Let P be the "fuzzy partial functions" from I to $\cup_{i\in I}\mu_i \doteq \{\lambda_i \mid \lambda_i \in \mu_i, i \in I\}$, where for $\phi \in P$ we mean that $\phi \subseteq I \times \cup_{i\in I}\mu_i$ such that for each $i \in I$, any element of ϕ with first component i has second component in μ_i, and for each i, there is at most one such element. Note that P is non-empty because the empty fuzzy partial function is a member of it. Let P be partially ordered by fuzzy order μ_R as follows: for all $\phi, \phi' \in P$

$$\mu_R(\phi, \phi') = \begin{cases} 1, & \phi \subseteq \phi' \\ 0, & \text{otherwise.} \end{cases}$$

Then we can check that (P, μ_R) is a fuzzy ordered set. Given any fuzzy chain $\{\phi_i\}_{i\in\Lambda}$ in P, we can verify that the union of $\Phi = \cup_{i\in\Lambda}\phi_i$ is a fuzzy upper bound to $\{\phi_i\}_{i\in\Lambda}$ in P. First we check that $\Phi \in P$. Clearly $\Phi \subseteq I \times \cup_{i\in I}\mu_i$. For $a \in \Phi$, then there is ϕ_i such that $a \in \phi_i$. If the first component of a is i, then it's second component is in μ_i and for such i, there is at most one such element. It follows that $a \in P$. Then we check that Φ is a fuzzy upper bound to $\{\phi_i\}_{i\in\Lambda}$. For any $\phi_i \in \{\phi_i\}_{i\in\Lambda}$, we have $\phi_i \subseteq \Phi$ and hence $R(\phi_i, \Phi) = 1$ by the definition of R. It follows that Φ is a fuzzy upper bound to $\{\phi_i\}_{i\in\Lambda}$. By the fuzzy Zorn's lemma, (P, μ_R) has a fuzzy maximal element ϕ. Therefore we can verify that such ϕ will be a fuzzy function, i.e, $\phi = \{(i, \lambda_i)\}$ with $\lambda_i \in \mu_i$, as required. In fact, if it is not a fuzzy function, then there is some $a \in \phi$ with the first component i_0 but it's second component is empty. Let ϕ_1 be a element in P, such that (i_0, λ_{i_0}) is in ϕ_1 but for other $i \in \Lambda$ the second component is same with one of ϕ, where $\lambda_{i_0} \in \mu_{i_0}$. Therefore $\phi \subseteq \phi_1$ and hence $\mu_R(\phi, \phi_1) = 1$. This contradict to that ϕ is a fuzzy maximal element. □

Proposition 3.5. *The following statements are equivalent:*
(1) The well-ordering principle holds.
(2) The fuzzy well-ordering principle holds.

Proof. (1)⇒(2) Let (WOP) hold. Then given a set X, there exists a binary relation $\leq$ on X, such that it is a well-ordered set. Define a fuzzy binary relation e by

$$e(x,y) = \begin{cases} 1, & x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

Then we can check that e is a fuzzy order on X. Since X is a well-ordered set, then it is a chain in $(X, \leq)$ and hence is also a fuzzy chain in (X, e). Now we verify that (X, e) a fuzzy well-ordered set. Let A be a subset of X. Since X is a well-ordered set in $(X, \leq)$, then there exists a least element $x_0 \in A$, that is, $x_0 \leq a$ for each $a \in A$. This means that $e(x_0, a) = 1$ for each $a \in A$. It follows that x_0 is a fuzzy least element of A in (X, e), and thus (X, e) is a fuzzy well-ordered set.
(2)⇒(1) Let (FWOP) hold. Then for any set X, there exists a fuzzy binary relation e, such that (X, e) is a fuzzy well-ordered set. Define a binary relation $\leq$ by $x \leq y$ iff $e(x, y) = 1$, for all $x, y \in X$. Then we can check that $\leq$ is a order on X. Since X is a fuzzy chain in (X, e), then it is also a chain in $(X, \leq)$. Let $A \subseteq X$. Since X is a fuzzy well-ordered set in $(X, \leq)$, then there exists a least element $x_1 \in A$, that is, $e(x_1, a) = 1$ for each $a \in A$. By the definition of $\leq$, we have $x_1 \leq a$ for each $a \in A$. This means that x_1 is a least element of A, and hence $(X, \leq)$ is a well-ordered set. □

By Theorem 2.4, Propositions 3.2, 3.3, 3.4 and 3.5, we can get the following theorem.

Theorem 3.6. *The following statements are equivalent:*
(1) Fuzzy axiom of choice holds;
(2) Axiom of choice holds;
(3) Zorn's lemma holds;
(4) Fuzzy Zorn's lemma holds;
(5) The well-ordering principle holds.
(6) The fuzzy well-ordering principle holds.

§4 The applications of fuzzy Zorn's lemma

In this section, we discuss the applications of fuzzy version of Zorn's lemma.

4.1 The application of the fuzzy Zorn's lemma for fuzzy ideals in rings

Firstly by use of the fuzzy Zorn's lemma, we will prove that in a ring R, any fuzzy ideal of R is contained in a maximal fuzzy ideal.

Definition 4.1. ([10]) Let R be a ring and A be a fuzzy set of R. A is called a fuzzy ideal of R if it satisfies the following conditions:
(1) $A(x+y) \geq A(x) \wedge A(y)$, for all $x, y \in R$,
(2) $A(-x) \geq A(x)$, for all $x \in R$,
(3) $A(xy) \geq A(x) \vee A(y)$, for all $x, y \in R$.

We denote the set of all fuzzy ideals of a ring R by $FI(R)$.

In the following we give a relation between fuzzy ideals and ideals in rings. For a fuzzy set A and $t \in [0,1]$, we denote $A_t = \{x \in R \mid A(x) \geq t\}$.

