

Quantile inference for nonstationary processes with infinite variance innovations

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Abstract. Based on the quantile regression, we extend Koenker and Xiao (2004) and Ling and McAleer (2004)'s works from finite-variance innovations to infinite-variance innovations. A robust t -ratio statistic to test for unit-root and a re-sampling method to approximate the critical values of the t -ratio statistic are proposed in this paper. It is shown that the limit distribution of the statistic is a functional of stable processes and a Brownian bridge. The finite sample studies show that the proposed t -ratio test always performs significantly better than the conventional unit-root tests based on least squares procedure, such as the Augmented Dick Fuller (ADF) and Phillips-Perron (PP) test, in the sense of power and size when infinite-variance disturbances exist. Also, quantile Kolmogorov-Smirnov (QKS) statistic and quantile Cramer-von Mises (QCM) statistic are considered, but the finite sample studies show that they perform poor in power and size, respectively. An application to the Consumer Price Index for nine countries is also presented.

§1 Introduction

An extensive literature in economics and finance suggests that many economic time series are well characterized as autoregressive processes with a unit root. The augmented Dickey-Fuller (ADF) test, proposed by Dickey and Fuller (1979, 1981), the Phillips and Perron (PP) (1988) test, and the Kwiatkowski, Phillips, Schmidt and Shin (KPSS) (1992) test are the most frequently used unit root tests, and are vital tools in time series econometrics. The aforementioned test statistics were all derived under the ordinary least squares (OLS) framework. However, as Koenker and Xiao (2004) pointed out, if an innovation distribution deviates from the normal distribution, these tests exhibit poor power performance.

For a (nearly) unit-root process with finite variance innovations, many robust approaches have been proposed to avoid imposing the restrictive normality assumption. For example, Cox

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and Llatas (1991) developed a test based on M -estimate for an AR(1) process with a (near) unit root; Lucas (1995) considered a unit root test based on a nonparametric modification of M -estimate, focusing on the Huber and Student t procedures; Breitung and Gourieroux (1997) and Hasan and Koenker (1997) proposed rank-type tests within the ADF framework; Koenker and Xiao (2004) proposed a t -ratio test at selected quantiles, a Kolmogorov-Smirnov test and a Cramer-von Mises test in a quantile autoregressive framework; Galvao (2009) extended the tests of Koenker and Xiao (2004) by including stationary covariates in the quantile autoregression.

However, large number empirical studies in macroeconomics and finance, indicate that time series with heavy tails provide better models for such data. For example, Fama (1965) and Mandelbrot (1963, 1967) argued that distributions of commodity and stock returns are often heavy-tailed with possible infinite variance, Rachev and Mittnik (2000) considered stable parietian models in finance, Lux and Marchesi (2000) studied agent-based models and Charemza, Hristova and Burrige (2005) studied the inflation data with heavy tails. More background information on heavy-tailed time series and their applications can be found in Finkenstädt and Rootzén (2003).

Due to its important applications in real data, unit-root tests for cases with infinite variance innovations have attracted more and more attentions in recent years. For example, Chan and Tran (1989) and Rachev, Mittnik and Kim (1998) considered the limit distribution for LSE-based tests, and Knight (1989) developed M and least absolute deviation (LAD) estimate based tests for case with i.i.d. heavy-tailed noise, while Knight (1991) extended these results to the case with infinite-order moving average dynamics; Phillips (1990) generalized the Phillips and Perron's test (1988) to the context of processes driven by weakly dependent shocks whose innovations display infinite variance; Samarakoon and Knight (2009) consider a M -estimate based test for finite-order autoregressive processes driven by infinite variance innovations; Chan and Zhang (2009) study the least squares estimate of the autoregressive coefficient of a nearly nonstationary autoregressive model with strong dependent and infinite variance innovations; Chan and Zhang (2010) consider the quantile estimate and the semi-parametric estimate of the autoregressive parameters with long- and short-range dependent innovations.

It is well known that when the noise is dependent, a more powerful unit-root test is based on the differenced process. A typical test is the augmented Dickey and Fuller (ADF) test. In this paper, the first contribution is that we extend Koenker and Xiao (2004) and Ling and McAleer (2004)'s works from finite-variance innovations to infinite-variance innovations by adopting this idea. More precisely, in this paper, we consider the following model

$$\Delta y_t = \delta y_{t-1} + \sum_{i=1}^p \phi_i \Delta y_{t-i} + u_t, \quad t = p+1, \dots, n, \quad (1.1)$$

where $\Delta y_t = y_t - y_{t-1}$, $\{u_t\}$ is an i.i.d. sequence lying in the domain of attraction of a stable law with tail index $\alpha \in (0, 2)$, i.e., there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$\frac{1}{a_n} \sum_{i=1}^n u_i - b_n \xrightarrow{d} Z_\alpha, \quad (1.2)$$

where \xrightarrow{d} denotes convergence in distribution, and Z_α is a stable variable with tail index α .

For simplicity, we also write such $\{u_t\}$ as $u_t \in \mathcal{N}\mathcal{D}(\alpha)$. By (1.1), the unit-root hypothesis is equivalent to test whether $\delta = 0$. The second contribution is that three statistics to test for unit-root and a re-sampling method to approximate the critical values of the t -ratio statistic are proposed. The first statistic is a t -ratio test using a given quantile level, and the other two tests are the quantile Kolmogorov-Smirnov (QKS) and quantile Cramer-von Mises (QCM) tests, which use all the quantiles information in some interval. It is shown that these tests converge to functionals of stable process and a Brownian bridge and perform better than that based on LSE procedure, especially the t -ratio statistics.

The paper is organized as follows. In Section 2, we introduce the model and estimation. In Section 3, the asymptotic results are given for the quantile autoregression parameters. The test statistics and the large sample distributions are presented in Section 4. The finite sample properties using a Monte Carlo simulation are showed in Section 5. A real data example is given in Section 6. The proofs are given in Section 7. Throughout the paper, the symbol “ \xrightarrow{d} ” denotes convergence in distribution, “ \implies ” denotes weak convergence on M-topology.

§2 Parameters Estimate

2.1 Quantile Regression

In this subsection, we consider the estimate for the parameter $(\delta, \phi_1, \dots, \phi_p)'$ in model (1.1). Let $Q_u(\tau)$ be the τ -th quantile of u_t and \mathcal{F}_t be the σ -field generated by $\{u_s, s \leq t\}$. The τ -th conditional quantile of Δy_t with respect to \mathcal{F}_{t-1} is given by

$$Q_{\Delta y_t}(\tau|\mathcal{F}_{t-1}) = Q_u(\tau) + \delta y_{t-1} + \sum_{i=1}^p \phi_i \Delta y_{t-i}. \quad (2.1)$$

Let $\phi(\tau) = (Q_u(\tau), \delta, \phi_1, \dots, \phi_p)'$ and $x_t = (1, y_{t-1}, \Delta y_{t-1}, \dots, \Delta y_{t-p})'$. Then

$$Q_{\Delta y_t}(\tau|\mathcal{F}_{t-1}) = x_t' \phi(\tau), \quad (2.2)$$

and the estimator $\hat{\phi}(\tau)$ for the $\phi(\tau)$ can be obtained as follows:

$$\hat{\phi}(\tau) = \arg \min_{\phi \in \mathbb{R}^{p+2}} \sum_{t=1}^n \rho_\tau(\Delta y_t - x_t' \phi), \quad (2.3)$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$. The above (2.2) is a linear conditional quantile function, and the solution to (2.3) can be obtained using usual linear programming.

2.2 Asymptotic properties

Let models (1.1) and (1.2) hold. Under the null hypothesis that y_t is a unit-root process (i.e. $\delta = 0$), the convergence rates of the components in $\hat{\phi}(\tau)$ are different. Specifically, $\hat{Q}_u(\tau)$ converges at rate \sqrt{n} , $\hat{\delta}$ converges at rate $\sqrt{n}a_n$, while other components converge at rate a_n , where $a_n = \inf\{x : P(u_t > x) < 1/n\}$ is given as in (1.2). Therefore, we introduce the standardization matrix $D_n = \text{diag}(\sqrt{n}, \sqrt{n}a_n, a_n, \dots, a_n)$, and denote $\hat{v} = D_n(\hat{\phi}(\tau) - \phi(\tau))$.

