Fractional sum and fractional difference on non-uniform lattices and analogue of Euler and Cauchy Beta formulas

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Abstract. As is well known, the definitions of fractional sum and fractional difference of \( f(z) \) on non-uniform lattices \( x(z) = c_1z^2 + c_2z + c_3 \) or \( x(z) = c_3q^z + c_2q^{-z} + c_3 \) are more difficult and complicated. In this article, for the first time we propose the definitions of the fractional sum and fractional difference on non-uniform lattices by two different ways. The analogue of Euler’s Beta formula, Cauchy’s Beta formula on non-uniform lattices are established, and some fundamental theorems of fractional calculus, the solution of the generalized Abel equation on non-uniform lattices are obtained etc.

§1 Introduction

The definitions of non-uniform lattices date back to the approximation of the following differential equation of hypergeometric type:

\[
\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0,
\]

(1)

where \( \sigma(z) \) and \( \tau(z) \) are polynomials of degrees at most two and one, respectively, and \( \lambda \) is a constant. Its solutions are some types of special functions of mathematical physics, such as the classical orthogonal polynomials, the hypergeometric and cylindrical functions, see G. E. Andrews, R. Askey, R. Roy [5, 6]. A. F. Nikiforov, V. B. Uvarov and S. K. Suslov [22, 23] generalized Eq. (1) to a difference equation of hypergeometric type case and studied the Nikiforov-Uvarov-Suslov difference equation on a lattice \( x(s) \) with variable step size \( \Delta x(s) = x(s) - x(s-1) \) as

\[
\bar{\sigma}[x(s)] \frac{\Delta}{\Delta x(s-1/2)} \left[ \frac{\nabla y(s)}{\nabla x(s)} \right] + \frac{1}{2} \bar{\tau}[x(s)] \left[ \frac{\Delta y(s)}{\Delta x(s)} + \frac{\nabla y(s)}{\nabla x(s)} \right] + \lambda y(s) = 0,
\]

(2)

where \( \bar{\sigma}(x) \) and \( \bar{\tau}(x) \) are polynomials of degrees at most two and one in \( x(s) \), respectively, \( \lambda \) is a constant, \( \Delta y(s) = y(s+1) - y(s) \), \( \nabla y(s) = y(s) - y(s-1) \), and \( x(s) \) is a lattice function that

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satisfies
\[
\frac{x(s+1)+x(s)}{2} = \alpha x(s+1) + \beta, \quad \alpha, \beta \text{ are constants,}
\]

\[
x^2(s+1) + x^2(s) \text{ is a polynomial of degree at most two w.r.t. } x(s+1/2).
\]

It should be pointed out that the difference equation (2) obtained as a result of approximating
the differential equation (1) on a non-uniform lattice is of independent importance and arises
in a number of other questions. Its solutions essentially generalized the solutions of the original
differential equation and are of interest in their own right [13–15, 21–25]. As it is known in
( [22], P59), the general solutions \(x(s)\) which satisfy the conditions in Eqs. (3) and (4) are
\[
x(s) = \hat{c}_1 s^2 + \hat{c}_2 s + \hat{c}_3;
\]
or
\[
x(s) = c_1 q^s + c_2 q^{-s} + c_3, q \neq 1
\]

**Definition 1.** ([22, 23]) Two kinds of lattice functions \(x(s)\) are called non-uniform lattices
which have the form (5) and (6), where \(c_i, \hat{c}_i\) are arbitrary constants and \(c_1 c_2 \neq 0, \hat{c}_1 \hat{c}_2 \neq 0\).
When \(c_1 = 1, c_2 = c_3 = 0\), or \(\hat{c}_2 = 1, \hat{c}_1 = \hat{c}_3 = 0\, these two kinds of lattice functions \(x(s)\)
\[
x(s) = s
\]
\[
x(s) = q^s,
\]
are called uniform lattices.

Let \(x(s)\) be a non-uniform lattice, where \(s \in \mathbb{C}\). For any real \(\gamma, x_\gamma(s) = x(s + \frac{\gamma}{2})\) is also
a non-uniform lattice. Given a function \(F(s)\), define the difference operator with respect to \(x_\gamma(s)\)
as
\[
\nabla_\gamma F(s) = \frac{\nabla F(s)}{\nabla x_\gamma(s)},
\]
and
\[
\nabla^k_\gamma F(z) = \frac{\nabla}{\nabla x_\gamma(z)} \frac{\nabla}{\nabla x_{\gamma+1}(z)} \cdots \frac{\nabla}{\nabla x_{\gamma+k-1}(z)} F(z) \quad (k = 1, 2, ...)
\]

Although the discrete fractional calculus on uniform lattice (7) and (8) are more current, but
great development has been made in this field [1–3, 8–12, 17, 18, 20]. In the recent monographs,
J. F. Cheng [10], C. Goodrich and A. Peterson [19] provided the comprehensive treatment of
the discrete fractional calculus with up-to-date references, and the developments in the theory
of fractional \(q\)-calculus had been well reported by M. H. Annaby and Z. S. Mansour [4].

But we should mention that, in the case of nonuniform lattices (5) or (6), even when \(n \in \mathbb{N}\),
the formula of \(n\)-order difference on non-uniform lattices is a remarkable job, since it is very
complicated and difficult to be obtained. In fact, in [22], A. Nikiforov, V. Uvarov, S. Suslov
obtained the formula of \(n - \text{th} \) difference \(\nabla_1^{(n)} f(s)\) as follows:

**Definition 2.** ([22]) Let \(n \in \mathbb{N}^+\), for nonuniform lattices (5) or (6), then
\[
\nabla_1^{(n)} [f(s)] = \sum_{k=0}^{n} \frac{(-n)_q k}{[k]_q !} \Gamma(2s - k + n + 1 + c) / q \Gamma(2s - k + n + 1 + c) / q f(s - k) \nabla x_{n+1}(s - k),
\]
where \(\Gamma(s)_q\) is modified \(q\)–gamma function which is defined as
\[
\Gamma(s)_q = q^{-(s-1)(s-2)/4} \Gamma(1)_q(s),
\]
and function $\Gamma_q(s)$ is called the $q$-gamma function; it is a generalization of Euler’s gamma function $\Gamma(s)$. It is defined by

\[
\Gamma_q(s) = \begin{cases} 
\frac{B_n(a(1-q^{k+1}))}{(1-q)^k B_n(1-q^{k+s})}, & \text{when } |q| < 1; \\
q^{-(s-1)(s-2)/2}\Gamma_{1/q}(s), & \text{when } |q| > 1.
\end{cases}
\]

and

\[
\varphi_n = \frac{q^{\frac{a}{2} - q^{\frac{a}{2}}}}{q^{\frac{a}{2} - q^{\frac{a}{2}}}}; \quad \mu, \text{ if } x(s) = c_1 s^2 + c_2 s + c_3,
\]

where

\[
c = \begin{cases} 
\frac{\log c_1}{\log q}, & \text{if } x(s) = c_1 s^2 + c_2 s + c_3; \\
\log c_1, & \text{if } x(s) = c_1 s^2 + c_2 s + c_3.
\end{cases}
\]

Now there exist two important and challenging problems that need to be further discussed:

(1) Assume that $g(s)$ be a given function, $f(s)$ be an unknown function, which satisfies the following generalized difference equation on non-uniform lattices

\[
\nabla_1^{(n)} [f(s)] = g(s).
\]

How to solve generalized difference equation (11)?

(2) The definitions of $-\alpha$-order fractional difference and $-\alpha$-order fractional sum on non-uniform lattices are very difficult and interesting problems. They have not appeared since the monographs [22, 23] were published. Can we give reasonable definitions of fractional sum and difference on non-uniform lattices?

We believe that as the most general discrete fractional calculus on non-uniform lattices, they should have an independent meaning and lead to many interesting new theories about them, which may be an important extension and development of the discrete fractional calculus.