Proposition 4.2. *([11]) Let R be a ring and A be a fuzzy subset of R. Then A is a fuzzy ideal of R if and only if A_t is a ideal of R, for all $t \in [0,1]$ and $A_t \neq \emptyset$.*

Proposition 4.3. *Let R be a ring and $\{A_i\}_{i\in I}$ be a family of fuzzy ideals of R such that $A_i \subseteq A_{i+1}$ for all $i \in I$. Then $\cup_{i\in I} A_i$ is also a fuzzy ideal.*

Proof. Let $J = \cup_{i\in I} A_i$ and $t \in [0,1]$ such that $J_t \neq \emptyset$. Now we can check that J_t is an ideal of R. Let $x, y \in J_t$, then $J(x) \geq t$ and $J(y) \geq t$. Let $J(x) = a$ and $J(y) = b$. By the definition of the supremum, for any $\epsilon > 0$, there is i_0 and j_0 such that $A_{i_0}(x) > a - \epsilon$ and $A_{j_0}(y) > b - \epsilon$. Take $k_0 = max\{i_0, j_0\}$, since $\{A_i\}_{i\in I}$ is a chain, we have $A_{k_0}(x) > a - \epsilon \geq t - \epsilon$ and $A_{k_0}(y) > b - \epsilon \geq t - \epsilon$. Since A_i is fuzzy ideal for each $i \in I$, then $J(x-y) = (\cup_{i\in I} A_i)(x-y) = sup_{i\in I}\{A_i(x-y)\} \geq sup_{i\in I}\{A_i(x) \wedge A_i(y)\} \geq A_{k_0}(x) \wedge A_{k_0}(y) \geq t - \epsilon$. By the arbitrariness of ϵ, we get $J(x-y) \geq t$, and thus $x - y \in J_t$. Similarly we can prove that for all $x \in R$ and $y \in J_t$, we have $xy \in J_t$. By Proposition 4.2, we get that J is a fuzzy ideal of R. □

Definition 4.4. Let R be a ring and A be a proper fuzzy ideal of R. A is called a maximal fuzzy ideal if there is no proper fuzzy ideal B such that $A \subset B \subset R$.

Theorem 4.5. *Let R be a ring with identity. Then every proper fuzzy ideal μ of R is contained in a maximal fuzzy ideal.*

Proof. Let $FI(\mu)$ be the set of proper fuzzy ideals in R containing μ. We define a fuzzy order e on $FI(\mu)$ by $e(\lambda, \nu) = 1$ iff $\lambda \subseteq \nu$ for all $\lambda, \nu \in FI(\mu)$. Let C be a fuzzy chain in $FI(\mu)$, that is, C is just a string of fuzzy ideals, $\cdots \mu_1 \subseteq \mu_2 \subseteq \cdots$.

Now we show that C has a fuzzy upper bound in $FI(\mu)$. Let $J = \cup_{\mu_i \in C} \mu_i$. Clearly $\mu \subseteq J$. Also we can check that J is a fuzzy ideal of R. Indeed, $J(x+y) = (\cup_{\mu_i\in C}\mu_i)(x+y) = sup_{\mu_i\in C}\{\mu_i(x+y)\} \geq sup_{\mu_i\in C}\{\mu_i(x) \wedge \mu_i(y)\} = sup_{\mu_i\in C}\{\mu_i(x)\} \wedge sup_{\mu_i\in C}\{\mu_i(y)\} = J(x) \wedge J(y)$. Similarly we can prove that $J(-x) \geq J(x)$ and $J(x \cdot y) \geq J(x) \vee J(y)$. By Definition 4.1, we get that J is a fuzzy ideal of R.

Moreover we prove that J is proper. Since μ_i is proper for each $\mu_i \in C$, then $\mu_i(x_i) = 0$ for some $x_i \in X$. Since μ_i is a fuzzy ideal, $\mu_i(x_i) = \mu_i(x_i \cdot 1) \geq \mu_i(1)$ and hence $\mu_i(1) = 0$ for each $\mu_i \in C$. It follows that $J(1) = 0$, and thus J is proper. We get $J \in FI(\mu)$. Note that $\mu_i \subseteq J$ and hence $e(\mu_i, J) = 1$ for each $\mu_i \in C$. This shows that J is a fuzzy upper bound of C in $FI(\mu)$. By the fuzzy Zorn's lemma, $FI(\mu)$ has a fuzzy maximal element, say it as ν. We can get that ν is a fuzzy maximal ideal of R. Otherwise, if there is a proper fuzzy ideal λ such that $\nu \subseteq \lambda$ and $\nu \neq \lambda$, then $\lambda \in FI(\mu)$ and $e(\nu, \lambda) = 1$, and so ν is not a fuzzy maximal element of $FI(\mu)$, a contradiction. □

Corollary 4.6. Every nonzero ring contains a maximal fuzzy ideal.

Proof. Let R be a nonzero ring. Define a fuzzy set A_0 as follwing:

$$A_0(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Then we can check that A_0 is a fuzzy ideal of R. Since R is a nonzero ring, then there is $a \in R$ and $a \neq 0$ such that $A_0(a) = 0$. This shows that A_0 is a proper fuzzy ideal. From Theorem 4.5, A_0 is contained in a maximal fuzzy ideal M. It follows that R contains a maximal fuzzy ideal M. □

Now, we discuss some properties of fuzzy ideals generated by fuzzy subsets, which will be used in the following discussions.

Definition 4.7. ([10]) Let R be a ring and A be a fuzzy set of R. The smallest fuzzy ideal containing A is called the fuzzy ideal generated by A, denoted by $[A]$.

Proposition 4.8. *([11]) Let R be a ring and A be a fuzzy subset of R. Then for any $x \in R$, we have*

$[A](x) = sup\{t \mid x \in [A_t], A_t \neq \emptyset\}$.

By use of the above representation of fuzzy ideal generated by a fuzzy subset, we give a representation of fuzzy ideal generated by a fuzzy point.

Proposition 4.9. *Let R be a ring and λ be a fuzzy point of R with supp$\{\lambda\} = \{x_0\}$. Then for any $x \in R$, we have*

$$[\lambda](x) = \begin{cases} \lambda(x_0), & x \in [x_0] \\ 0, & otherwise. \end{cases}$$

Moreover, supp$\{[\lambda]\} = [x_0]$.