Then,

$$\Delta y_t - x_t' \widehat{\phi}(\tau) = \Delta y_t - x_t' \phi(\tau) - x_t' (\widehat{\phi}(\tau) - \phi(\tau)) \equiv u_{t\tau} - (D_n^{-1} \widehat{v})' x_t,$$

where $u_{t\tau} = \Delta y_t - x_t' \phi(\tau)$ and minimizing the right-hand side of (2.3) is equivalent to

$$\min_v \sum_{t=1}^n [\rho_\tau(u_{t\tau} - (D_n^{-1} v)' x_t) - \rho_\tau(u_{t\tau})] =: \min_v \Omega_n(v).$$

If \widehat{v} minimizes $\Omega_n(v)$, then $\widehat{v} = D_n(\widehat{\phi}(\tau) - \phi(\tau))$ and the original function in (2.3) is represented as a convex objective function of $\Omega_n(v)$. As a result, we can use the convexity lemma in Knight (1989, 1991) and Pollard (1991) to show the asymptotic properties of the function. By the convexity lemma, if the finite-dimensional distributions of $\Omega_n(v)$ converge weakly to those of $\Omega(v)$, and $\Omega(v)$ has a unique minimum, then the convexity of $\Omega_n(v)$ implies that \widehat{v} converges in distribution to the minimizer of $\Omega(v)$. By Knight's identity (Knight, 1989, 1998), $\Omega_n(v)$ can be divided into two parts as follows:

$$\Omega_n(v) = - \sum_{t=1}^n v' D_n^{-1} x_t \psi_\tau(u_{t\tau}) + \sum_{t=1}^n \int_0^{v' D_n^{-1} x_t} [I(u_{t\tau} \leq s) - I(u_{t\tau} \leq 0)] ds, \tag{2.4}$$

where $\psi_\tau(u) = \tau - I(u \leq 0)$. To show the weak convergence of $\Omega_n(v)$, we need to show the convergence of each part. Once $\Omega_n(v)$ is shown to converge weakly to $\Omega(v)$, then $\widehat{v} = D_n(\widehat{\phi}(\tau) - \phi(\tau))$ will converge in distribution to the minimizer of $\Omega(v)$.

To show the asymptotic properties, we impose the following assumptions.

Assumption 1. $\{u_t\}$ is an i.i.d. symmetric sequence belonging to $\mathcal{ND}(\alpha)$ with $\alpha \in (0, 2)$.

Assumption 2. u_t has a positive continuous density $f(u)$ on \mathbb{R} .

Assumption 3. $|\sum_{i=1}^p \phi_i z^i - 1| \neq 0$ if $|z| < 1$.

Let $B_\tau(r) = B(\tau, r)$ be a two-parameter rescaled Brownian bridge [see Goodman (1976)], that is, (a) for given r , $B(\tau, r)$ is a Brownian bridge with respect to $\tau \in [0, 1]$. Hence, the two-parameter process, $B(\tau, r)$, is partially a Brownian motion and partially a Brownian bridge; (b) for given τ , $(\tau(1 - \tau))^{-1/2} B(\tau, r)$ is a Brownian motion. Under the assumption that y_t is a unit-root process, we have the following asymptotic results.

Theorem 1. Let y_t be determined by (1.1) with $\delta = 0$. Under Assumption 1-3, we have

$$D_n^{-1} \sum_{t=1}^n x_t \psi_\tau(u_{t\tau}) \Rightarrow \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha(r) dB_\tau(r) \\ \Phi_\tau \end{bmatrix},$$

$$\sum_{t=1}^n \int_0^{v' D_n^{-1} x_t} [I(u_{t\tau} \leq s) - I(u_{t\tau} \leq 0)] ds \Rightarrow \frac{1}{2} f(F^{-1}(\tau)) v' \Gamma v,$$

where $\bar{\eta}_\alpha(r) = [1, (1 - \sum_{i=1}^p \phi_i)^{-1} \eta_\alpha(r)]'$, $\eta_\alpha(r)$ is a stable process with index α and $\Phi_\tau = [\eta_\alpha(1), \dots, \eta_\alpha(1)]'$ is a p -dimensional stable process vector. In addition,

$$\Gamma \equiv \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha(r) \bar{\eta}_\alpha'(r) dr & \mathbf{0}_{2 \times p} \\ \mathbf{0}_{p \times 2} & \Psi \end{bmatrix},$$

where $\Psi_{ij} = \text{diag}(\eta_{\alpha/2}, \dots, \eta_{\alpha/2})$ and $\eta_{\alpha/2}$ is a stable variable with index $\alpha/2$.

Corollary 1. *Under the Assumption of Theorem 1, we have*

(i) for the τ -quantile $\phi(\tau)$,

$$D_n(\widehat{\phi}(\tau) - \phi(\tau)) \Rightarrow \frac{1}{f(F^{-1}(\tau))} \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha(r) \bar{\eta}'_\alpha(r) dr & \mathbf{0}_{2 \times p} \\ \mathbf{0}_{p \times 2} & \Psi \end{bmatrix}^{-1} \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha(r) dB_\tau(r) \\ \Phi_\tau \end{bmatrix}.$$

(ii) for the unit-root parameter δ ,

$$\sqrt{na_n} \widehat{\delta} \Rightarrow \frac{1}{f(F^{-1}(\tau))} \left[\int_0^1 \underline{\eta}_\alpha^2(r) dr \right]^{-1} \int_0^1 \underline{\eta}_\alpha(r) dB_\tau(r), \tag{2.5}$$

where $\underline{\eta}_\alpha(r) = (1 - \sum_{i=1}^p \phi_i)^{-1} (\eta_\alpha(r) - \int_0^1 \eta_\alpha(r) dr)$.

§3 Quantile Inference for Unit-root

In this section, we consider the unit root test with the null hypothesis $H_0 : \delta = 0$, and the alternative hypothesis $H_1 : \delta < 0$. To this end, we propose three statistics based on t -ratio, quantile Kolmogorov-Smirnov (QKS) and quantile Cramer-von Mises (QCM).

3.1 t -ratio Statistic

Inference based on the quantile regression provides a more robust approach to testing the unit root hypothesis. Based on the asymptotic distribution of $\widehat{\delta}$, we can construct a t -ratio test statistic as follows:

$$t_n(\tau) = (\tau(1 - \tau))^{-1/2} f(\widehat{F^{-1}(\tau)}) (Y'_{-1} P_X Y_{-1})^{1/2} \widehat{\delta},$$

where $f(\widehat{F^{-1}(\tau)})$ is a consistent estimator of $f(F^{-1}(\tau))$. Here, Y_{-1} is the vector of lagged dependent variables (y_{t-1}) , and P_X is the projection matrix onto the space orthogonal to $X_t = (1, \Delta y_{t-1}, \dots, \Delta y_{t-p})'$. In particular,

$$Y'_{-1} P_X Y_{-1} = \sum_{t=1}^n y_{t-1}^2 - \sum_{t=1}^n y_{t-1} X'_t \left(\sum_{t=1}^n X_t X'_t \right)^{-1} \sum_{t=1}^n X_t y_{t-1}. \tag{3.1}$$

By Lemma 1 below, it holds that $\sum_{t=1}^n y_{t-1}^2 = O(na_n^2)$, the elements of $\sum_{t=1}^n y_{t-1} X'_t (\sum_{t=1}^n X_t X'_t)^{-1} \sum_{t=1}^n X_t y_{t-1}$ are $O(na_n^2)$ or $O(a_n^2)$, so $Y'_{-1} P_X Y_{-1} = O(na_n^2)$.

Theorem 2. *Under the null hypothesis, and using the result for the asymptotic distribution of $\widehat{\delta}$, we have*

$$t_n(\tau) \Rightarrow \left[\int_0^1 \underline{\eta}_\alpha^2(t) dt \right]^{-1/2} \int_0^1 \underline{\eta}_\alpha(t) dB(t),$$

where $B(t)$ is a standard Brownian motion.

We consider the following simulation procedure generating the asymptotic critical values of $t_n(\tau)$ similarly to Koenker and Xiao (2004) and Li and Park (2018):

- 1 Let $w_t = \Delta y_t$, $t = 2, \dots, n$, then fit w_t by the following q th order autoregression

$$w_t = \sum_{j=1}^q \beta_j w_{t-j} + u_t, \quad t = q + 1, \dots, n,$$

estimate β_1, \dots, β_q by quantile regression (QR), and obtain the residuals \hat{u}_t .