The purpose of this paper is to inquire into the feasibility of establishing discrete fractional calculus on nonuniform lattices. In this article, for the first time we propose the definitions of the fractional sum and fractional difference on non-uniform lattices. In order to keep this paper to a reasonable length, we have chosen to restrict ourselves to some fundamental theorems of discrete fractional calculus, such as the analogue of Euler Beta formula, Cauchy Beta formula on non-uniform lattices, and the solution of the generalized Abel equation on non-uniform lattices etc. The other important results such as Taylor formula and Leibnize formula on non-uniform lattices will be given in a future. The results we obtain here are essentially new and have not been found in other literature.

§2 Integer Sum and Fractional Sum on Non-uniform Lattices

Let $x(s)$ be a non-uniform lattice, where $s \in \mathbb{C}$. Let $\nabla_\gamma F(s) = f(s)$, then

\[
F(s) - F(s - 1) = f(s) [x_\gamma(s) - x_\gamma(s - 1)].
\]

Choose $z, a \in \mathbb{C}$, and $z - a \in \mathbb{N}$. Summing from $s = a + 1$ to $z$, we have

\[
F(z) - F(a) = \sum_{s=a+1}^{z} f(s) \nabla x_\gamma(s).
\]
Thus, we define
\[
\int_{a+1}^{z} f(s) \, d\nabla x_{\gamma}(s) = \sum_{s=a+1}^{z} f(s) \, d\nabla x_{\gamma}(s).
\]
It is easy to verify that

**Proposition 3.** Given two functions \( F(z), f(z) \) with complex variable \( z, a \in C \), and \( z - a \in N \), we have

1. \( \nabla_{\gamma} \left[ \int_{a+1}^{z} f(s) \, d\nabla x_{\gamma}(s) \right] = f(z) \),
2. \( \int_{a+1}^{z} \nabla_{\gamma} F(s) \, d\nabla x_{\gamma}(s) = F(z) - F(a) \).

A generalized power \([x(s) - x(z)]^{(\alpha)}\) on nonuniform lattice is given by
\[
[x(s) - x(z)]^{(\alpha)} = \prod_{k=0}^{n-1} [x(s) - x(z - k)], \quad (n \in N^{+}),
\]
and a more formal definition and further properties of the generalized powers \([x_\nu(s) - x_\nu(z)]^{(\alpha)}\) on nonuniform lattice are very important, which are defined as follows:

**Definition 4.** (See [7, 25]) Let \( \alpha \in C \), the generalized powers \([x_\nu(s) - x_\nu(z)]^{(\alpha)}\) are defined by

\[
[x_\nu(s) - x_\nu(z)]^{(\alpha)} = \begin{cases} 
\frac{\Gamma(s-z+\alpha)}{\Gamma(s-z)} c_1 q^{\alpha} \left( \frac{\Gamma(s-z+\alpha)\Gamma(s+z+\nu+\alpha+1)}{\Gamma(s-z)\Gamma(s+z+\nu+\alpha+1)} \right), & \text{if } x(s) = c_1 s^2 + \tilde{c}_2 s + \tilde{c}_3, \\
(q-1)^{\alpha} \frac{\Gamma(s-z+\alpha)}{\Gamma(q(s-z))}, & \text{if } x(s) = q^s, \\
[c_1 (1-q)^2]^{\alpha} q^{-\alpha(s+z)} \frac{\Gamma(s+z+\nu+\alpha+1)}{\Gamma(q(s-z))}, & \text{if } x(s) = c_1 q^s + c_2 q^{-s} + c_3.
\end{cases}
\]

**Proposition 5.** [7, 25]. For \( x(s) = c_1 q^s + c_2 q^{-s} + c_3 \) or \( x(s) = \tilde{c}_1 s^2 + \tilde{c}_2 s + \tilde{c}_3 \), the generalized power \([x_\nu(s) - x_\nu(z)]^{(\alpha)}\) satisfy the following properties:

\[
[x_\nu(s) - x_\nu(z)]^{(\alpha)} [x_\nu(s) - x_\nu(z - 1)]^{(\mu)} = [x_\nu(s) - x_\nu(z)]^{(\alpha)} [x_\nu(s) - x_\nu(z - \mu)]
\]
\[
= [x_\nu(s) - x_\nu(z)]^{(\alpha+1)},
\]
\[
[x_\nu(s) - x_\nu(z)]^{(\mu)} [x_\nu(s) - x_\nu(z - \mu)]
\]
\[
= [x_\nu(s) - x_\nu(z)]^{(\mu)} [x_\nu(s) - x_\nu(z - \mu + 1)]
\]
\[
= [x_\nu(s) - x_\nu(z)]^{(\mu)} [x_\nu(s) - x_\nu(z - \mu + 1)]
\]
\[
= [x_\nu(s) - x_\nu(z)]^{(\mu)} [x_\nu(s) - x_\nu(z + 1)]^{(\mu+1)},
\]
\[
\nabla_{\nu} \left[ x_\nu(s) - x_\nu(z) \right]^{(\mu)} = \nabla_{\nu} \left[ x_\nu(s) - x_\nu(z) \right]^{(\mu)},
\]
\[
= -[\mu]_{\nu} [x_\nu(s) - x_\nu(z)]^{(\mu-1)},
\]
\[
\nabla_\nu \left[ x_\nu(s) - x_\nu(z) \right]^{(\mu)} = -[\mu]_{\nu} [x_\nu(s) - x_\nu(z)]^{(\mu-1)}
\]
\[
= -[\mu]_{\nu} [x_\nu(s) - x_\nu(z)]^{(\mu-1)},
\]
\[
\nabla_\nu \left[ x_\nu(s) - x_\nu(z) \right]^{(\mu)} = -[\mu]_{\nu} [x_\nu(s) - x_\nu(z)]^{(\mu-1)}
\]
\[
= -[\mu]_{\nu} [x_\nu(s) - x_\nu(z)]^{(\mu-1)}.
\]

where \([\mu]_{\nu}\) is defined as (10).
Now let us first define the integer sum on non-uniform lattices $x_\gamma(s)$ in detail, which is very helpful for us to define fractional sum on non-uniform lattices $x_\gamma(s)$.

For $\gamma \in R$, the 1-th order sum of $f(z)$ over \{1, 2, ..., $z$\} on non-uniform lattices $x_\gamma(s)$ is defined by

$$y_1(z) = \nabla_\gamma^{-1}f(z) = \int_{a+1}^{z} f(s)d\nabla x_\gamma(s),$$

(21)

then by Proposition 3, we have

$$\nabla_\gamma^{-1}\nabla_\gamma^{-1}f(z) = \frac{\nabla y_1(z)}{\nabla x_\gamma(z)} = f(z),$$

(22)

and 2-th order sum of $f(z)$ over \{1, 2, ..., $z$\} on non-uniform lattices $x_\gamma(s)$ is defined by

$$y_2(z) = \nabla_\gamma^{-2}f(z) = \nabla_\gamma^{-1}[\nabla_\gamma^{-1}f(z)] = \int_{a+1}^{z} y_1(s)d\nabla x_\gamma+1(s)$$

$$= \int_{a+1}^{z} d\nabla x_\gamma+1(s) \int_{a+1}^{s} f(t)d\nabla x_\gamma(t)$$

$$= \int_{a+1}^{z} f(t)d\nabla x_\gamma(t) \int_{t}^{z} d\nabla x_\gamma+1(s)$$

$$= \int_{a+1}^{z} [x_\gamma+1(z) - x_\gamma+1(t - 1)]f(s)d\nabla x_\gamma(s).$$

(23)