Proof. By Proposition 4.8, we have $[\lambda](x) = sup\{t \mid x \in [\lambda_t], \lambda_t \neq \emptyset\}$. Note that $\lambda_t \neq \emptyset$ iff $\lambda_t = \{x_0\}$ iff $\lambda(x_0) \geq t$. It follows that for all $x \in [x_0]$, $[\lambda](x) = sup\{t \mid x \in [x_0], \lambda(x_0) \geq t\} = \lambda(x_0)$. Otherwise, $[\lambda](x) = 0$. □

In the following, by use of fuzzy Zron's lemma, we discuss the connection between fuzzy prime ideals and the fuzzy nilpotent elements in the rings. First we recall notion of nilpotent elements and an important theorem concerns nilpotent elements in commutative rings.

Definition 4.10. Let R be a commutative ring. An element x of R is called nilpotent if $x^n = 0$ for some $n \geq 1$.

Let R be a commutative ring. Denote the set of all prime ideals in R by $PI(R)$ and the set of all nilpotent elements of R by $NE(R)$.

Theorem 4.11. *Let R be a commutative ring. Then the intersection of all prime ideals in R is the set of nilpotent elements in R, that is, $\cap\{P \mid P \in PI(R)\} = NE(R)$.*

For fuzzy subsets A, B of a set X, define $A \cdot B$, $A + B$ and A_* as follows: for any $x \in X$,

$$(A \cdot B)(x) = \begin{cases} sup\{A(x_1) \wedge B(x_2) \mid x = x_1x_2\}, & if\ \exists t_1, t_2 \in R, s.t.,\ x = t_1t_2 \\ 0, & otherwise \end{cases}$$

$$(A + B)(x) = \begin{cases} sup\{A(x_1) \wedge B(x_2) \mid x = x_1 + x_2\}, & if\ \exists t_1, t_2 \in R, s.t.,\ x = t_1 + t_2 \\ 0, & otherwise \end{cases}$$

$$A_* = \{x \in X | A(x) = A(0)\}.$$

We denote $AB = A \cdot B$ and $A^1 = A$, $A^n = A^{n-1}A$ for $n > 1$. Firstly, we give a property of A^n for a fuzzy subset A of R.

Lemma 4.12. *Let A, B be fuzzy subsets and λ, ν be fuzzy points with $supp\{\lambda\} = \{x_0\}$ and $supp\{\nu\} = \{y_0\}$ in a ring R. Then we have the following:*
(1) $supp\{AB\} = supp\{A\}supp\{B\}$;
(2) $supp\{[\lambda][\nu]\} = [x_0][y_0]$;
(3) $supp\{A + B\} = supp\{A\} + supp\{B\}$;
(4) $supp\{[\lambda] + [\nu]\} = [x_0] + [y_0]$;
(5) $A \neq 0$ and $B \neq 0$ iff $AB \neq 0$ iff $A + B \neq 0$;
(6) $A \neq 0$ iff $A^n \neq 0$ for all $n \in \mathbf{N}$;
where $A + B = \{x + y \mid x \in A, y \in B\}$ and $AB = \{xy \mid x \in A, y \in B\}$ for all $A, B \subseteq R$.

Proof. We can directly check that the above statements hold. □

Proposition 4.13. *([10]) Let A, B, C be fuzzy ideals of a ring R. Then the following hold:*
(1) $A(B + C) \subseteq AB + AC$, $(B + C)A \subseteq BA + CA$;
(2) $AB \subseteq A \cap B \subseteq A, B$;
(3) $A \subseteq B$ implies $AC \subseteq BC$, $CA \subseteq CB$ and $A + C \subseteq B + C$;
(4) AB and $A + B$ are also fuzzy ideals of R;
(5) $A(0) \geq A(x)$ for all $x \in R$;
(6) $A(0) > 0$ for any non-zero fuzzy ideal of R.

Proposition 4.14. *Let A, B be nonzero fuzzy ideals of a ring R. Then we have the following:*
(1) $A \subseteq B$ implies $supp\{A + B\} = supp\{B\}$;
(2) $A \subseteq nA$, where $nA = \underbrace{A + \cdots + A}_{n\ times}$;
(3) $supp\{nA\} = supp\{A\}$.

Proof. We can directly check that the above statements hold. □

Definition 4.15. ([11]) Let μ_P be a fuzzy ideal of a ring R and $|Im\mu_P| > 1$. If for any fuzzy points λ and ν, $\lambda\nu \subseteq \mu_P$ implies $\lambda \subseteq \mu_P$ or $\mu \subseteq \mu_P$, then μ_P is called a fuzzy prime ideal.

We denote the set of all fuzzy prime ideals of R by $FPI(R)$.

Proposition 4.16. *([11]) Let μ_P be a fuzzy prime ideal of a ring R. Then we have the following:*
(1) $|Im\mu_P| = 2$*;*
(2) $\mu_P(0) = 1$*;*
(3) $(\mu_P)_*$ *is a prime ideal of R.*

Proposition 4.17. *Let μ_P be a fuzzy prime ideal of a ring R. Then for any fuzzy ideal A of R, we have $A \subseteq A + \mu_P$.*

Proof. Note that $(A + \mu_P)(x) \geq A(x) \wedge \mu_P(0)$ for all $x \in R$. By Proposition 4.16(2), we get $\mu_P(0) = 1$, and thus $(A + \mu_P)(x) \geq A(x) \wedge 1 = A(x)$ for all $x \in R$. That is $A + \mu_P \supseteq A$. □

Proposition 4.18. *Let R be a ring and A be a fuzzy set of R. Define a fuzzy point x_A for $x \in R$ as follows:*

$$x_A(t) = \begin{cases} A(x), & t{=}x \\ 0, & otherwise. \end{cases}$$

Then we have $A = \cup\{x_A \mid x \in R\}$, that is, any fuzzy set of R can be represented by a union of some fuzzy points.

Proof. We can directly check it. □

Definition 4.19. Let R be a ring and λ be a fuzzy point of R. λ is called a fuzzy nilpotent element if $\lambda^n(0) > 0$ for some $n \geq 1$.