- 2 Draw i.i.d variables $\{u_t^*\}_{t=q+1}^n$ by Bootstrap method from the centered residuals $\hat{u}_t - \frac{1}{n-q} \sum_{j=q+1}^n \hat{u}_j$ and generate w_t^* from u_t^* using the fitted autoregression

$$w_t^* = \sum_{j=1}^q \hat{\beta}_j w_{t-j}^* + \hat{u}_t^*, \quad t = q+1, \dots, n,$$

with $w_j^* = \Delta y_j$ for $j = 1, \dots, q$.

- 3 Generate y_t^* under the null restriction of a unit root, $y_t^* = y_{t-1}^* + w_t^*$, with $y_1^* = y_1$.
4 Generate random normal vector e_t for $t = 1, \dots, n$.

- 5 Approximate $t(\tau)$ by

$$\hat{t}(\tau) \equiv \left[\sum_t (y_t^* - \bar{y}^*)^2 \right]^{-1/2} \left[\sum_t (y_t^* - \bar{y}^*) e_t \right],$$

where $\bar{y}^* = n^{-1} \sum_t y_t^*$.

- 6 Repeat steps 2 to 5 many times.

Let $CV(\tau, \theta)$ be the 100θ quantiles (i.e., $P(\hat{t}(\tau) \leq CV(\tau, \theta)) = \theta$). Then, the unit root hypothesis will be rejected at the $(1 - \theta)$ level if $t_n(\tau) \leq CV(\tau, \theta)$. To obtain the asymptotic critical values, we let the sample size, n , be 1000. The Monte Carlo simulation is repeated 10,000 times. In order to obtain the feasible t -ratio test, we should obtain consistent estimators for $f(F^{-1}(\tau))$. Noting that $dF^{-1}/dt = (f(F^{-1}(t)))^{-1}$, so $f(F^{-1}(\tau))$ can be estimated consistently by

$$f(\widehat{F^{-1}(\tau)}) = \frac{2h_n}{\widehat{F}_n^{-1}(\tau + h_n) - \widehat{F}_n^{-1}(\tau - h_n)},$$

where h_n is the bandwidth and \widehat{F}_n^{-1} is the estimator for $F^{-1}(s)$.

In order to select the bandwidth, Koenker and Xiao (2006) suggest two choices. One is the bandwidth rule suggested by Hall and Sheather (1988):

$$h_{HS} = n^{-1/3} z_\alpha^{2/3} \left[\frac{1.5\phi^2(\Phi^{-1}(\tau))}{2(\Phi^{-1}(\tau))^2 + 1} \right]^{1/3},$$

where z_α satisfies $\Phi(z_\alpha) = 1 - \alpha/2$ for the α -level test. The other is Bofinger's (1975) rule:

$$h_B = n^{-1/5} \left[\frac{4.5\phi^4(\Phi^{-1}(t))}{(2\Phi^{-1}(t))^2 + 1} \right]^{1/5},$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and cumulative distribution function of the standard normal distribution, respectively.

3.2 QKS and QCM

In addition to the t -ratio statistic $t_n(\tau)$, which uses only a given quantile τ , just as the ADF coefficient test Z_δ , we may also use the coefficient-based statistic in the QAR model. Define the coefficient-based statistic

$$U_n(\tau) = \sqrt{na_n} \hat{\delta}.$$

By Corollary 1, under the unit-root hypothesis and Assumptions 1-3,

$$U_n(\tau) \Rightarrow U(\tau) = \frac{1}{f(F^{-1}(\tau))} \left[\int_0^1 \eta_\alpha^2 \right]^{-1} \int_0^1 \eta_\alpha dB_\tau,$$

Let $\tau \in \mathcal{T} = [\tau_0, 1 - \tau_0]$, for small $\tau_0 > 0$. We define the QKS and QCM tests similarly to Koenker and Xiao (2004) and Galvao (2009) as follows:

$$QKS \equiv \sup_{\tau \in \mathcal{T}} |U_n(\tau)| \quad \text{and} \quad QCM \equiv \int_{\tau \in \mathcal{T}} |U_n(\tau)|^2 d\tau.$$

Theorem 3. Suppose that $\delta = 0$ and Assumptions 1-3 hold, then

$$QKS \xrightarrow{d} \sup_{\tau \in \mathcal{T}} |U(\tau)|, \quad \text{and} \quad QCM \xrightarrow{d} \int_{\tau \in \mathcal{T}} |U(\tau)|^2 d\tau.$$

In practice, we may calculate $U_n(\tau)$ at $\{\tau_i = i/n\}_{i=1}^n$, and thus the statistics QKS can be constructed by taking maximum over $\tau_i \in \mathcal{T}$ and QCM are obtained using numerical integration. The critical values of the QKS and QCM tests can be obtained by resample method, which was used above to calculating critical value of $t_n(\tau)$.

§4 Finite sample evaluation

To study the finite sample properties of the proposed tests, we consider the following design, which is the leading case studied in the literature:

$$y_t = \phi y_{t-1} + u_t,$$

where $\{u_t\}$ are i.i.d random variables. We examined different values of ϕ and different types of error distributions in the experiment. In particular, we consider four values of $\phi \in \{1.0, 0.95, 0.90, 0.85\}$, and $u_t \sim S(\alpha, 0, 1, 0)$, with $\alpha \in \{0.4, 0.8, 1.0, 1.2, 1.6, 2\}$. Also, $u_t \sim t(2)$ is included.

For the following tests: (1) the classical ADF coefficient-based test; (2) the Phillips-Perron semiparametric Z_ϕ test; (3) Breitung's variance ratio test (\mathcal{VRT}); (4) \mathcal{M} unit root test (\mathcal{MZ}_ϕ); (5) the KS-type test (QKS) based on QAR with $\mathcal{T} = [0.1, 0.9]$; (6) the CM-type test (QCM) with $\mathcal{T} = [0.1, 0.9]$; (7) the t-ratio test $t_n(\tau)$ based on QAR at $\tau = 0.5$, the empirical size and the empirical power are given in Table 1-4.

Table 1 reports the size performance of the tests. First, the ADF and PP tests show severe size distortion for all sample sizes and all distributions. Second, the \mathcal{VRT} and \mathcal{MZ}_ϕ tests show good size performance, similar to that of the $t_n(0.5)$ test. In addition, when the sample size is $n = 50$, the QKS test has severe size distortion, while, it has good size performance when $n = 200$. Finally, the size performance of the QKS test is better than that of the QCM test. Moreover, as the sample size increases, the size performances of the QKS and QCM tests improve.

Table 2-4 give the empirical power of the tests. First, the ADF and PP tests have low power for different ϕ . second, the power performance of the \mathcal{VRT} test is poor but improves with the sample size increasing. In addition, the \mathcal{MZ}_ϕ , QCM and $t_n(0.5)$ tests show good performance. However, the power of QKS test is poor, the possible reason is that when τ

tends to 0 or 1, the sample used to calculate $U_n(\tau)$ could be quite small, which may lead to huge error in calculating the critical values of QKS .

Table 1. Size of the tests ($\phi = 1$).