Meanwhile, we have

$$\nabla_\gamma^{-1}\nabla_\gamma^{-1}y_1(z) = \frac{\nabla y_2(z)}{\nabla x_\gamma+1(z)} = y_1(z),$$

$$\nabla_\gamma^{-2}\nabla_\gamma^{-2}f(z) = \frac{\nabla}{\nabla x_\gamma(z)} \frac{\nabla y_2(z)}{\nabla x_\gamma+1(z)} = \frac{\nabla y_1(z)}{\nabla x_\gamma(z)} = f(z),$$

(24)

More generalaly, by the induction, we can define the $k$-th order sum of $f(z)$ over \{1, 2, ..., $z$\} on non-uniform lattices $x_\gamma(s)$ as

$$y_k(z) = \nabla\nabla^{-k}f(z) = \nabla^{-1}\nabla^{-1}[\nabla^{-1}(k-1)f(z)] = \int_{a+1}^{z} y_{k-1}(s)d\nabla x_\gamma+k-1(s)$$

$$= \frac{1}{\Gamma(k)} \int_{a+1}^{z} [x_\gamma+k-1(z) - x_\gamma+k-1(t - 1)]^{(k-1)}f(t)d\nabla x_\gamma(t), (k = 1, 2, ...),$$

(25)

And then we have

$$\nabla\nabla^{-k}\nabla^{-k}f(z) = \frac{\nabla}{\nabla x_\gamma(z)} \frac{\nabla}{\nabla x_\gamma+1(z)} \frac{\nabla}{\nabla x_\gamma+k-1(z)} \frac{\nabla y_k(z)}{\nabla x_\gamma+k-1(z)} = f(z), (k = 1, 2, ...)$$

(26)

It is noted that the right hand side of (25) is still meaningful when $k \in C$, so we can give the definition of fractional sum of $f(z)$ on non-uniform lattices $x_\gamma(s)$ as follows

**Definition 6. (Fractional sum on non-uniform lattices)** For any Re $\alpha \in R^+$, the $\alpha$-th order sum of $f(z)$ over \{1, 2, ..., $z$\} on non-uniform lattices (5) and (6) is defined by

$$\nabla\nabla^{-\alpha}f(z) = \frac{1}{\Gamma(\alpha)} \int_{a+1}^{z} [x_\gamma+\alpha-1(z) - x_\gamma+\alpha-1(t - 1)]^{(\alpha-1)}f(s)d\nabla x_\gamma(s),$$

(27)

where

$$\Gamma(\alpha) = \begin{cases} q^{-\alpha}(s-1)(s-2)\Gamma_q(\alpha), & \text{if } x(s) = c_1 q^s + c_2 q^{-s} + c_3; \\ \Gamma(\alpha), & \text{if } x(s) = c_1 s^2 + c_2 s + c_3, \end{cases}$$

(28)
which satisfy the following

\[ [\Gamma(\alpha + 1)]_q = [\alpha]_q [\Gamma(\alpha)]_q. \]

§3 The Analogue of Euler Beta Formula on Non-uniform Lattices

Euler Beta formula is well known as

\[ \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}dt = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (\text{Re} \alpha > 0, \text{Re} \beta > 0) \]

or

\[ \int_a^z (z-t)^{\alpha-1}(t-a)^{\beta-1}dt = \frac{(z-a)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} \quad (\text{Re} \alpha > 0, \text{Re} \beta > 0) \]

In this section, we obtain the analogue Euler Beta formula on non-uniform lattices, which is very crucial for us to propose several new definitions in this manuscript, and is also of independent importance.

**Theorem 7.** (Euler Beta formula on non-uniform lattices) For any \( \beta \in \mathbb{C} \); then for non-uniform lattices \( x_\beta(s) \), we have

\[
\int_{x_0}^{x_{z+1}} \frac{[x_\beta(z) - x_\beta(t-1)]^{(\beta-1)} [x(t) - x(a)]^{(\alpha)}}{[\Gamma(\beta)]_q [\Gamma(\alpha + 1)]_q} d\nabla x_1(t) = \frac{[x_\beta(z) - x_\beta(a)]^{(\alpha+\beta)}}{[\Gamma(\alpha + \beta + 1)]_q}. \tag{28}
\]

The proof of **Theorem 7** should use some lemmas.

**Lemma 8.** For any \( \alpha, \beta \), we have

\[ [\alpha + \beta]_q x(t) - [\alpha]_q x_\beta(t) - [\beta]_q x_\alpha(t) = \text{const}. \tag{29} \]

**Proof.** If we set \( x(t) = c_1 t^2 + c_2 t + c_3 \), then the left hand side of Eq.(29) is

\[
\text{LHS} = c_1 [(\alpha + \beta) t^2 - \alpha(t - \beta)^2 - \beta(t + \frac{\alpha}{2})^2] + c_2 [(\alpha + \beta) t - \alpha(t - \beta) - \beta(t + \frac{\alpha}{2})] = - \frac{\alpha \beta}{4} (\alpha + \beta) c_3 = \text{const}. \tag{30}
\]

If we set \( x(t) = c_1 t^2 + c_2 q^{-t} + c_3 \), then the left hand side of Eq.(29) is

\[
\text{LHS} = c_1 \left[ \frac{q^{\alpha+\beta}-q^{-\alpha} q^{\frac{\alpha}{2}}}{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}} - \frac{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}}{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}} q^t - \frac{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}}{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}} q^{-t} \right] + c_2 \left[ \frac{q^{\alpha+\beta}-q^{-\alpha} q^{\frac{\alpha}{2}}}{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}} - \frac{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}}{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}} q^t - \frac{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}}{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}} q^{-t} \right] = 0. \tag{32}
\]

\[
\square
\]
Lemma 9. For any $\alpha, \beta$, we have
\[
[\alpha + 1]q[x_\beta(z) - x_\beta(t - \beta)] - [\beta]q[x_{1-\alpha}(t + \alpha) - x_{1-\alpha}(a)] \\
= [\alpha + 1]q[x_\beta(z) - x_\beta(a - \alpha - \beta)] \\
- [\alpha + \beta + 1]q[x(t) - x(a - \alpha)].
\]  
(34)

Proof. (34) is equivalent to
\[
[\alpha + \beta + 1]q[x(t) - [\alpha + 1]q[x_\beta(t - \beta)] - [\beta]q[x_{1-\alpha}(t + \alpha) \\
= [\alpha + \beta + 1]q[x_\beta(z) - x_\beta(a - \alpha - \beta)] - [\beta]q[x_{1-\alpha}(a)].
\]  
(35)

Set $\alpha + 1 = \tilde{\alpha}$, we only need to prove that
\[
[\tilde{\alpha} + \beta]q[x_\beta(t) - [\tilde{\alpha}]q[x_\beta(t - \beta)] - [\beta]q[x_{\tilde{\alpha}}(t + \tilde{\alpha} - 1) \\
= [\tilde{\alpha} + \beta]q[x_\beta(z) - \tilde{\alpha} + 1] - [\tilde{\alpha}]q[x_\beta(a - \tilde{\alpha} + 1 - \beta)] - [\beta]q[x_{\tilde{\alpha}}(a - \tilde{\alpha} + 1)].
\]  
(36)

That is
\[
[\tilde{\alpha} + \beta]q[x(t)] - [\tilde{\alpha}]q[x_{\tilde{\alpha}}(t - \beta)] - [\beta]q[x_{\tilde{\alpha}}(t) \\
= [\tilde{\alpha} + \beta]q[x_\beta(a - \tilde{\alpha} + 1)] - [\tilde{\alpha}]q[x_\beta(a - \tilde{\alpha} + 1)] - [\beta]q[x_{\tilde{\alpha}}(a - \tilde{\alpha} + 1)].
\]  
(37)

By Lemma 8, Eq. (37) holds, and then Eq. (34) holds.