Clearly, any fuzzy nilpotent element is not equal to 0. We denote the set of all fuzzy nilpotent elements of R by $FN(R)$.

Proposition 4.20. *Let R be a commutative ring and λ, ν be fuzzy points of R with $supp\{\lambda\} = \{x_0\}$ and $supp\{\nu\} = \{y_0\}$. Then we have the following:*
(1) $\lambda\nu$ is also a fuzzy point of R with $supp\{\lambda\nu\} = x_0y_0$;
(2) λ^n is also a fuzzy point of R with $supp\{\lambda^n\} = x_0^n$, for any $n \in \mathbf{N}$;
(3) λ is a fuzzy nilpotent elements of R iff x_0 is a nilpotent elements of R;
(4) $\lambda^n(0) > 0$ iff $x_0^n = 0$;
(5) $supp\{[\lambda][\nu]\} = supp\{[\lambda\nu]\}$.

Proof. We can directly check that the above statements hold. □

Example 1. Let $R^{2\times 2} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R\}$, where R is the set of all real numbers. Then $(R^{2\times 2}, +, \cdot, \mathbf{0})$ forms a commutative ring, where $\mathbf{0}$ denotes zero matrix, $+$ and $\cdot$ are additions and multiplications of matrix, respectively. It is easy to see that $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is a nilpotent element of R, since $B^2 = \mathbf{0}$. Let 1_B be the characteristic function of $\{B\}$. We can see $supp\{1_B\} = \{B\}$. By Proposition 4.20(2), we have $supp\{1_B^2\} = \{B^2\}$, and hence $1_B^2(B^2) = 1_B^2(0) > 0$. It follows that 1_B is a fuzzy nilpotent element of $R^{2\times 2}$. Moreover, let $a \neq 0$ and $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, then 1_A is not a fuzzy nilpotent element of $R^{2\times 2}$.

Now by use of the fuzzy Zorn's lemma, we give a result to show a relation between fuzzy prime ideals and fuzzy nilpotent elements on a commutative ring. Indeed, it is a fuzzy version of Theorem 4.11.

Theorem 4.21. *The intersection of all fuzzy prime ideals in a commutative ring R is the union of all fuzzy nilpotent elements in R, that is,* $\cap\{\mu_P \mid \mu_P \in FPI(R)\} = \cup\{\lambda \mid \lambda \in FN(R)\}$.

Proof. Take a fuzzy nilpotent element λ with $supp\{\lambda\} = x_0$ and a fuzzy prime ideal μ_P in R. Then we have that $\lambda^n(0) > 0$ for some $n \geq 1$. By Proposition 4.20(2), λ^n is also a fuzzy point of R with $supp\{\lambda^n\} = x_0^n = 0$, and hence, $\lambda^n(x) = 0$ for any $x \neq 0$. It follows from Proposition 4.16(2) that $\lambda^n \subseteq \mu_P$. Since μ_P is prime, we get $\lambda \subseteq \mu_P$, and so, $\cup\{\lambda \mid \lambda \in FN(R)\} \subseteq \mu_P$. It follows that $\cup\{\lambda|\lambda \in FN(R)\} \subseteq \cap\{\mu_P \mid \mu_P \in FPI(R)\}$.

Now we prove the inverse inclusion relation. Let λ be a fuzzy point of R such that $\lambda \subseteq \mu_P$ for all fuzzy prime ideals. We claim that λ is a fuzzy nilpotent element. Otherwise, if λ is not a fuzzy nilpotent element, then $\lambda^n(0) = 0$ for each $n \geq 1$. Define a set S as follows:

$$S = \{A \in FI(R) \mid \lambda^n \notin A, \forall n \geq 1\}.$$

Consider the fuzzy ideal A_0 given in the proof of Corollary 4.6. Since λ is not a fuzzy nilpotent element, then $supp\{\lambda^n\} \neq \{0\}$, and so $A_0 \in S$. This shows that S is nonempty. We define a fuzzy relation on S by the following: for all $A, B \in S$,

$$R(A, B) = \begin{cases} 1, & A \subseteq B \\ 0, & \text{otherwise.} \end{cases}$$

Then R is a fuzzy ordered relation on S. Let $C = \{A_i\}_{i\in I}$ is a fuzzy chain in (S, R), where $I = \{1, 2, \cdots\}$ and $R(A_i, A_{i+1}) = 1$ for $i = 1, 2, \cdots$. Then we can check that $A = \cup_{i\in I}A_i$ is a fuzzy upper bound of C in S. By Proposition 4.3, A is a fuzzy idea of R. In order to prove $A \in S$, it is only to check $\lambda^n \notin A$, for all $n \geq 1$. If it is not, then $\lambda^n \in A$ for some $n \in \mathbf{N}$. By Definition 2.3(2), we have that $x_0 = supp\{\lambda^n\} \in supp\{A\}$, or $A(x_0) > 0$. It follows that $sup_{i\in I}\{A_i(x_0)\} > 0$. Hence we can get $A_i(x_0) > 0$ for some $i \in I$. This means $supp\{\lambda^n\} = x_0 \in supp\{A_i\}$, or $\lambda^n \in A_i$. This contradict to $A_i \in S$. So that we get that $\lambda^n \notin A$, for all $n \geq 1$. This shows that A is a fuzzy upper bound of S, and thus the condition of fuzzy Zorn's lemma holds in S. By the fuzzy Zorn's lemma, there is a maximal element in S, say M. In the following, we prove that M is a fuzzy prime ideal. First we claim that $\mid ImM \mid > 1$. Otherwise if $\mid ImM \mid = 1$, then from $A_0 \subseteq M$ we get $M = 1$, and so $supp\{M\} = R$. This contradict to $\lambda^n \notin M$. Moreover let $\sigma\nu \subseteq M$ for $\sigma, \nu \in FP(R)$. To prove $\sigma \subseteq M$ or $\nu \subseteq M$, we assume otherwise. Then $[\sigma] + M$ and $[\nu] + M$ are both ideals by Proposition 4.14 and strictly larger than M, so that they can't be in S. This means that $\lambda^n \in [\sigma] + M$ and $\lambda^m \in [\nu] + M$ for some $n, m \in \mathbf{N}$. Moreover, we have $\lambda^{n+m} \in ([\sigma] + M)([\nu] + M)$, by 4.12(1). Note that the following inclusions:

$\lambda^{n+m} \in ([\sigma] + M)([\nu] + M)$
$\subseteq [\sigma][\nu] + [\sigma]M + [\nu]M + MM$ (by Proposition 4.13(1))
$\subseteq [\sigma][\nu] + 3M.$ (by Proposition 4.13(2), (3))

Hence $\lambda^{n+m} \in [\sigma][\nu] + 3M$, and hence $x_0^{n+m} \in supp\{[\sigma][\nu] + 3M\} = supp\{[\sigma][\nu]\} + supp\{3M\}$. By proposition 4.20(5) and 4.14(3), $supp\{[\sigma][\nu]\} + supp\{3M\} = supp\{[\sigma\nu]\} + supp\{3M\} =$

$supp\{[\sigma\nu]\} + supp\{M\} = supp\{[\sigma\nu] + M\}$. Since $\sigma\nu \subseteq M$, then we have $[\sigma\nu] \subseteq M$, and hence $supp\{[\sigma\nu] + M\} = supp\{M\}$, by Proposition 4.14(1). From the above arguments, we get $x_0^{n+m} \in supp\{M\}$, that is $\lambda^{n+m} \in M$. It is a contradiction to $M \in S$. This shows that M is a fuzzy prime ideal. Since $M \in S$, we have $\lambda \notin M$, and hence we get a contradiction to the choice of λ. Therefore, if a fuzzy point $\lambda \subseteq \mu_P$ for all prime fuzzy ideals, then λ must be a fuzzy nilpotent element. Let $F = \cap\{\mu_P \mid \mu_P \in FPI(R)\}$. By Proposition 4.18, we have $F = \cup\{x_F \mid x \in R\}$, where x_F is a fuzzy point and $x_F \subseteq F$. It follows that $x_F \subseteq \mu_P$, for each fuzzy prime ideal μ_P. By the above arguments, we get x_F is a fuzzy nilpotent element, and so

$$\cup\{x_F \mid x \in R\} \subseteq \cup\{\lambda \mid \lambda \in FN(R)\},$$

or $F \subseteq \cup\{\lambda \mid \lambda \in FN(R)\}$. Therefore the inverse inclusion relation holds. We complete the proof. □

From the following example, we can see that in Theorem 4.21, if fuzzy prime ideals are replaced by fuzzy ideals, it does not hold.

Example 2. Consider the ring $R^{2\times 2}$ as given in Example 1. For any $\mu \in FI(R^{2\times 2})$, we have $1_{\{\mathbf{0}\}} \subseteq \mu$, and thus $\cap\{\mu_P \mid \mu_P \in FPI(R^{2\times 2})\} = 1_{\{\mathbf{0}\}}$. From Example 1, we know that $1_{\{\mathbf{B}\}}$ is a fuzzy nilpotent element of $R^{2\times 2}$ for $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ and $b \neq 0$. Note that $1_{\{\mathbf{0}\}} \subsetneq 1_{\{\mathbf{B}\}}$, and hence $\cap\{\mu \mid \mu \in FI(R^{2\times 2})\} \neq \cup\{\lambda \mid \lambda \in FN(R^{2\times 2})\}$.

4.2 Fuzzy Tychonoff theorem and the applications of fuzzy Zorn's lemma

In this subsection, we discuss the fuzzy version of Tychonoff Theorem and give it's proof by fuzzy Zorn's lemma. Firstly we present Tychonoff Theorem.

Tychonoff Theorem. ([14]) For an arbitrary family of compact topological spaces $\{X_i\}_{i\in I}$, the product space $\prod_{i\in I} X_i$ with the product topology is compact.

Definition 4.22. A set F_μ consisting of fuzzy subsets of a set X is called a fuzzy filter on X, if the following conditions are satisfied:
(1) $0 \notin F_\mu$, $1 \in F_\mu$,
(2) for any fuzzy subsets U, V of X, if $U \in F_\mu$ and $U \subseteq V$, then $V \in F_\mu$,
(3) for any $U, V \in F_\mu$, we have $U \cap V \in F_\mu$.

Definition 4.23. A fuzzy filter F_μ on a set X is called a fuzzy ultrafilter or a fuzzy maximal filter, if F_μ is maximal with respect to the inclusion, that is, any fuzzy filter including F_μ must be equal to F_μ.

Proposition 4.24. *Let X, Y be two sets and $f : X \to Y$ be a map.*
(1) For a fuzzy filter F_μ on X, we define
$f(F_\mu) = \{V \in F(Y) \mid f^{-1}(V) \in F_\mu\}$.
Then $f(F_\mu)$ is the smallest fuzzy filter on Y including the set of the images $\{f(U) \mid U \in F_\mu\}$.
(2) For a fuzzy filter G_μ on Y such that $f^{-1}(V) \neq 0$ for all $V \in G_\mu$, we define

$f^{-1}(G_\mu) = \{U \subseteq F(X) \mid \exists V \in G_\mu(f^{-1}(V) \subseteq U)\}$.
Then $f^{-1}(G_\mu)$ is the smallest fuzzy filter on X including the set of the inverse images $\{f^{-1}(V) \mid V \in G_\mu\}$.

Proof. (1) Let F_μ be a fuzzy filter on X. Clearly $f(F_\mu) \supseteq \{f(U) \mid U \in F_\mu\}$.