	<i>ADF</i>	<i>PP</i>	<i>VRT</i>	MZ_ϕ	<i>QKS</i>	<i>QCM</i>	$t_n(0.5)$
$n = 50$							
$t(2)$	0.200	0.189	0.043	0.037	0.066	0.232	0.046
$S(0.4, 0, 1, 0)$	0.169	0.352	0.036	0.056	0.137	0.209	0.025
$S(0.8, 0, 1, 0)$	0.270	0.241	0.039	0.043	0.108	0.225	0.020
$S(1, 0, 1, 0)$	0.270	0.253	0.057	0.042	0.083	0.193	0.033
$S(1.2, 0, 1, 0)$	0.229	0.257	0.038	0.052	0.058	0.184	0.029
$S(1.6, 0, 1, 0)$	0.229	0.267	0.034	0.053	0.074	0.251	0.044
$S(2, 0, 1, 0)$	0.185	0.193	0.042	0.034	0.148	0.476	0.055
$n = 100$							
$t(2)$	0.210	0.223	0.045	0.042	0.030	0.173	0.034
$S(0.4, 0, 1, 0)$	0.328	0.328	0.060	0.061	0.089	0.121	0.032
$S(0.8, 0, 1, 0)$	0.243	0.253	0.036	0.044	0.053	0.117	0.038
$S(1, 0, 1, 0)$	0.255	0.292	0.046	0.062	0.037	0.106	0.045
$S(1.2, 0, 1, 0)$	0.218	0.259	0.039	0.052	0.032	0.115	0.030
$S(1.6, 0, 1, 0)$	0.248	0.224	0.034	0.041	0.039	0.182	0.025
$S(2, 0, 1, 0)$	0.211	0.218	0.042	0.046	0.080	0.341	0.045
$n = 200$							
$t(2)$	0.185	0.230	0.053	0.063	0.036	0.145	0.041
$S(0.4, 0, 1, 0)$	0.324	0.307	0.057	0.054	0.023	0.040	0.030
$S(0.8, 0, 1, 0)$	0.251	0.261	0.060	0.049	0.036	0.066	0.037
$S(1, 0, 1, 0)$	0.253	0.247	0.051	0.046	0.022	0.089	0.047
$S(1.2, 0, 1, 0)$	0.225	0.283	0.043	0.051	0.012	0.072	0.036
$S(1.6, 0, 1, 0)$	0.252	0.239	0.054	0.053	0.033	0.139	0.034
$S(2, 0, 1, 0)$	0.188	0.215	0.038	0.039	0.032	0.233	0.053

To gain a further understanding of these phenomena, the empirical density functions of $t_n(0.5)$ under the null hypothesis H_0 and the alternative H_1 with $t(2)$ noises are plotted in figure 1(a). The black line is the approximated distribution of the limiting distribution $t(0.5)$. The red line is the empirical density of $t_n(0.5)$ under null hypothesis H_0 , and the rest are the empirical density of $t_n(0.5)$ for different ϕ . It shows that the smaller of ϕ , the bigger of the empirical power. Similarly, the empirical density functions of QKS and QCM are plotted in figure 1(b) and figure 2. In figure 2(a), empirical density of QCM under H_0 and H_1 are plotted, and the asymptotic distribution of $\int_{\tau \in \mathcal{T}} |U(\tau)|^2 d\tau$ is also plotted in figure 2(b). Figure 1(b) shows that QKS has a good size performance, but performs poor in power. Figure 2 shows that QCM has severe size distortion, but performs well in power.

§5 Real example

In this section, we study the unit root properties of the monthly Consumer Price Index (CPI) for several countries. The series vary in length from 241 to 723 observations, and cover various

Table 2. Power of the tests ($\phi = 0.95$).

	<i>ADF</i>	<i>PP</i>	<i>VRT</i>	MZ_ϕ	<i>QKS</i>	<i>QCM</i>	$t_n(0.5)$
$n = 50$							
$t(2)$	0.195	0.192	0.076	0.091	0.039	0.162	0.309
$S(0.4, 0, 1, 0)$	0.272	0.295	0.053	0.137	0.252	0.652	0.982
$S(0.8, 0, 1, 0)$	0.187	0.202	0.048	0.109	0.067	0.374	0.870
$S(1, 0, 1, 0)$	0.222	0.226	0.069	0.155	0.036	0.264	0.666
$S(1.2, 0, 1, 0)$	0.184	0.230	0.068	0.113	0.027	0.180	0.503
$S(1.6, 0, 1, 0)$	0.229	0.222	0.070	0.106	0.047	0.166	0.277
$S(2, 0, 1, 0)$	0.220	0.252	0.060	0.110	0.120	0.316	0.251
$n = 100$							
$t(2)$	0.201	0.207	0.113	0.263	0.029	0.284	0.640
$S(0.4, 0, 1, 0)$	0.152	0.202	0.036	0.802	0.540	0.932	0.995
$S(0.8, 0, 1, 0)$	0.164	0.193	0.061	0.284	0.145	0.802	0.995
$S(1, 0, 1, 0)$	0.171	0.236	0.102	0.366	0.062	0.644	0.951
$S(1.2, 0, 1, 0)$	0.177	0.254	0.088	0.359	0.036	0.444	0.837
$S(1.6, 0, 1, 0)$	0.227	0.226	0.109	0.317	0.041	0.274	0.535
$S(2, 0, 1, 0)$	0.227	0.247	0.100	0.296	0.063	0.290	0.452
$n = 200$							
$t(2)$	0.196	0.202	0.242	0.725	0.098	0.750	0.904
$S(0.4, 0, 1, 0)$	0.131	0.162	0.076	0.988	0.904	0.999	1
$S(0.8, 0, 1, 0)$	0.174	0.188	0.111	0.958	0.491	0.996	1
$S(1, 0, 1, 0)$	0.168	0.203	0.156	0.935	0.296	0.986	1
$S(1.2, 0, 1, 0)$	0.180	0.198	0.155	0.902	0.163	0.938	0.991
$S(1.6, 0, 1, 0)$	0.224	0.251	0.232	0.842	0.141	0.724	0.841
$S(2, 0, 1, 0)$	0.231	0.259	0.234	0.751	0.045	0.542	0.723

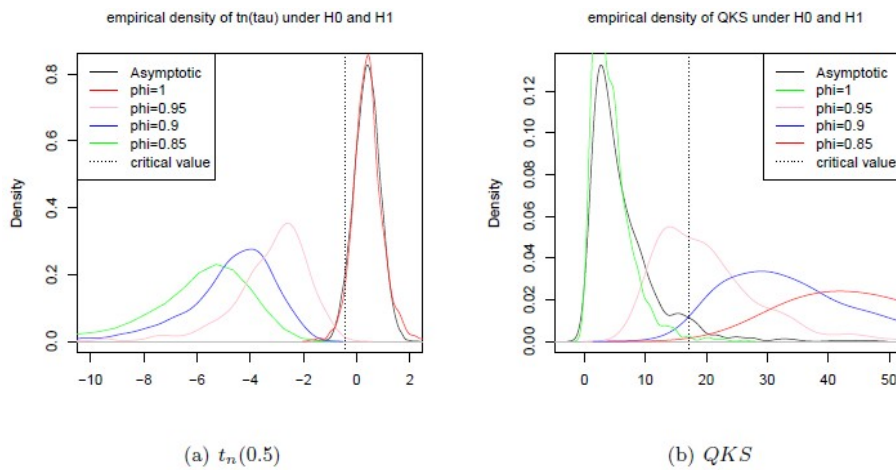


Figure 1. Empirical density functions of $t_n(\tau)$ and QKS under the null hypothesis H_0 and the alternative H_1 .

Table 3. Power of the tests ($\phi = 0.9$).

	<i>ADF</i>	<i>PP</i>	<i>VRT</i>	MZ_ϕ	<i>QKS</i>	<i>QCM</i>	$t_n(0.5)$
$n = 50$							
$t(2)$	0.182	0.169	0.146	0.245	0.041	0.303	0.583
$S(0.4, 0, 1, 0)$	0.165	0.204	0.041	0.772	0.339	0.772	0.979
$S(0.8, 0, 1, 0)$	0.151	0.161	0.065	0.248	0.095	0.615	0.961
$S(1, 0, 1, 0)$	0.167	0.201	0.098	0.290	0.050	0.467	0.865
$S(1.2, 0, 1, 0)$	0.184	0.189	0.095	0.265	0.022	0.359	0.729
$S(1.6, 0, 1, 0)$	0.208	0.198	0.101	0.249	0.036	0.284	0.507
$S(2, 0, 1, 0)$	0.213	0.226	0.123	0.275	0.105	0.445	0.482
$n = 100$							
$t(2)$	0.178	0.217	0.231	0.769	0.096	0.758	0.897
$S(0.4, 0, 1, 0)$	0.156	0.185	0.063	0.980	0.688	0.986	0.997
$S(0.8, 0, 1, 0)$	0.143	0.164	0.097	0.928	0.342	0.956	0.999
$S(1, 0, 1, 0)$	0.158	0.192	0.159	0.918	0.177	0.928	0.993
$S(1.2, 0, 1, 0)$	0.181	0.234	0.178	0.885	0.118	0.852	0.962
$S(1.6, 0, 1, 0)$	0.236	0.204	0.194	0.793	0.127	0.699	0.774
$S(2, 0, 1, 0)$	0.211	0.253	0.217	0.721	0.191	0.716	0.725
$n = 200$							
$t(2)$	0.250	0.238	0.396	1	0.429	0.990	0.992
$S(0.4, 0, 1, 0)$	0.125	0.164	0.099	0.992	0.969	1	1
$S(0.8, 0, 1, 0)$	0.157	0.197	0.183	0.996	0.755	1	1
$S(1, 0, 1, 0)$	0.170	0.225	0.281	0.999	0.605	1	1
$S(1.2, 0, 1, 0)$	0.208	0.226	0.284	0.997	0.469	0.997	1
$S(1.6, 0, 1, 0)$	0.239	0.270	0.360	0.998	0.509	0.983	0.965
$S(2, 0, 1, 0)$	0.260	0.270	0.413	0.999	0.651	0.928	0.911