Proof of Theorem 7: Set
\[
\rho(t) = [x(t) - x(a)]^{(\alpha)}[x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)},
\]  
(38)
and
\[
\sigma(t) = [x_{1-\alpha}(t + \alpha) - x_{1-\alpha}(a)] [x_\beta(z) - x_\beta(t)].
\]  
(39)

By Proposition 5, since
\[
[x_{1-\alpha}(t + \alpha) - x_{1-\alpha}(a)] [x(t) - x(a)]^{(\alpha)} = [x_1(t) - x_1(a)]^{(\alpha + 1)}
\]  
(40)
and
\[
[x_\beta(z) - x_\beta(t)] [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} = [x_\beta(z) - x_\beta(t)]^{(\beta)}
\]  
(41)
so that we obtain
\[
\sigma(t) \rho(t) = [x_1(t) - x_1(a)]^{(\alpha + 1)} [x_\beta(z) - x_\beta(t)]^{(\beta)}.
\]  
(42)

Making use of
\[
\nabla_t[f(t)g(t)] = g(t - 1)\Delta_t[f(t)] + f(t)\nabla_t[g(t)],
\]
where
\[
f(t) = [x_1(t) - x_1(a)]^{(\alpha + 1)}, g(t) = [x_\beta(z) - x_\beta(t)]^{(\beta)},
\]
let’s calculate the $\nabla_{x_1}[\sigma(t) \rho(t)]$.

From Proposition 5, we have
\[
\nabla_{x_1} [][x_1(t) - x_1(a)]^{(\alpha + 1)} = [\alpha + 1]q[x(t) - x(a)]^{(\alpha)}
\]
and

\[ \nabla_t \frac{\Delta_t}{\Delta x_1(t-1)} \{ [x_\beta(t) - x_\beta(t)][^\beta] \} \]

\[ = \frac{\Delta_t}{\Delta x_1(t-1)} \{ [x_\beta(z) - x_\beta(t-1)][^\beta] \} \]

\[ = -[\beta]_q [x_\beta(z) - x_\beta(t-1)]^{(\beta-1)}. \]

These yield

\[ \nabla_t \frac{\Delta_t}{\Delta x_1(t)} \{ [x_1(t) - x_1(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t)][^\beta] \} \]

\[ = [\alpha + 1]_q [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t-1)][^\beta] \]

\[ - [\beta]_q [x_1(t) - x_1(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t-1)][^{(\beta-1)}] \]

\[ = ([\alpha + 1]_q [x_\beta(z) - x_\beta(t-\beta)] - [\beta]_q [x_1-a(t+\alpha) - x_1-a(a)]) \rho(t) \]

\[ \equiv \tau(t) \rho(t), \] (43)

where

\[ \tau(t) = [\alpha + 1]_q [x_\beta(z) - x_\beta(t-\beta)] - [\beta]_q [x_1-a(t+\alpha) - x_1-a(a)], \] (44)

this is due to

\[ [x_\beta(z) - x_\beta(t-1)][^\beta] = [x_\beta(z) - x_\beta(t-\beta)][x_\beta(z) - x_\beta(t-1)]^{(\beta-1)}. \]

Then from Lemma 9, it yields

\[ \tau(t) = [\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)] - [\alpha + \beta + 1]_q [x(t) - x(a - \alpha)]. \] (45)

So that we get

\[ \nabla_t \frac{\Delta_t}{\Delta x_1(t)} \{ [x_1(t) - x_1(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t)][^\beta] \} \]

\[ = ([\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)] \]

\[ - [\alpha + \beta + 1]_q [x(t) - x(a - \alpha)]) \]

\[ \cdot [x(t) - x(a)]^{(\alpha)} [y_\beta(z) - x_\beta(t-1)]^{(\beta-1)} \nabla x_1(t). \] (46)

Summing from \( a + 1 \) to \( z \), we have

\[ \sum_{t=a+1}^{z} \nabla_t \frac{\Delta_t}{\Delta x_1(t)} \{ [x_1(t) - x_1(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t)][^\beta] \} \]

\[ = \int_{a+1}^{z} \{ [\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)] \] (47)
From (49), one has
\[ - [\alpha + \beta + 1] q [x(t) - x(a - \alpha)] \cdot [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} d\nabla x_1(t). \] 
\[ \tag{48} \]

Set
\[ I(\alpha) = \int_{a+1}^{z} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} [x(t) - x(a)]^{(\alpha)} d\nabla x_1(t), \] 
\[ \tag{49} \]

and
\[ I(\alpha + 1) = \int_{a+1}^{z} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} [x(t) - x(a)]^{(\alpha + 1)} d\nabla x_1(t). \] 
\[ \tag{50} \]

Then from (48) and by the use of Proposition 5, one has
\[
\sum_{t=a+1}^{z} \nabla_t \{ [x_1(t) - x_1(a)]^{(\alpha + 1)} [x_\beta(z) - x_\beta(t)]^{(\beta)} \} \\
= [\alpha + 1] q [x_\beta(z) - x_\beta(a - \alpha - \beta)] \int_{a+1}^{z} [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} d\nabla x_1(t) \\
- [\alpha + \beta + 1] q \int_{a+1}^{z} [x(t) - x(a)][x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} d\nabla x_1(t) \\
= [\alpha + 1] q [x_\beta(z) - x_\beta(a - \alpha - \beta)] \int_{a+1}^{z} [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} d\nabla x_1(t) \\
- [\alpha + \beta + 1] q \int_{a+1}^{z} [x(t) - x(a)]^{(\alpha + 1)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} d\nabla x_1(t) \\
= [\alpha + 1] q [x_\beta(z) - x_\beta(a - \alpha - \beta) I(\alpha) - [\alpha + \beta + 1] q I(\alpha + 1). \]

Since
\[
\sum_{t=a+1}^{z} \nabla_t \{ [x_1(t) - x_1(a)]^{(\alpha + 1)} [x_\beta(z) - x_\beta(t)]^{(\beta)} \} = 0, \] 
\[ \tag{51} \]

therefore, we have prove that
\[
\frac{I(\alpha + 1)}{I(\alpha)} = \frac{[\alpha + 1] q}{[\alpha + \beta + 1] q} [x_\beta(z) - x_\beta(a - \alpha - \beta)]. \] 
\[ \tag{52} \]

From (52), one has
\[
\frac{I(\alpha + 1)}{I(\alpha)} = \frac{\Gamma(\alpha + 2) q}{\Gamma(\alpha + \beta + 2) q} [x_\beta(z) - x_\beta(a)]^{(\alpha + \beta + 1)} \frac{\Gamma(\alpha + 1) q}{\Gamma(\alpha + \beta + 1) q} [x_\beta(z) - x_\beta(a)]^{(\alpha + \beta)} . \]

So that we can set
\[
I(\alpha) = k \frac{\Gamma(\alpha + 1) q}{\Gamma(\alpha + \beta + 1) q} [x_\beta(z) - x_\beta(a)]^{(\alpha + \beta)}, \] 
\[ \tag{53} \]

where \( k \) is undetermined.

Set \( \alpha = 0 \), then
\[
I(0) = k \frac{1}{\Gamma(\beta + 1) q} [x_\beta(z) - x_\beta(a)]^{(\beta)}, \] 
\[ \tag{54} \]

From (49), one has
\[ I(0) = \int_{a+1}^{z} [x_{\beta}(z) - x_{\beta}(t-1)](\beta-1) d\nabla x_1(t) \]
\[ = \frac{1}{[\beta]_q} [x_{\beta}(z) - x_{\beta}(a)]^{(\beta)}, \tag{55} \]

From (54) and (55), one gets
\[ k = \frac{[\Gamma(\beta + 1)]_q}{[\beta]_q} = [\Gamma(\beta)]_q. \]
Hence, we obtain that
\[ I(\alpha) = \frac{[\Gamma(\beta)]_q [\Gamma(\alpha + 1)]_q [x_{\beta}(z) - x_{\beta}(a)]^{(\alpha+\beta)}}, \tag{56} \]
and the proof of Theorem 7 is completed.