Now we prove that $f(F_\mu)$ is a fuzzy filter on Y. We claim that $0 \notin F_\mu$. If $0 \in F_\mu$, then $f^{-1}(0)(x) = 0(f(x)) = 0 \in F_\mu$, a contradiction. Similarly we can prove that $1 \in f(F_\mu)$. Moreover let $U \in f(F_\mu)$ and $U \subseteq V$. Then $f^{-1}(U) \subseteq f^{-1}(V)$ and $f^{-1}(U) \in F_\mu$. Since F_μ is a fuzzy filter, we get $f^{-1}(V) \in F_\mu$ and hence $V \in f(F_\mu)$. Lastly we let $U, V \in f(F_\mu)$. Then $f^{-1}(U) \in F_\mu$ and $f^{-1}(V) \in F_\mu$. Since F_μ is a fuzzy filter, we have $f^{-1}(U) \cap f^{-1}(V) \in F_\mu$. Note that for all $x \in X$, we have $(f^{-1}(U) \cap f^{-1}(V))(x) \leq (f^{-1}(U))(x) = U(f(x))$ and $(f^{-1}(U) \cap f^{-1}(V))(x) \leq (f^{-1}(V))(x) = V(f(x))$. Thus $(f^{-1}(U) \cap f^{-1}(V))(x) \leq U(f(x)) \wedge V(f(x)) = (U \cap V)(f(x)) = f^{-1}(U \cap V)(x)$, and hence $f^{-1}(U) \cap f^{-1}(V) \subseteq f^{-1}(U \cap V)$. Combining the above arguments, we get that $f(F_\mu)$ is a fuzzy filter on Y.

Let G_μ be a fuzzy filter and $G_\mu \supseteq \{f(U) \mid U \in F_\mu\}$. Taking $U \in f(F_\mu)$, then $f^{-1}(U) \in F_\mu$ and hence $f(f^{-1}(U)) \in \{f(V) \mid V \in F_\mu\} \subseteq G_\mu$. Note that $f(f^{-1}(U)) \subseteq U$ and thus $U \in G_\mu$. It follows that $f(F_\mu) \subseteq G_\mu$.
(2) Let G_μ be a fuzzy filter on Y such that $f^{-1}(V) \neq \emptyset$ for all $V \in G_\mu$. Let $U \in \{f^{-1}(V) \mid V \in G_\mu\}$. Then $U = f^{-1}(V)$ for some $V \in G_\mu$. By the definition of $f^{-1}(G_\mu)$, we have $U \in f^{-1}(G_\mu)$, that is, $f^{-1}(G_\mu) \supseteq \{f^{-1}(V) \mid V \in G_\mu\}$.

Next we prove $0 \notin f^{-1}(G_\mu)$. If $0 \in f^{-1}(G_\mu)$, then $f^{-1}(V) \subseteq 0$ for some $V \in G_\mu$, and hence $f^{-1}(V) = 0$. But by hypothesis, $f^{-1}(V) \neq 0$, a contradiction. Clearly $1 \in f^{-1}(G_\mu)$. Moreover for $U, V \in F(X)$, if $U \in f^{-1}(G_\mu)$ and $U \subseteq V$, then $f^{-1}(W) \subseteq U$ for some $W \in G_\mu$. By $U \subseteq V$, we have $f^{-1}(W) \subseteq V$ for some $W \in G_\mu$, and so $V \in f^{-1}(G_\mu)$. At last let $U, V \in f^{-1}(G_\mu)$. Then $f^{-1}(W_1) \subseteq U$ and $f^{-1}(W_2) \subseteq V$ for some $W_1, W_2 \in G_\mu$. Hence $f^{-1}(W_1 \cap W_2) = f^{-1}(W_1) \cap f^{-1}(W_2) \subseteq U \cap V$. It follows from $W_1 \cap W_2 \in G_\mu$ that $U \cap V \in f^{-1}(G_\mu)$. Combining the above arguments, we get that $f^{-1}(G_\mu)$ is a fuzzy filter on X.

Let F_μ be a fuzzy filter and $F_\mu \supseteq \{f^{-1}(V) \mid V \in G_\mu\}$. Let $U \in f^{-1}(G_\mu)$. Then $f^{-1}(V) \subseteq U$ for some $V \in G_\mu$. Since $F_\mu \supseteq \{f^{-1}(V) \mid V \in G_\mu\}$, $f^{-1}(V) \in F_\mu$. It follows from F_μ being a fuzzy filter that $U \in F_\mu$. That is, $f^{-1}(G_\mu) \subseteq F_\mu$. □

Proposition 4.25. *Let X, Y be two sets, and $f : X \to Y$ be a onto map. For any $U \in F(X)$ and $V, Z \in F(Y)$, we have the following equivalence:*
$U \cap f^{-1}(V) \subseteq f^{-1}(Z) \Leftrightarrow f(U) \cap V \subseteq Z$.

Proof. ($\Rightarrow$) Note that for all $x \in X$ and $y = f(x)$,
$U \cap f^{-1}(V) \subseteq f^{-1}(Z)$
$\Rightarrow U(x) \leq f^{-1}(Z)(x)$ *or* $f^{-1}(V)(x) \leq f^{-1}(Z)(x)$
$\Rightarrow U(x) \leq Z(f(x))$ *or* $V(f(x)) \leq Z(f(x))$
$\Rightarrow f(U)(y) = sup\{U(x) \mid f(x) = y\}$ *or* $V(y) \leq Z(y)$
$\Rightarrow f(U)(y) \leq sup\{Z(f(x)) \mid f(x) = y\}$ *or* $V(y) \leq Z(y), \quad (\because U(x) \leq Z(f(x)))$
$\Rightarrow f(U)(y) \leq sup\{Z(y) \mid f(x) = y\}$ *or* $V(y) \leq Z(y)$
$\Rightarrow f(U)(y) \leq Z(y)$ *or* $V(y) \leq Z(y)$
$\Rightarrow (f(U) \cap V)(y) \leq Z(y)$
$\Rightarrow f(U) \cap V \subseteq Z$.