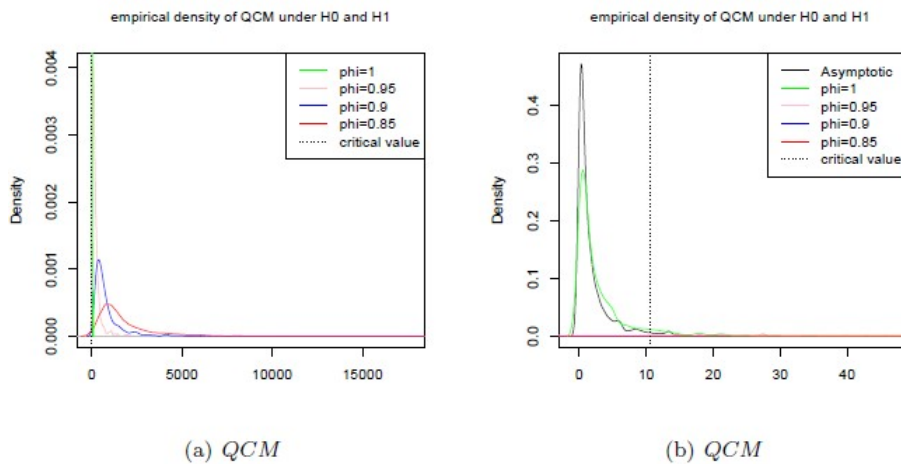


Figure 2. Empirical density functions of QCM under the null hypothesis H_0 and the alternative H_1 .

Table 4. Power of the tests ($\phi = 0.85$).

	<i>ADF</i>	<i>PP</i>	<i>VRT</i>	<i>MZ_φ</i>	<i>QKS</i>	<i>QCM</i>	<i>t_n(0.5)</i>
<i>n</i> = 50							
<i>t</i> (2)	0.198	0.153	0.176	0.408	0.050	0.553	0.764
<i>S</i> (0.4, 0, 1, 0)	0.133	0.169	0.064	0.948	0.410	0.814	0.989
<i>S</i> (0.8, 0, 1, 0)	0.163	0.143	0.076	0.710	0.146	0.756	0.983
<i>S</i> (1, 0, 1, 0)	0.146	0.181	0.116	0.693	0.085	0.674	0.940
<i>S</i> (1.2, 0, 1, 0)	0.152	0.193	0.108	0.641	0.059	0.597	0.866
<i>S</i> (1.6, 0, 1, 0)	0.205	0.161	0.154	0.471	0.076	0.515	0.641
<i>S</i> (2, 0, 1, 0)	0.193	0.212	0.195	0.471	0.181	0.687	0.653
<i>n</i> = 100							
<i>t</i> (2)	0.216	0.213	0.337	0.960	0.232	0.938	0.963
<i>S</i> (0.4, 0, 1, 0)	0.124	0.145	0.063	0.989	0.757	0.992	1
<i>S</i> (0.8, 0, 1, 0)	0.149	0.142	0.140	0.986	0.462	0.990	1
<i>S</i> (1, 0, 1, 0)	0.170	0.204	0.224	0.987	0.314	0.979	0.999
<i>S</i> (1.2, 0, 1, 0)	0.163	0.222	0.230	0.991	0.226	0.960	0.989
<i>S</i> (1.6, 0, 1, 0)	0.198	0.216	0.310	0.9763	0.298	0.925	0.926
<i>S</i> (2, 0, 1, 0)	0.210	0.246	0.330	0.967	0.573	0.905	0.873
<i>n</i> = 200							
<i>t</i> (2)	0.248	0.236	0.510	1	0.768	1	0.999
<i>S</i> (0.4, 0, 1, 0)	0.124	0.140	0.105	0.995	0.973	1	1
<i>S</i> (0.8, 0, 1, 0)	0.157	0.148	0.287	0.994	0.888	1	1
<i>S</i> (1, 0, 1, 0)	0.174	0.236	0.399	0.998	0.810	1	1
<i>S</i> (1.2, 0, 1, 0)	0.181	0.256	0.421	0.997	0.717	1	1
<i>S</i> (1.6, 0, 1, 0)	0.249	0.264	0.505	0.999	0.814	0.999	0.988
<i>S</i> (2, 0, 1, 0)	0.243	0.280	0.557	1	0.956	0.990	0.974

time periods between January 1947 and January 2020. The time series and their ACF are showed in Figure 3. The distribution of CPI (defined as the first difference of the logarithm of prices) are often skewed and leptocurtic, differing markedly in shape from the normal, therefore, an alternative assumption about the behaviour of the innovations is that they are draws from a symmetric stable Paretian distribution. Table 5 gives the test results for a unit root using $t_n(0.5)$, QKS and QCM.

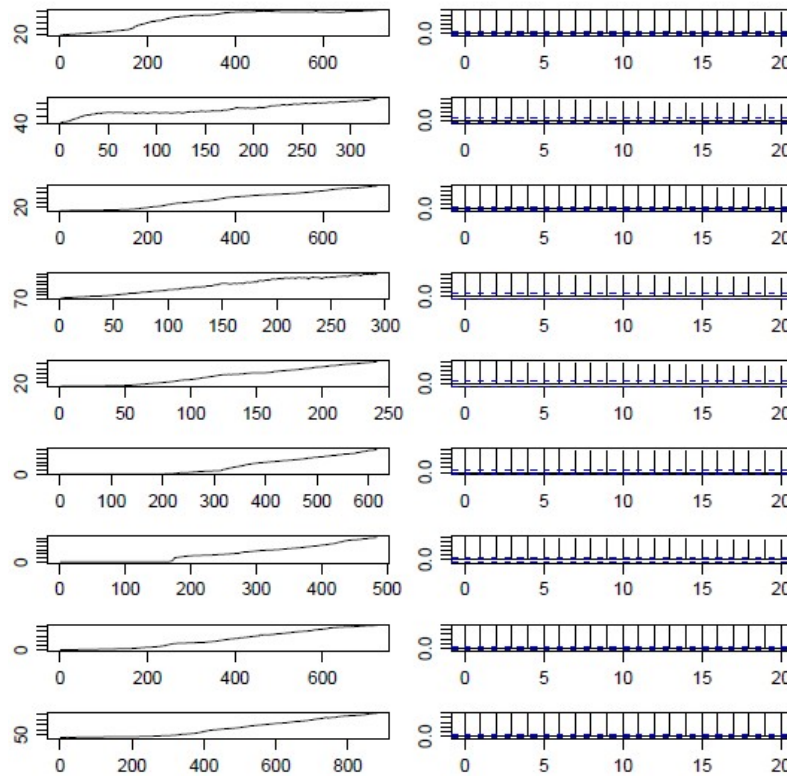


Figure 3. Time series plots of the monthly CPI and their ACF.

The columns of 3, 5 and 7 are the values of test statistics $t_n(0.5)$, QKS and QCM, and the column of 2, 4 and 6 are the critical values at the 5% significance level, except for the row corresponding to Australia, which are the critical values at the 10% significance level. Table 5 shows the CPI for Japan, China, UK, Euro, Mexico, Brazil, Korea and U.S. are unit root processes for different test statistics. These critical values are calculated under the unit root null using the resampling procedure described in Section 4.1. While, for the CPI of Australia, the unit root hypothesis is marginally rejected by the KS-type and CM-type tests at the 10% level, but not rejected by $t_n(0.5)$, while the unit root hypothesis for the CPI of Australia shouldn't be rejected from the Figure 3. This also indicates that the t_n test performs better in testing unit-root.