§4 Generalized Abel Equation and Fractional Difference on Non-uniform Lattices

The definition of fractional difference of \( f(z) \) on non-uniform lattices \( x_\gamma(s) \) seems more difficult and complicated. Our idea is to start by solving the generalized Abel equation on non-uniform lattices. In detail, an important question is: Let \( m - 1 < \text{Re}\, \alpha \leq m \), \( f(z) \) over \( \{a+1, a+2, \ldots, z\} \) be a given function, \( g(z) \) over \( \{a+1, a+2, \ldots, z\} \) be an unknown function, which satisfies the following generalized Abel equation
\[ \nabla_{\gamma}^{-\alpha} g(z) = \int_{a+1}^{z} \frac{x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(t-1)}{[\Gamma(\alpha)]_q} g(t) d\nabla x_\gamma(t) = f(t). \tag{57} \]
How to solve generalized Abel equation (57)?

In order to solve equation (57), we should use the fundamental analogue of Euler Beta Theorem 7 on non-uniform lattices.

**Theorem 10.** (Solution for Abel equation) Set functions \( f(z) \) and \( g(z) \) over \( \{a+1, a+2, \ldots, z\} \) satisfy
\[ \nabla_{\gamma}^{-\alpha} g(z) = f(z), \ 0 < m - 1 < \text{Re}\, \alpha \leq m. \]
Then
\[ g(z) = \nabla_{\gamma}^{m} \nabla_{\gamma+\alpha}^{m-\alpha} f(z) \tag{58} \]
holds.

**Proof.** We only need to prove that
\[ \nabla_{\gamma}^{-m} g(z) = \nabla_{\gamma+\alpha}^{-(m-\alpha)} f(z), \]
that is
\[ \nabla_{\gamma+\alpha}^{-(m-\alpha)} f(z) = \nabla_{\gamma+\alpha}^{-(m-\alpha)} \nabla_{\gamma}^{-\alpha} g(z) = \nabla_{\gamma}^{-m} g(z). \]
In fact, by Definition 6, we have
\[
\nabla_{\gamma + \alpha}^{-(m - \alpha)} f(z) = \int_{\alpha + 1}^{\gamma} \frac{[x_{\gamma + m - 1}(z) - x_{\gamma + m - 1}(t - 1)]^{(m - \alpha - 1)}}{[\Gamma(m - \alpha)]_q} f(t) d\nabla x_{\gamma + \alpha}(t) \\
= \int_{\alpha + 1}^{\gamma} \frac{[x_{\gamma + m - 1}(z) - x_{\gamma + m - 1}(t - 1)]^{(m - \alpha - 1)}}{[\Gamma(m - \alpha)]_q} d\nabla x_{\gamma + \alpha}(t) \\
\cdot \int_{\alpha + 1}^{\gamma} \frac{[x_{\gamma + \alpha - 1}(t) - x_{\gamma + \alpha - 1}(s - 1)]^{(\alpha - 1)}}{[\Gamma(\alpha)]_q} g(s) d\nabla x_{\gamma}(s) \\
= \int_{\alpha + 1}^{\gamma} g(s) \nabla x_{\gamma}(s) \int_{\alpha + 1}^{\gamma} \frac{[x_{\gamma + m - 1}(z) - x_{\gamma + m - 1}(t - 1)]^{(m - \alpha - 1)}}{[\Gamma(m - \alpha)]_q} d\nabla x_{\gamma + \alpha}(t) \\
\cdot \frac{[x_{\gamma + \alpha - 1}(t) - x_{\gamma + \alpha - 1}(s - 1)]^{(\alpha - 1)}}{[\Gamma(\alpha)]_q} d\nabla x_{\gamma + \alpha}(t).
\]

In Theorem 7, replacing \(a + 1\) with \(s\), \(\alpha\) with \(\alpha - 1\), \(\beta\) with \(m - \alpha\), and replacing \(x(t)\) with \(x_{\nu + m - 1}(t)\), we can obtain the following equality
\[
\int_{s}^{\gamma} \frac{[x_{\gamma + m - 1}(z) - x_{\gamma + m - 1}(t - 1)]^{(m - \alpha - 1)}}{[\Gamma(m - \alpha)]_q} \frac{[x_{\gamma + \alpha - 1}(t) - x_{\gamma + \alpha - 1}(s - 1)]^{(\alpha - 1)}}{[\Gamma(\alpha)]_q} d\nabla x_{\gamma + \alpha}(t) \\
= \frac{[x_{\gamma + m - 1}(z) - x_{\gamma + m - 1}(s - 1)]^{(-m - 1)}}{[\Gamma(m)]_q},
\]

therefore, we have
\[
\nabla_{\gamma + \alpha}^{-(m - \alpha)} f(z) = \int_{\alpha + 1}^{\gamma} \frac{[x_{\gamma + m - 1}(z) - x_{\gamma + m - 1}(s - 1)]^{(-m - 1)}}{[\Gamma(m)]_q} g(s) d\nabla x_{\gamma}(s) = \nabla_{\gamma}^{m} g(z),
\]
which yields
\[
\nabla_{\gamma}^{m} \nabla_{\gamma + \alpha}^{-(m - \alpha)} f(z) = \nabla_{\gamma}^{m} \nabla_{\gamma}^{-m} g(z) = g(z).
\]

Inspired by Theorem 10, this is natural that we give the \(\alpha\)-th order (\(0 < m - 1 < \text{Re} \alpha \leq m\)) Riemann-Liouville difference of \(f(z)\) as follows:

**Definition 11.** (Riemann-Liouville fractional difference) Let \(m\) be the smallest integer exceeding \(\text{Re} \alpha\), \(\alpha\)-th order Riemann-Liouville difference of \(f(z)\) over \(\{a + 1, a + 2, \ldots, z\}\) on non-uniform lattices is defined by

\[
\nabla_{\gamma}^{\alpha} f(z) = \nabla_{\gamma}^{\alpha} (\nabla_{\gamma + \alpha}^{-(m - \alpha)} f(z)).
\]

Formally, in Definition 6, if \(\alpha\) is replaced by \(-\alpha\), then the RHS of (27) become

\[
\int_{\alpha + 1}^{\gamma} \frac{[x_{\gamma - \alpha - 1}(z) - x_{\gamma - \alpha - 1}(t - 1)]^{(-\alpha - 1)}}{[\Gamma(-\alpha)]_q} f(t) d\nabla x_{\gamma}(t) \\
= \nabla_{\gamma - \alpha}(t) \nabla_{\gamma - \alpha}(t) \nabla_{\gamma - \alpha}(t) \nabla_{\gamma - \alpha}(t) \nabla_{\gamma - \alpha}(t) \nabla_{\gamma - \alpha}(t) \\
\cdot \int_{\alpha + 1}^{\gamma} \frac{[x_{\gamma + n - \alpha - 1}(z) - x_{\gamma + n - \alpha - 1}(t - 1)]^{(n - \alpha - 1)}}{[\Gamma(n - \alpha)]_q} f(t) d\nabla x_{\gamma}(t) \\
= \nabla_{\gamma - \alpha}^{n} \nabla_{\gamma}^{-n + \alpha} f(z) = \nabla_{\gamma - \alpha}^{n} f(z).
\]
From (61), we can also obtain $\alpha$-th order difference of $f(z)$ as follows

**Definition 12.** (Riemann-Liouville fractional defference2) Let $\Re \alpha > 0$, $\alpha$-th order Riemann-Liouville difference of $f(z)$ over $\{a + 1, a + 2, ..., z\}$ on non-uniform lattices can be defined by

$$\nabla_{a+1}^\alpha f(z) = \int_{a+1}^z \frac{[x_{\gamma}-1(z) - x_{\gamma-\alpha-1}(t-1)]^{(-\alpha-1)} f(t)d\gamma [z]}{\Gamma(-\alpha)_q},$$

Replacing $x_{\gamma-\alpha}(t)$ with $x_{\gamma}(t)$. Then

$$\nabla_{a+1}^\alpha f(z) = \int_{a+1}^z \frac{[x_{\gamma-1}(z) - x_{\gamma-1}(t-1)]^{(-\alpha-1)} f(t)d\gamma [z]}{\Gamma(-\alpha)_q},$$

where $\alpha \notin N$.