($\Leftarrow$) For any $y \in Y$ and $y = f(x)$, we have the following:
$f(U) \cap V \subseteq Z$
$\Rightarrow f(U)(y) \leq Z(y)$ *or* $V(y) \leq Z(y)$
$\Rightarrow f(U)(f(x)) \leq Z(f(x))$ *or* $V(f(x)) \leq Z(f(x))$
$\Rightarrow f(U)(f(x)) \leq f^{-1}(Z)(x)$ *or* $f^{-1}(V)(x) \leq f^{-1}(Z)(x)$
$\Rightarrow U(x) \leq f(U)(f(x)) \leq f^{-1}(Z)(x)$ *or* $f^{-1}(V)(x) \leq f^{-1}(Z)(x)$
$\Rightarrow U(x) \leq f^{-1}(Z)(x)$ *or* $f^{-1}(V)(x) \leq f^{-1}(Z)(x)$
$\Rightarrow (U \cap f^{-1}(V))(x) \leq f^{-1}(Z)(x)$
$\Rightarrow U \cap f^{-1}(V) \subseteq f^{-1}(Z)$. □

Proposition 4.26. *Let X, Y be two sets, and $f : X \to Y$ be a map. For a fuzzy filter F_μ on X and a fuzzy filter G_μ on Y such that $f^{-1}(V) \neq 0$ for all $V \in G_\mu$, we have the following equivalence:*

$f^{-1}(G_\mu) \subseteq F_\mu \Leftrightarrow G_\mu \subseteq f(F_\mu)$.

Proof. The equivalence follows since we have:
$f^{-1}(G_\mu) \subseteq F_\mu$
$\Leftrightarrow \forall U \in F(X)(\exists V \in G_\mu)(f^{-1}(V) \subseteq U) \Rightarrow U \in F_\mu$
$\Leftrightarrow \forall V \in G_\mu(f^{-1}(V) \in F_\mu)$
$\Leftrightarrow G_\mu \subseteq f(F_\mu)$. □

This is essentially the Galois connection between the power fuzzy subsets $P(F(X))$ and $P(F(Y))$ induced by the inverse image map $f^{-1} : F(Y) \to F(X)$.

Proposition 4.27. *Let X, Y be two sets and $f : X \to Y$ be a onto map. Let F_μ and G_μ be fuzzy filters on X and Y, respectively. If $f(F_\mu) \subseteq G_\mu$, then there exists a fuzzy filter H_μ on X such that $F_\mu \subseteq H_\mu$ and $G_\mu = f(H_\mu)$.*

Proof. For any $U \in F_\mu$ we have $f(U) \in f(F_\mu) \subseteq G_\mu$. Hence for any $V \in G_\mu$, we have $f(U) \cap V \neq 0$. This implies that $U \cap f^{-1}(V) \neq 0$. Otherwise if $U \cap f^{-1}(V) = 0$, then we have the following implications: for all $x \in X$ and $y \in Y$,
$(U \cap f^{-1}(V))(x) = 0$
$\Rightarrow U(x) = 0$ *or* $f^{-1}(V)(x) = 0$
$\Rightarrow U(x) = 0$ *or* $V(f(x)) = 0$
$\Rightarrow f(U)(x) = 0$ *or* $V(y) = 0$ (by f being onto)
$\Rightarrow f(U) \cap V = 0$,
a contradiction.

Let H_μ be the smallest fuzzy filter on X including $F_\mu \cup f^{-1}(G_\mu)$, namely,

$H_\mu = \{W \subseteq F(X) \mid \exists U \in F_\mu, \exists V \in G_\mu, (U \cap f^{-1}(V) \subseteq W)\}$.

Then we have $F_\mu \subseteq H_\mu$ and $f^{-1}(G_\mu) \subseteq H_\mu$, which implies $G_\mu \subseteq f(H_\mu)$ by Lemma 4.25.

Conversely we have

$f(H_\mu) = \{Z \in F(Y) \mid \exists U \in F_\mu, \exists V \in G_\mu(U \cap F^{-1}(V) \subseteq f^{-1}(Z))\}$.

For any $U \in F_\mu$, $V \in G_\mu$, and $Z \in F(Y)$, we have the following:
$Z \in f(H_\mu)$
$\Rightarrow U \cap f^{-1}(V) \subseteq f^{-1}(Z)$
$\Rightarrow f(U) \cap V \subseteq Z$ (by Proposition 4.25)

$\Rightarrow Z \in G_\mu, \quad (\because f(F_\mu) \subseteq G_\mu)$
and hence $f(H_\mu) \subseteq G_\mu$. □

Proposition 4.28. *Let X, Y be two sets and $f : X \to Y$ be a onto map. For any fuzzy ultrafilter F_μ on X, $f(F_\mu)$ is also a fuzzy ultrafilter on Y.*

Proof. It follows from Proposition 4.27 that $f(F_\mu)$ is also a fuzzy filter on Y. If $f(F_\mu)$ is not maximal, then there is a fuzzy filter G_μ on Y such that $f(F_\mu) \subseteq G_\mu$ and $f(F_\mu) \neq G_\mu$. From Proposition 4.27, there is a fuzzy filter H_μ on X, such that $F_\mu \subseteq H_\mu$ and $G_\mu = f(H_\mu)$. Since $f(F_\mu) \neq G_\mu$, we have $F_\mu \neq H_\mu$, contradicting to the maximality of F_μ. □

Definition 4.29. A fuzzy topological space X_μ is a set equipped with a map $N : FP(X) \to P(F(X))$, which satisfies the following conditions:
(1) $N(\lambda)$ is a fuzzy filter on X for all $\lambda \in FP(X)$,
(2) For all $\lambda \in FP(X)$, λ belongs to the intersection $\cap N(\lambda)$. That is, $\lambda \in U$ for all $U \in N(\lambda)$,
(3) For all $\lambda \in FP(X)$ and $U \in N(\lambda)$, there exists an element $V \in N(\lambda)$ such that $V \subseteq U$ and $U \in N(\nu)$ for each $\nu \in V$.

An element of $N(\lambda)$ is called a neighborhood of λ.

Definition 4.30. Let X_μ be a fuzzy topological space with the neighborhood filter $N(\lambda)$ for $\lambda \in FP(X)$. A fuzzy filter F_μ on X is said to converge to an element λ of $FP(X)$, and we write $F_\mu \to \lambda$, if $N(\lambda) \subseteq F_\mu$.

Definition 4.31. A fuzzy topological space X_μ is called compact if for any filter F_μ on X, there exists a fuzzy filter G_μ on X and an element λ of $FP(X)$ such that $F_\mu \subseteq G_\mu$ and $G \to \lambda$.

By Definitions 4.30 and 4.31, we have following corollary.

Corollary 4.32. Any fuzzy ultrafilter on a compact space converges.