Table 5. Unit root tests for CPI.

	$t_n(0.5)$	QKS		QCM		
Japan	-8.030	0.418	0.313	0.128	0.029	0.001
China	-4.294	0.304	0.150	0.097	0.010	0.001
UK	-6.504	0.387	0.150	0.082	0.011	0.001
Euro	-3.560	0.397	0.153	0.071	0.016	0.001
Australia	-5.728	0.460	0.072	0.151	0.001	0.002
Mexico	-16.371	-0.119	0.205	0.167	0.007	0.001
Brazil	-27.263	-0.535	0.699	0.171	0.060	0.002
Korea	-8.618	0.061	0.877	0.266	0.178	0.004
U.S.	-9.223	0.206	0.081	0.043	0.002	0.0002

§6 Proofs

Before the proof of Theorem 1, we first give a weak convergence result need to be used.

Lemma 1. *Under the assumptions of Theorem 1, we have*

- (i) $\left[\frac{1}{a_n} \sum_{t=1}^n \Delta y_{t-1} \psi_\tau(u_{t\tau}), \dots, \frac{1}{a_n} \sum_{t=1}^n \Delta y_{t-p} \psi_\tau(u_{t\tau}) \right]' \xrightarrow{d} \Phi_\tau,$
- (ii) $\left(\frac{y_s}{a_n}, \frac{1}{\sqrt{n}} \sum_{t=1}^{[ns]} \psi_\tau(u_{t\tau}) \right) \implies (\eta_\alpha(s), B(\tau, s)),$ on $[0, 1] \times [0, 1],$
- (iii) $\left(\frac{1}{a_n^2} \sum_{t=1}^n \Delta y_{t-i}^2, \frac{1}{a_n^2} \sum_{t=1}^n \Delta y_{t-i} \Delta y_{t-j} \right) \xrightarrow{d} (\eta_{\alpha/2}, 0),$ for $1 \leq i \neq j \leq p,$

where $\eta_{\alpha/2}$ is a stable variable with tail index $\alpha/2$. Further,

$$\left(\frac{1}{a_n} \sum_{t=1}^n \Delta y_{t-1} \psi_\tau(u_{t\tau}), \dots, \frac{1}{a_n} \sum_{t=1}^n \Delta y_{t-p} \psi_\tau(u_{t\tau}), \frac{y_s}{a_n}, \frac{1}{\sqrt{n}} \sum_{t=1}^{[ns]} \psi_\tau(u_{t\tau}), \frac{1}{a_n^2} \sum_{t=1}^n \Delta y_{t-i}^2 \right)$$

converge jointly to their limit processes.

Proof. (ii) can be shown similar to Lemma A.1. of Chan and Zhang (2009). (iii) and the joint convergence follows from Lemma 3.3 of Chan and Zhang (2010). We will only give the proof of (i) in the following.

Note that $\psi_\tau(u_{t\tau})$ are independent with $\Delta y_{t-i}, (i = 1, \dots, p),$ and $nP(|\Delta y_t| > a_n) \sim 1,$ then

$$nP(|\Delta y_{t-i} \psi_\tau(u_{t\tau})| > a_n) \sim E|\psi_\tau(u_{t\tau})|^\alpha. \tag{6.1}$$

And following Resnick (1986), it holds

$$\sum_{j=1}^n \delta_{a_n^{-1} \Delta y_{t-i} \psi_\tau(u_{t\tau})} \xrightarrow{d} N, \tag{6.2}$$

where N is a Poisson random process.

When $0 < \alpha < 1,$ by Theorem 3.1 of Davis and Hsing (1995), (6.1) and (6.2), we obtain

$$\frac{1}{a_n} \sum_{t=1}^n \Delta y_{t-i} \psi_\tau(u_{t\tau}) \xrightarrow{d} \eta_\alpha(1). \tag{6.3}$$

Next, we consider the case $\alpha \geq 1.$ Denote $S_n = \frac{1}{a_n} \sum_{t=1}^n \Delta y_{t-i} \psi_\tau(u_{t\tau}),$ where $\Delta y_t =$

$\sum_{j=0}^{\infty} c_j u_{t-j}$. And $S_n^{(M)} = \frac{1}{a_n} \sum_{t=1}^n \Delta y_{t-i, M} \psi_{\tau}(u_{t\tau})$, where $\Delta y_{t, M} = \sum_{j=0}^M c_j u_{t-j}$. Then

$$\begin{aligned} |S_n - S_n^{(M)}| &= a_n^{-1} \left| \sum_{t=1}^n \sum_{j=M+1}^{\infty} c_j u_{t-1-j} \psi_{\tau}(u_{t\tau}) \right| \\ &\leq a_n^{-1} \left| \sum_{t=1}^n \sum_{j=M+1}^{\infty} c_j \psi_{\tau}(u_{t\tau}) [u_{t-1-j} I(u_{t-1-j} \leq a_n) - E u_1 I(u_1 \leq a_n)] \right| \\ &\quad + a_n^{-1} \left| \sum_{t=1}^n \sum_{j=M+1}^{\infty} c_j \psi_{\tau}(u_{t\tau}) u_{t-1-j} I(u_{t-1-j} > a_n) \right| \\ &\quad + a_n^{-1} \left| \sum_{t=1}^n \sum_{j=M+1}^{\infty} c_j \psi_{\tau}(u_{t\tau}) E u_1 I(u_1 \leq a_n) \right| \\ &=: V_1 + V_2 + V_3. \end{aligned}$$

Let $b = \sup_{t, \tau} |\psi_{\tau}(u_{t\tau})|$. By Karamata's theorem, we have

$$E|V_2| \leq b a_n^{-1} n \sum_{j=M+1}^{\infty} |c_j| E|u_{t-1-j} I(u_{t-1-j} > a_n)| \rightarrow \frac{b\alpha}{\alpha - 1} \sum_{j=M+1}^{\infty} |c_j| \tag{6.4}$$

and

$$|V_3| \leq b a_n^{-1} n \sum_{j=M+1}^{\infty} |c_j| E|u_{t-1-j} I(u_{t-1-j} > a_n)| \rightarrow \frac{b\alpha}{\alpha - 1} \sum_{j=M+1}^{\infty} |c_j|. \tag{6.5}$$

Also,

$$\begin{aligned} E|V_1|^2 &= a_n^{-2} \sum_{t=1}^n \sum_{j=M+1}^{\infty} c_j^2 E[\psi_{\tau}^2(u_{t\tau})] Var(u_1 I(u_1 \leq a_n)) \\ &\leq \tau(1 - \tau) n a_n^{-2} (E u_1^2 I(u_1 \leq a_n)) \left(\sum_{j=M+1}^{\infty} |c_j|^2 \right) \\ &\rightarrow \frac{\tau(1 - \tau)\alpha}{2 - \alpha} \left(\sum_{j=M+1}^{\infty} |c_j|^2 \right). \end{aligned} \tag{6.6}$$

By Markov's inequality and (6.4)-(6.6), we then have for any $\epsilon > 0$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} P[|S_n - S_n^{(M)}| > \epsilon] & \\ &\leq \epsilon^{-1} [(E|V_1|^2)^{1/2} + E|V_2| + |V_3|] \\ &\leq \frac{2b\alpha}{\alpha - 1} \sum_{j=M+1}^{\infty} |c_j| + \left(\frac{\tau(1 - \tau)\alpha}{2 - \alpha} \sum_{j=M+1}^{\infty} |c_j|^2 \right)^{1/2} \\ &\rightarrow 0, \quad \text{as } M \rightarrow \infty. \end{aligned} \tag{6.8}$$