## §5 Caputo fractional Difference on Non-uniform Lattices

In this section, we give suitable definition of Caputo fractional difference on non-uniform lattices. By the use of $\nabla_{\nu}(f(s)g(s)) = f(s-1)\nabla_{\nu}g(s) + g(s)\nabla_{\nu}f(s)$, the following theorem can be verified straight forwardly.

**Theorem 13.** (Sum by parts formula) Given two functions $f(s), g(s)$ with complex variable $s$, then

$$\int_{a+1}^z g(s)\nabla_{\gamma} f(s)d\gamma [z] = f(z)g(z) - f(a)g(a) - \int_{a+1}^z f(s-1)\nabla_{\gamma} g(s)d\gamma [z],$$

where $z, a \in C$, and $z - a \in N$.

The idea of the definition of Caputo fractional difference on non-uniform lattices is also inspired by the the solution of generalized Abel equation (57). In section 4, we have obtained that the solution of the generalized Abel equation

$$\nabla_{\gamma}^\alpha g(z) = f(z), 0 < m - 1 < \alpha \leq m$$

is

$$g(z) = \nabla_{\gamma}^\alpha f(z) = \nabla_{\gamma}^{-m} \nabla_{\gamma+\alpha}^{-m+\alpha} f(z).$$

Now we will give a new expression of (64) by parts formula. In fact, we have

$$\nabla_{\gamma}^\alpha f(z) = \nabla_{\gamma}^{-m} \nabla_{\gamma+\alpha}^{-m+\alpha} f(z)$$

$$= \nabla_{\gamma}^{-m} \int_{a+1}^z \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha-1)} f(s)d\gamma [x_{\gamma+m-1}(s)]}{\Gamma(m - \alpha)_q}.$$ 

In view of the identity

$$\nabla_{\gamma}[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s)]^{(m-\alpha-1)} f(s)d\gamma [x_{\gamma+m-1}(s)] = [m - \alpha]_q [x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s)]^{(m-\alpha-1)},$$

then the expression

$$\int_{a+1}^z \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha-1)} f(s)d\gamma [x_{\gamma+m-1}(s)]}{\Gamma(m - \alpha)_q}.$$
can be written as
\[
\int_{a+1}^{z} f(s) \nabla_{\gamma+\alpha-1} \left\{ \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q} \right\} d\varphi s
\]
\[
= \int_{a+1}^{z} f(s) \nabla_{\gamma+\alpha-1} \left\{ \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q} \right\} d\varphi x_{\gamma+\alpha-1}(s).
\]

Summing by parts formula, we get
\[
\int_{a+1}^{z} f(s) \nabla_{\gamma+\alpha-1} \left\{ \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q} \right\} d\varphi x_{\gamma+\alpha-1}(s)
\]
\[
= f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q}
\]
\[
+ \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q} \nabla_{\gamma+\alpha-1}[f(s)] d\varphi x_{\gamma+\alpha-1}(s).
\]

Therefore, we lead to
\[
\int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q} f(s) d\varphi x_{\gamma+\alpha}(s)
\]
\[
= f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q}
\]
\[
+ \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q} \nabla_{\gamma+\alpha-1}[f(s)] d\varphi x_{\gamma+\alpha-1}(s).
\]

By mathematical induction we can obtain
\[
\int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q} \nabla_{\gamma+\alpha-k}[f(s)] d\varphi x_{\gamma+\alpha-k}(s)
\]
\[
= \nabla_{\gamma+\alpha-k}^{k} f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha+k)}}{[\Gamma(m - \alpha + k + 1)]_q}
\]
\[
+ \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha+k)}}{[\Gamma(m - \alpha + k + 1)]_q} \nabla_{\gamma+\alpha-(k+1)}^{k+1}[f(s)] d\varphi x_{\gamma+\alpha-(k+1)}(s).
\]

\((k = 0, 1, ..., m - 1)\)

Substituting (66) and (67) into (65), we get
\[
\nabla_{\gamma}^{m} f(z) = \nabla_{\gamma}^{m} \left\{ f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha)}}{[\Gamma(m - \alpha + 1)]_q} \right\}
\]
\[
+ \nabla_{\gamma+\alpha-1}^{m} f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha+1)}}{[\Gamma(m - \alpha + 2)]_q}
\]
\[
+ \nabla_{\gamma+\alpha-k}^{m} f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha+k)}}{[\Gamma(m - \alpha + k + 1)]_q}
\]
\[
+ \cdots + \nabla_{\gamma+\alpha-(m-1)}^{m-1} f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(2m-\alpha-1)}}{[\Gamma(2m - \alpha)]_q}
\]
\[
+ \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(2m-\alpha-1)}}{[\Gamma(2m - \alpha)]_q} \nabla_{\gamma+\alpha-m}^{m} f(s) d\varphi x_{\gamma+\alpha-m}(s)
\]
Let
\[ f = \text{set functions} \]

Since the set of points
\[ \{ \text{Caputo fractional difference} \} \]

Let
\[ \Gamma \]

we obtain
\[ \text{and it yields} \]
\[ \Gamma \]

in simple connected domain
\[ D \]

Theorem 16.

on non-uniform lattices in terms of complex integration.

Theorem 14. (Solution2 for Abel equation) Set functions \( f(z) \) and \( g(z) \) over \( \{ a+1, a+2, ..., z \} \) satisfy
\[ \nabla_{\gamma}^{-\alpha} g(z) = f(z), \quad 0 < m - 1 < \Re \alpha \leq m, \]
then
\[ g(z) = \sum_{k=0}^{m-1} \nabla_{\gamma+\alpha-k} f(a) \frac{|x_{\gamma+1}(z)|-|x_{\gamma+1}(a)|}{\Gamma(m-\alpha+k+1)} + \nabla_{\gamma+\alpha-m}^{m} f(z) \]  \quad (68)
holds.

As a result, we have the following

**Theorem 14.** (Solution2 for Abel equation) Set functions \( f(z) \) and \( g(z) \) over \( \{ a+1, a+2, ..., z \} \) satisfy
\[ \nabla_{\gamma}^{-\alpha} g(z) = f(z), \quad 0 < m - 1 < \Re \alpha \leq m, \]
then
\[ g(z) = \sum_{k=0}^{m-1} \nabla_{\gamma+\alpha-k} f(a) \frac{|x_{\gamma+1}(z)|-|x_{\gamma+1}(a)|}{\Gamma(m-\alpha+k+1)} + \nabla_{\gamma+\alpha-m}^{m} f(z) \]  \quad (68)
holds.

Inspired by **Theorem 14**, this is also natural that we give the \( \alpha \)-th order \((0 < m < \Re \alpha \leq m - 1)\) Caputo fractional difference of \( f(z) \) as follows:

**Definition 15.** (Caputo fractional difference) Let \( m \) be the smallest integer exceeding \( \Re \alpha \), \( \alpha \)-th order Caputo fractional difference of \( f(z) \) over \( \{ a+1, a+2, ..., z \} \) on non-uniform lattices is defined by
\[ ^C \nabla_{\gamma}^{\alpha} f(z) = \nabla_{\gamma+\alpha-m}^{m} f(z). \]  \quad (69)

§6 Complex Variable Approach for Riemann-Liouville Fractional Difference On Non-uniform Lattices

In this section, we represent \( k \in N^+ \) order difference and \( \alpha \in C \) order fractional difference on non-uniform lattices in terms of complex integration.