Definition 4.33. For an arbitrary family of sets $\{x_i\}_{i \in I}$, the product set is defined as

$$\prod_{i \in I} X_i = \{x : I \to \cup_{i \in I} X_i \mid \forall i \in I (x(i) \in X_i)\}$$
.

An element of the product set is called a choice map of $\{X_i\}_{i \in I}$.

For each index $i \in I$, the projection map pr_i from the product $X = \prod_{i \in I} X_i$ onto X_i is defined by $pr_i(x) = x(i)$ for all $x \in X$.

Definition 4.34. For an arbitrary family of fuzzy topological spaces $\{X_{\mu_i}\}_{i \in I}$, the product fuzzy topology on the product set $X = \prod_{i \in I} X_i$ is the weakest topology making the projections $\{pr_i : X \to X_{\mu_i}\}_{i \in I}$ continuous. Namely, the neighborhood filter of $\lambda \in X$ is the smallest fuzzy filter including the union:

$\cup_{i \in I}\{pr_i^{-1}(U) \mid U \in N(pr_i(\lambda))\}$, or $\cup_{i \in I} pr_i^{-1}(N(pr_i(\lambda)))$.

Lemma 4.35. *For any fuzzy filter F_μ on the product space $X = \prod_{i \in I} X_i$, and an element λ of $FP(X)$, the following two conditions are equivalent:*
(1) $F_\mu \to \lambda$;
(2) $pr_i(F_\mu) \to pr_i(\lambda)$ for all $i \in I$.

Proof. The lemma follows from Definition 4.29 and Proposition 4.26. □

In following, we prove Tychonoff theorem as a direct application of Zorns lemma. The idea is to construct an ultrafilter and an element of the product space simultaneously by taking a suitable partially ordered set.

Theorem 4.36. *For an arbitrary family of compact fuzzy topological spaces $\{X_{\mu_i}\}_{i\in I}$, any fuzzy filter on the product space $\prod_{i\in I} X_{\mu_i}$ with the product topology is included in a convergent fuzzy ultrafilter.*

Proof. Let F_μ be a fuzzy filter on the product set $\prod_{i\in I} X_{\mu_i}$. Let P be the set of pairs (G_μ, x), where G_μ is a fuzzy filter on $\prod_{i\in I} X_{\mu_i}$ including F_μ, and $x : J \to \cup_{i\in I} X_{\mu_i}$ is a map with $J \subseteq I$ satisfying $x(j) \in X_{\mu_j}$ and $pr_j(G_\mu) \to x(j)$ for all $j \in J$. If we define a fuzzy binary relation R_μ on P by:

$$R_\mu((G_\mu, x), (H_\mu, y)) = \begin{cases} 1, & G_\mu \subseteq H_\mu, x \subseteq y \\ 0, & \text{otherwise} \end{cases}$$

Then we can check that R_μ is a fuzzy order on P. Now we check that the fuzzy ordered set (P, R_μ) satisfies the assumption of fuzzy Zorns lemma. Namely, $(F_\mu, 0) \in P$, and for any non-empty fuzzy chain $C \subseteq P$, define $H_\mu = \cup_{(G_\mu,x)\in C}\{G_\mu\}$ and $y = \cup_{(G_\mu,x)\in C}\{x\}$. Then we can check that $(H_\mu, y) \in P$. Clearly (H_μ, y) is a fuzzy upper bound for C in (P, R_μ). By the fuzzy Zorn's lemma, a maximal element (G_μ, x) of P exists. Note that if G_μ is included in a fuzzy filter H_μ, then $R_\mu((G_\mu, x), (H_\mu, x))) = 1$. Since (G_μ, x) is maximal, we have $(G_\mu, x) = (H_\mu, x)$, and hence G_μ must be equal to H_μ. Thus G_μ is a fuzzy ultrafilter. If $x : J \to \cup_{i\in I} X_{\mu_i}$ with $J \neq I$, then there is an element $i \in I$ with $i \notin J$. Since $pr_i(G_\mu)$ is a fuzzy ultrafilter by Proposition 4.28, and X_{μ_i} is compact, $pr_i(G_\mu)$ converges to an element p of X_{μ_i} by Corollary 4.32. This implies that the pair $(G_\mu, x) \cup \{i, p\}$ is an element of P and is strictly bigger than (G_μ, x), contradicting its maximality. Thus $J = I$ and therefore, x is an element of the product space $\prod_{i\in I} X_{\mu_i}$. As $pr_i(G_\mu) \to pr_i(x)$ for all $i \in I$, the fuzzy ultrafilter G_μ converges to x, by Lemma 4.35. □

Theorem 4.37. *(Fuzzy Tychonoff Theorem) For an arbitrary family of compact fuzzy topological spaces $\{X_{\mu_i}\}_{i\in I}$, the product space $\prod_{i\in I} X_{\mu_i}$ with the product fuzzy topology is compact.*

Proof. Since any fuzzy filter on $\prod_{i\in I} X_{\mu_i}$ is included in a convergent fuzzy filter, the product space is compact, by Definition 4.31. □

§5 Conclusion

As well-known, there are some interesting and profound statements and results, such as, axiom of choice, Zorn's lemma, well-ordering principle and comparability principle. It is important to discuss the fuzzy versions of them. In this paper, by use of Zadeh's fuzzy order, we introduce the fuzzy axiom of choice, fuzzy Zorn's lemma and fuzzy well-ordering principle, and prove that they are equivalent. As an application of fuzzy Zorn's lemma, we prove that every proper fuzzy ideal of a ring is contained in a maximal fuzzy ideal. Moreover we give the fuzzy version of Tychonoff Theorem. By use of the fuzzy Zorn's lemma, we prove that fuzzy

Tychonoff Theorem. But as a equivalent statement to axiom of choice, it's fuzzy version have not introduced in this paper. In the future, we will consider giving the fuzzy version of the comparability principle and discuss the relations between it and other fuzzy versions. Moreover we can consider the applications of the fuzzy axiom of choice, for example, we can discuss the existence of the bases of fuzzy victor spaces by the fuzzy axiom of choice.

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