Note that

$$S_n^{(M)} = \sum_{j=0}^M c_j \left(a_n^{-1} \sum_{t=1}^n u_{t-1-j} \psi_{\tau}(u_{t\tau}) \right) \xrightarrow{d} S^{(M)},$$

where $S^{(M)}$ is a stable process with index α . Together with (6.8), it holds that

$$\frac{1}{a_n} \sum_{t=1}^n \Delta y_{t-i} \psi_{\tau}(u_{t\tau}) \xrightarrow{d} \eta_{\alpha}(1),$$

where $\eta_\alpha(1) = \lim_{M \rightarrow \infty} S^{(M)}$. □

The proof of Theorem 1. By Lemma 1, it follows that

$$\begin{aligned}
 D_n^{-1} \sum_{t=1}^n x_t \psi_\tau(u_{t\tau}) &= \begin{bmatrix} \frac{1}{\sqrt{n}} \sum_t \psi_\tau(u_{t\tau}) \\ \frac{1}{\sqrt{na_n}} \sum_t y_{t-1} \psi_\tau(u_{t\tau}) \\ \frac{1}{a_n} \sum_t \Delta y_{t-1} \psi_\tau(u_{t\tau}) \\ \vdots \\ \frac{1}{a_n} \sum_t \Delta y_{t-p} \psi_\tau(u_{t\tau}) \end{bmatrix} \\
 &\Rightarrow \begin{bmatrix} \int_0^1 dB_\tau \\ \int_0^1 (1 - \sum_{i=1}^p \phi_i)^{-1} \eta_\alpha dB_\tau \\ \eta_\alpha(1) \\ \vdots \\ \eta_\alpha(1) \end{bmatrix} \\
 &= \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha dB_\tau \\ \Phi_\tau \end{bmatrix} \equiv \Phi^*.
 \end{aligned}$$

We now consider the limit of

$$\begin{aligned}
 U_n(v) &\equiv \sum_{t=1}^n \int_0^{v' D_n^{-1} x_t} [I(u_{t\tau} \leq s) - I(u_{t\tau} \leq 0)] ds \\
 &= \sum_{t=1}^n (u_{t\tau} - v' D_n^{-1} x_t) I(v' D_n^{-1} x_t < u_{t\tau} < 0) \\
 &\quad + \sum_{t=1}^n (v' D_n^{-1} x_t - u_{t\tau}) I(0 < u_{t\tau} < v' D_n^{-1} x_t) \\
 &\equiv U_{1n}(v) + U_{2n}(v).
 \end{aligned}$$

First, we consider the limit of

$$U_{1n} = \sum_{t=1}^n \xi_t(v),$$

where $\xi_t(v) = (u_{t\tau} - v' D_n^{-1} x_t) I(v' D_n^{-1} x_t < u_{t\tau} < 0)$. To avoid technical problems in taking conditional expectations, following Knight (1989), we consider truncation of $v_2 a_n^{-1} y_{t-1}$ at some finite number $m < 0$ and define

$$U_{1nm}(v) = \sum_{t=1}^n \xi_{tm}(v),$$

where $\xi_{tm}(v) = (u_{t\tau} - v' D_n^{-1} x_t) I(v' D_n^{-1} x_t < u_{t\tau} < 0) M_t$, and $M_t = I(m < v_2 a_n^{-1} y_{t-1} < 0)$.

We further define

$$\bar{\xi}_{tm}(v) = E[\xi_{tm}(v) | \mathcal{F}_{t-1}] \quad \text{and} \quad \bar{U}_{1nm}(v) = \sum_{t=1}^n \bar{\xi}_{tm}(v).$$

Then $\{\xi_{tm}(v) - \bar{\xi}_{tm}(v)\}$ is a martingale difference sequence.

Noting that

$$\bar{U}_{1nm}(v) = \sum_{t=1}^n E[(u_{t\tau} - v' D_n^{-1} x_t) I(v' D_n^{-1} x_t < u_{t\tau} < 0) M_t | \mathcal{F}_{t-1}]$$

$$\begin{aligned}
 &= \sum_{t=1}^n \int_{[v' D_n^{-1} x_t + F^{-1}(\tau)]M_t}^{F^{-1}(\tau)} [r - [v' D_n^{-1} x_t + F^{-1}(\tau)]M_t] f(r) dr \\
 &= \sum_{t=1}^n \int_{[v' D_n^{-1} x_t + F^{-1}(\tau)]M_t}^{F^{-1}(\tau)} \left[\int_{[v' D_n^{-1} x_t + F^{-1}(\tau)]M_t}^r ds \right] f(r) dr \\
 &= \sum_{t=1}^n \int_{[v' D_n^{-1} x_t + F^{-1}(\tau)]M_t}^{F^{-1}(\tau)} \left[\int_s^{F^{-1}(\tau)} f(r) dr \right] ds \\
 &= \sum_{t=1}^n \int_{[v' D_n^{-1} x_t + F^{-1}(\tau)]M_t}^{F^{-1}(\tau)} [F^{-1}(\tau) - s] \cdot \frac{F(F^{-1}(\tau)) - F(s)}{F^{-1}(\tau) - s} ds.
 \end{aligned}$$

Under Assumption 2,

$$\frac{F(F^{-1}(\tau)) - F(s)}{F^{-1}(\tau) - s} = f(F^{-1}(\tau)) + o_p(1).$$

It follows that

$$\begin{aligned}
 \bar{U}_{1nm}(v) &= \sum_{t=1}^n \int_{[v' D_n^{-1} x_t + F^{-1}(\tau)]M_t}^{F^{-1}(\tau)} [F^{-1}(\tau) - s] f(F^{-1}(\tau)) ds + o_p(1) \\
 &= \sum_{t=1}^n f(F^{-1}(\tau)) \left[\frac{(F^{-1}(\tau) - s)^2}{2} \Big|_{[v' D_n^{-1} x_t + F^{-1}(\tau)]M_t}^{F^{-1}(\tau)} \right] + o_p(1) \\
 &= \frac{1}{2} \sum_{t=1}^n f(F^{-1}(\tau)) v' [D_n^{-1} x_t x_t' D_n^{-1}] v M_t + o_p(1).
 \end{aligned}$$

Thus,

$$\bar{U}_{1nm}(v) \Rightarrow \frac{1}{2} f(F^{-1}(\tau)) v' \Gamma_{1m} v,$$

where

$$\Gamma_{1m} = \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha \bar{\eta}'_\alpha M_t & \mathbf{0}_{2 \times p} \\ \mathbf{0}_{p \times 2} & \Psi \end{bmatrix}.$$

In addition, $(v' D_n^{-1} x_t) I(m < v_2 a_n^{-1} y_{t-1} < 0) \xrightarrow{p} 0$ uniformly in t , we have

$$\sum_{t=1}^n E[\xi_{tm}(v)^2 | \mathcal{F}_{t-1}] \leq \max(v' D_n^{-1} x_t) I(m < v_2 a_n^{-1} y_{t-1} < 0) \sum_{t=1}^n \bar{\xi}_{tm}(v) \xrightarrow{p} 0.$$

Thus the summation of martingale difference sequence

$$\sum_{t=1}^n [\xi_{tm}(v) - \bar{\xi}_{tm}(v)] \xrightarrow{p} 0.$$

Therefore, the limiting distribution of $\sum_{t=1}^n \xi_{tm}(v)$ is the same as that of $\sum_{t=1}^n \bar{\xi}_{tm}(v)$, that is,

$$U_{1nm} \Rightarrow \frac{1}{2} f(F^{-1}(\tau)) v' \Gamma_{1m} v.$$

For any small number $\epsilon > 0$,

$$\begin{aligned}
 &P[|U_{1n}(v) - U_{1nm}(v)| > \epsilon] \\
 &< P[|U_{1n}(v) - U_{1nm}(v)| > 0] \\
 &= P\left[\sum_{t=1}^n (u_{t\tau} - v' D_n^{-1} x_t) I(v' D_n^{-1} x_t < u_{t\tau} < 0) I(v_2 a_n^{-1} y_{t-1} \leq m) > 0\right] \\
 &\leq P\left[\bigcup_{t=1}^n \{v_2 a_n^{-1} y_{t-1} \leq m\}\right] = P\left(\min_{1 \leq t \leq n} v_2 a_n^{-1} y_{t-1} \leq m\right),
 \end{aligned}$$

which implies that

$$\lim_{m \rightarrow -\infty} \limsup_{n \rightarrow \infty} P[|U_{1n}(v) - U_{1nm}(v)| > \epsilon] \leq \lim_{m \rightarrow -\infty} P(\inf_{0 \leq r \leq 1} v_2(1 - \sum_{i=1}^p \phi_i)^{-1} \eta_\alpha(r) \leq m) = 0.$$