**Theorem 16.** Let \( n \in N, \Gamma \) be a simple closed positively oriented contour. If \( f(s) \) is analytic in simple connected domain \( D \) bounded by \( \Gamma \) and \( z \) is any nonzero point lies inside \( D \), then
\[ \nabla_{\gamma-n+1} f(z) = \left( n! \frac{\log q}{q^z} \frac{1}{q^z} \right) \int_{\Gamma} \frac{f(s) \nabla x_{\gamma+1}(s) ds}{x_{\gamma}(s) - x_{\gamma}(z)^{(n+1)}}, \]  \quad (70)
where \( \Gamma \) enclosed the simple poles \( s = z, z-1, ..., z-n \) in the complex plane.

**Proof.** Since the set of points \( \{ z - i, i = 0, 1, ..., n \} \) lie inside \( D \). Hence, from the generalizated Cauchy’s integral formula, we obtain
\[ f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)x_{\gamma}'(s) ds}{x_{\gamma}(s) - x_{\gamma}(z)}, \]  \quad (71)
and it yields
\[ f(z - 1) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)x_{\gamma}'(s) ds}{x_{\gamma}(s) - x_{\gamma}(z-1)}. \]  \quad (72)
Substituting with the value of \( f(z) \) and \( f(z-1) \) into \( \nabla f(z) = \frac{f(z)-f(z-1)}{x_\gamma(z)-x_\gamma(z-1)}, \) then we have
\[
\nabla f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s) - x_\gamma(z)][x_\gamma(s) - x_\gamma(z-1)]}.
\]
Substituting with the value of \( \nabla f(z) \) and \( \nabla f(z-1) \) into \( \nabla f(z) = \frac{\nabla f(z) - \nabla f(z-1)}{x_\gamma(z) - x_\gamma(z-1)}, \) then we have
\[
\nabla f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s) - x_\gamma(z)][x_\gamma(s) - x_\gamma(z-1)][x_\gamma(z) - x_\gamma(z-2)]}.
\]
In view of \( x_\gamma(z) - x_\gamma(z-2) = [2]_q \nabla x_\gamma-1(z), \)
we obtain
\[
\nabla \left( \nabla x_{\gamma-1}(z) \right) = \frac{[2]_q}{2\pi i} \int_{\Gamma} \frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s) - x_\gamma(z)]}\frac{[n]_q!}{[n+1]_q!} \int_{\Gamma} \frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s) - x_\gamma(z)][x_\gamma(s) - x_\gamma(z)]},
\]
where
\[
[x_\gamma(s) - x_\gamma(z)]^{n+1} = \prod_{i=0}^{n} [x_\gamma(s) - x_\gamma(z-i)].
\]
And last, by the use of identity
\[
x'_\gamma(s) = \frac{\log q}{q^2 - q^2 \frac{1}{z}} \nabla x_\gamma+1(s),
\]
we have
\[
\nabla^{n}_{\gamma-n+1} f(z) = \frac{[n]_q!}{2\pi i} \int_{\Gamma} \frac{f(s)\nabla x_{\gamma+1}(s)ds}{[x_\gamma(s) - x_\gamma(z)]^{n+1}}.
\]
Inspired by formula (73), so we can give the definition of fractional difference of \( f(z) \) over \( \{a+1, a+2, ..., z\} \) on non-uniform lattices as follows

**Definition 17. (Complex fractional difference on non-uniform lattices)** Let \( \Gamma \) be a simple closed positively oriented contour. If \( f(s) \) is analytic in simple connected domain \( D \) bounded by \( \Gamma, \)
assume that \( z \) is any nonzero point inside \( D, \)
\( a+1 \) is a point inside \( D, \)
and \( z - a \in N, \)
then for any \( \alpha \in R^+, \)
the \( \alpha \)-th order fractional difference of \( f(z) \) over \( \{a+1, a+2, ..., z\} \) on non-uniform lattices is defined by
\[
\nabla^\alpha_{\gamma-a+1} f(z) = \frac{[\Gamma(\alpha +1)]_q}{2\pi i} \int_{\Gamma} \frac{f(s)\nabla x_{\gamma+1}(s)ds}{[x_\gamma(s) - x_\gamma(z)]^{\alpha+1}},
\]
where \( \Gamma \) enclosed the simple poles \( s = z, z-1, ..., a+1 \) in the complex plane.

We can calculate the integral (74) by Cauchy’s residue theorem. In detail, we have

**Theorem 18. (Fractional difference on non-uniform lattices)** Assume \( z, a \in C, \)
\( z - a \in N, \)
\( \alpha \in R^+. \)
(1) Let \( x(s) \) be quadratic lattices (5), then the \( \alpha \)-th order fractional difference of \( f(z) \) over \( \{a + 1, a + 2, \ldots, z\} \) on non-uniform lattices can be rewritten by

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \sum_{k=0}^{z-(\alpha+1)} f(z-k) \frac{\Gamma(2z-k+\gamma-\alpha)\nabla x_{\gamma+1}(z-k)}{\Gamma(2z+\gamma+1-k)} (-\alpha)_k \cdot k!.
\]

(75)

(2) Let \( x(s) \) be quadratic lattices (6), then the \( \alpha \)-th order fractional difference of \( f(z) \) over \( \{a + 1, a + 2, \ldots, z\} \) on non-uniform lattices can be rewritten by

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \sum_{k=0}^{z-(\alpha+1)} f(z-k) \frac{\Gamma(2z-k+\gamma-\alpha)\nabla x_{\gamma+1}(z-k)}{|\Gamma(2z+\gamma+1-k)|_q} \frac{(-\alpha)_k}{|k|_q!}.
\]

(76)

Proof. From (74), in the case of the quadratic lattices (5), one has

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\gamma} f(s)\nabla x_{\gamma+1}(s) ds.
\]

According to the assumption of Definition 17, \( \Gamma(s-z) \) has simple poles at \( s = z-k, k = 0, 1, 2, \ldots, z-(\alpha + 1) \). The residue of \( \Gamma(s-z) \) at the point \( s = z-k \) is

\[
\lim_{s \to z-k} (s-z)\Gamma(s-z) = \frac{(-\alpha)_k}{k!}.
\]

Then by the use of Cauchy’s residue theorem, we have

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \Gamma(\alpha+1) \sum_{k=0}^{z-(\alpha+1)} f(z-k) \frac{\Gamma(2z-k+\gamma-\alpha)\nabla x_{\gamma+1}(z-k)}{\Gamma(\alpha+1-k)\Gamma(2z+\gamma+1-k)} (-\alpha)_k \cdot k!.
\]

Since

\[
\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)} = \alpha(\alpha-1)\ldots(\alpha-k+1),
\]

and

\[
\alpha(\alpha-1)\ldots(\alpha-k+1)(-\alpha)_k = (-\alpha)_k,
\]

therefore, we get

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \sum_{k=0}^{z-(\alpha+1)} f(z-k) \frac{\Gamma(2z-k+\gamma-\alpha)\nabla x_{\gamma+1}(z-k)}{\Gamma(2z+\gamma+1-k)} (-\alpha)_k \cdot k!.
\]

From (74), in the case of the quadratic lattices (6), we have

\[
\nabla_{\gamma-\alpha+1}^\alpha [f(z)] = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\gamma} f(s)\nabla x_{\gamma+1}(s) ds.
\]