Therefore, as $m \rightarrow -\infty$,

$$U_{1n}(v) \Rightarrow \frac{1}{2} f(F^{-1}(\tau)) v' \Gamma_1 v I(v_2(1 - \sum_{i=1}^p \phi_i)^{-1} \eta_\alpha(r) < 0),$$

where

$$\Gamma_1 = \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha \bar{\eta}'_\alpha I(v_2(1 - \sum_{i=1}^p \phi_i)^{-1} \eta_\alpha(r) < 0) & \mathbf{0}_{2 \times p} \\ \mathbf{0}_{p \times 2} & \Psi \end{bmatrix}.$$

Similarly, we can show

$$U_{2n}(v) \Rightarrow \frac{1}{2} f(F^{-1}(\tau)) v' \Gamma_2 v I(v_2(1 - \sum_{i=1}^p \phi_i)^{-1} \eta_\alpha(r) \geq 0),$$

where

$$\Gamma_2 = \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha \bar{\eta}'_\alpha I(v_2(1 - \sum_{i=1}^p \phi_i)^{-1} \eta_\alpha(r) \geq 0) & \mathbf{0}_{2 \times p} \\ \mathbf{0}_{p \times 2} & \Psi \end{bmatrix}.$$

As a result,

$$U_n(v) \Rightarrow \frac{1}{2} f(F^{-1}(\tau)) v' \Gamma v,$$

where $\Gamma = \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha \bar{\eta}'_\alpha & \mathbf{0}_{2 \times p} \\ \mathbf{0}_{p \times 2} & \Psi \end{bmatrix}$. □

The proof of Corollary 1. By Theorem 1 and (2.4), it holds that

$$\Omega_n(v) \Rightarrow -v' \Phi^* + \frac{1}{2} f(F^{-1}(\tau)) v' \Gamma v := \Omega(v).$$

By the convexity Lemma of Pollard (1991) and arguments of Knight (1989), note that $\Omega_n(v)$ and $\Omega(v)$ are minimized at $\hat{v} = D_n(\hat{\phi}(\tau) - \phi(\tau))$ and

$$\frac{1}{f(F^{-1}(\tau))} \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha \bar{\eta}'_\alpha & \mathbf{0}_{2 \times p} \\ \mathbf{0}_{p \times 2} & \Psi \end{bmatrix}^{-1} \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha dB_\psi^\tau \\ \Phi_\tau \end{bmatrix}.$$

Furthermore,

$$D_n(\hat{\phi}(\tau) - \phi(\tau)) \Rightarrow \frac{1}{f(F^{-1}(\tau))} \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha \bar{\eta}'_\alpha & \mathbf{0}_{2 \times p} \\ \mathbf{0}_{p \times 2} & \Psi \end{bmatrix}^{-1} \begin{bmatrix} \int_0^1 \bar{\eta}_\alpha dB_\psi^\tau \\ \Phi_\tau \end{bmatrix},$$

which also implies (2.5). □

Theorems 2-3 follows easily from Theorem 1, Lemma 1 and the continuous mapping theorem. We omit the details here.

References

[1] E Bofinger. *Estimation of a density function using order statistics*, Australian Journal of Statistics, 1975, 17: 1-7.
 [2] J Breitung, C Gouriéroux . *Rank tests for unit roots*, Econometrics, 1997, 81(1): 7-27.
 [3] N H Chan, L T Tran. *On the first-order autoregressive process with infinite variance*, Econometric Theory, 1989, 5(3): 354-362.

- [4] N H Chan, R M Zhang. *Quantile inference for near-integrated autoregressive time series under infinite variance and strong dependence*, Stochastic Processes and their Applications, 2009, 119(12): 4124-4148.
- [5] N H Chan, R M Zhang. *Inference for nearly nonstationary processes under strong dependence with infinite variance*, Statistica Sinica, 2010, 19: 925-947.
- [6] W W Chatemza, D Hristova, P Burridge. *Is inflation stationary?*, Applied Economics, 2005, 37(8): 901-903.
- [7] D D Cox, I Llatas. *Maximum likelihood type estimation for nearly nonstationary autoregressive time series*, The Annals of Statistics, 1991, 19(3): 1109-1129.
- [8] R A Davis, T Hsing. *Point process and partial sum convergence for weakly dependent random variables with infinite variance*, The Annals of Probability, 1995, 23(2): 879-917.
- [9] D A Dickey, W A Fuller. *Distribution of the estimators for autoregressive time series with a unit root*, Journal of the American Statistical Association, 1979, 74(366a): 427-431.
- [10] D A Dickey, W A Fuller. *Likelihood ratio statistics for autoregressive time series with a unit root*, Econometrica, 1981, 49(4): 1057-1072.
- [11] B Falk, C H Wang. *Testing long-run PPP with infinite-variance returns*, Journal of Applied Econometrics, 2003, 18(4): 471-484.
- [12] E F Fama. *The Behavior of Stock-Market Prices*, Journal of Business, 1965, 38(1): 34-105.
- [13] B Finkenstädt, H Rootzén. *Extreme Values in Finance, Telecommunications, and the Environment*, New York: Chapman and Hall/CRC, 2003.
- [14] A F Galvao. *Unit root quantile autoregression testing using covariates*, Journal of Econometrics, 2009, 152(2): 165-178.
- [15] I Georgiev, P M M Rodrigues, A M R Taylor. *Unit Root Tests and Heavy-Tailed Innovations*, Journal of Time Series Analysis, 2017, 38(5): 733-768.
- [16] V Goodman. *Distribution Estimates for Functionals of the Two-Parameter Wiener Process*, The Annals of Probability, 1976, 4(6): 977-982.
- [17] P Hall, S J Sheather. *On the Distribution of a Studentized Quantile*, Journal of the Royal Statistical Society, 1988, 50(3): 381-391.
- [18] M N Hasan, R W Koenker. *Robust rank tests of the unit root hypothesis*, Econometrica, 1997, 65(1): 133-161.
- [19] K Knight. *Limit theory for autoregressive-parameter estimates in an infinite-variance random walk*, The Canadian Journal of Statistics, 1989, 17: 261-278.
- [20] K Knight. *Limit theory for M-estimates in an integrated infinite variance processes*, Econometric Theory, 1991, 7(2): 200-212.
- [21] K Knight. *Limiting distributions for L_1 regression estimators under general conditions*, The Annals of Statistics, 1998, 26(2): 755-770.
- [22] R Koenker, Z J Xiao. *Unit root quantile autoregression inference*, Journal of the American Statistical Association, 2004, 99(464): 775-787.
- [23] D Kwiatkowski, P C B Phillips, P Schmidt, Y Shin. *Testing for the null hypothesis of stationary against the alternative of a unit root: How sure are we that economic time series have a unit root*, Journal of Econometrics, 1992, 54(1-3): 159-178.
- [24] H Q Li, S Y Park. *Testing for a unit root in a nonlinear quantile autoregression framework*, Econometric Reviews, 2018, 37(8): 867-892.
- [25] S Q Ling, M McAleer. *Regression quantiles for unstable autoregressive models*, Journal of Multivariate Analysis, 2004, 89(2): 304-328.

- [26] A Lucas. *Unit root tests based on M estimators*, *Econometric Theory*, 1995, 11(2): 331-346.
- [27] B B Mandelbrot. *The variation of certain speculative prices*, *Journal of Business*, 1963, 36(4): 394-419.
- [28] B B Mandelbrot. *How long is the coast of Britain? Statistical self similarity and fractional dimension*, *Science*, 1967, 156(3775): 636-638.
- [29] P C B Phillips, P Perron. *Testing for a unit root in time series regression*, *Biometrika Trust*, 1988, 75(2): 335-346.
- [30] P C B Phillips. *Time series regression with a unit root and infinite-variance errors*, *Econometric Theory*, 1990, 6(1): 44-62.
- [31] D Pollard. *Asymptotics for least absolute deviation regression estimators*, *Econometric Theory*, 1991, 7(2): 186-199.
- [32] S Rachev, S Mittnik. *Stable Paretian Models in Finance*, *Quantitative Finance*, 2000.
- [33] S T Rachev, S Mittnik, J R Kim. *Time series with unit roots and infinite-variance disturbances*, *Applied Mathematics Letters*, 1998, 11(5): 69-74.
- [34] S I Resnick. *Point Processes, Regular Variation and Weak Convergence*, *Advances in Applied Probability*, 1986, 18(1): 66-138.
- [35] D M M Samarakoon, K Knight. *A note on unit root tests with infinite variance noise*, *Econometric Reviews*, 2009, 28(4): 314-334.

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