(77)
From the assumption of Definition 17, \([\Gamma(s-z)]_q\) has simple poles at \(s = z-k, k = 0, 1, 2, \ldots, z-(a+1)\). The residue of \([\Gamma(s-z)]_q\) at the point \(s = z = -k\) is
\[
\lim_{s \to z-k} (s-z+k)[\Gamma(s-z)]_q
= \lim_{s \to z-k} \frac{s-z+k}{s-z-k}[\Gamma(s-z)]_q
= \frac{q^k - q^{-\frac{1}{2}}}{\log q} \lim_{s \to z-k} \frac{s-z+k}{s-z-k}[\Gamma(s-z)]_q
= \frac{q^k - q^{-\frac{1}{2}}}{\log q} \lim_{s \to z-k} \frac{[s-z]_q([s-z+1]_q \ldots [s-z+k-1]_q[s-z+k]_q)[\Gamma(s-z)]_q}{(z-k)(s-z+1)\ldots(s-z+k-1)}
= \frac{q^k - q^{-\frac{1}{2}}}{\log q} \frac{1}{[k]_q}.
\]
Then by the use of Cauchy’s residue theorem, we have
\[
\nabla_{\gamma+1-a}^\alpha [f(z)] = [\Gamma(\alpha + 1)]_q \sum_{k=0}^{z-(a+1)} f(z-k) \frac{[\Gamma(2z-k+\gamma-a)]_q}{[\Gamma(\alpha + 1-k)]_q} \frac{\nabla x_{\gamma+1}(z-k)}{[\Gamma(2z+\gamma+1-k)]_q} (-1)^k [k]!.
\]
Since
\[
\frac{[\Gamma(\alpha + 1)]_q}{[\Gamma(\alpha + 1-k)]_q} = [\alpha]_q [\alpha-1]_q \ldots [\alpha-k+1]_q,
\]
and
\[
[\alpha]_q [\alpha-1]_q \ldots [\alpha-k+1](-1)^k = (-1)_k,
\]
therefore, we obtain that
\[
\nabla_{\gamma+1-a}^\alpha [f(z)] = \sum_{k=0}^{z-(a+1)} f(z-k) \frac{[\Gamma(2z-k+\gamma-a)]_q}{[\Gamma(2z+\gamma+1-k)]_q} \frac{\nabla x_{\gamma+1}(z-k)}{[\Gamma(2z+\gamma+1-k)]_q} (-1)_k.
\]

So far, with respect to the definition of the R-L fractional difference on non-uniform lattices, we have given two kinds of definitions, such as Definition 11 or Definition 12 in section 4 and Definition 17 or Theorem 18 in section 6 through two different ideas and methods. Now let’s compare Definition 12 in section 4 and Theorem 18 in section 6.

Here follows a theorem connecting the R-L fractional difference (63) and the complex generalization of fractional difference (74):

**Theorem 19.** For any \(\alpha \in R^+\), let \(\Gamma\) be a simple closed positively oriented contour. If \(f(s)\) is analytic in simple connected domain \(D\) bounded by \(\Gamma\), assume that \(z\) is any nonzero point inside \(D, a+1\) is a point inside \(D\), such that \(z-a \in N\), then the complex generalization fractional integral (74) equals the R-L fractional difference (62) or (63):
\[
\nabla_{\gamma+1-a}^\alpha [f(z)] = \sum_{k=a+1}^{z} \frac{[x_{\gamma-a}(z) - x_{\gamma-a}(k-1)](-\alpha-1)}{[\Gamma(-\alpha)]_q} f(k) \nabla x_{\gamma+1}(k).
\]
Proof. By Theorem 18, we have
\[
\nabla_{\gamma+1}^{\alpha}[f(z)] = \sum_{k=0}^{z-(a+1)} \frac{([-\alpha]_q)_k [\Gamma(2z - k + \gamma - \alpha)]_q}{[k]_q! [\Gamma(2z - k + \gamma + 1)]_q} f(z-k) \nabla x_{\gamma+1}(z-k).
\]

So that the two Theorem 12 and Theorem 18 are consistent. \(\square\)

Set \(\alpha = \gamma\) in Theorem 18, we obtain

Corollary 20. Assume that conditions of Definition 17 and Theorem 18 hold, then
\[
\nabla_{1}^{\gamma}[f(z)] = \frac{[\Gamma(\gamma + 1)]_q}{2\pi i} \int f(s) \nabla x_{\gamma+1}(s) ds,
\]

where \(\Gamma\) enclosed the simple poles \(s = z, z-1, \ldots, a + 1\) in the complex plane.

Remark 21. When \(\gamma = n \in N^+\), we have
\[
\nabla_{1}^{\gamma}[f(z)] = \frac{[\Gamma(n + 1)]_q}{2\pi i} \int f(s) \nabla x_{\gamma+1}(s) ds,
\]

where \(\Gamma\) enclosed the simple poles \(s = z, z-1, \ldots, z - n\) in the complex plane.

This is consistent with Definition 2 proposed by Nikiforov, A., Uvarov, V., Suslov, S in [22].

Finally, for complex integral of Riemann-Liouville fractional difference on non-uniform lattices, we can establish an analogue of Cauchy Beta formula on non-uniform lattices, which is also of independent importance:

Theorem 22. (Cauchy Beta formula) Let \(\alpha, \beta \in C\), and assume that
\[
\int_{\Gamma} \Delta_{t}^{\alpha} \left\{ \frac{1}{[x_\beta(z) - x_\beta(t)][\beta]_q} \right\} dt = 0,
\]

then
\[
1 = \frac{[\Gamma(\beta + 1)]_q}{2\pi i} \int_{\Gamma} f(t) \nabla x_{\beta+1}(t) dt = \frac{[\Gamma(\alpha + \beta)]_q}{[\Gamma(\alpha + \beta + 1)]_q} \int_{\Gamma} f(t) \nabla x_{\beta+1}(t) dt,
\]

where \(\Gamma\) be a simple closed positively oriented contour, \(\alpha\) lies inside \(C\).
Lemma 23. For any $\alpha, \beta$, then we have
\[
[1 - \alpha]q[x_\beta(z) - x_\beta(t - \beta)] + [\beta]q[x_{1 - \alpha}(t + \alpha - 1) - x_{1 - \alpha}(a)]
= [1 - \alpha]q[x_\beta(z) - x_\beta(a + 1 - \alpha - \beta)] + [\alpha + \beta - 1]q[x(t) - x(a + 1 - \alpha)].
\] (79)

Proof. (79) is equivalent to
\[
[\alpha + \beta - 1]q x(t) + [1 - \alpha]q x_\beta(t - \beta) - [\beta]q x_{1 - \alpha}(t + \alpha - 1)
= [\alpha + \beta - 1]q x(a + 1 - \alpha) + [1 - \alpha]q x_\beta(a + 1 - \alpha - \beta) - [\beta]q x_{1 - \alpha}(a).
\] (80)
Set $\alpha - 1 = \tilde{\alpha}$, then (80) can be written as
\[
[\tilde{\alpha} + \beta]q x(t) - [\tilde{\alpha}]q x_{1 - \beta}(t) - [\beta]q x_{1 - \alpha}(t)
= [\tilde{\alpha} + \beta]q x(a - \tilde{\alpha}) - [\tilde{\alpha}]q x_{1 - \beta}(a - \tilde{\alpha}) - [\beta]q x_{1 - \alpha}(a - \tilde{\alpha}).
\] (81)
By the use of Lemma 9, then Eq. (81) holds, and then Eq. (79) holds. \(\square\)

Proof of Theorem 22: Set
\[
\rho(t) = \frac{1}{[x_\beta(z) - x_\beta(t)]^{(\beta + 1)}} \frac{1}{[x(t) - x(a)]^{(\alpha)}},
\]
and
\[
\sigma(t) = [x_{\alpha - 1}(t + \alpha - 1) - x_{\alpha - 1}(a)][x_\beta(z) - x_\beta(t)].
\]
Since
\[
[x_\beta(z) - x_\beta(t)]^{(\beta + 1)} = [x_\beta(z) - x_\beta(t - 1)]^{(\beta)}[x_\beta(z) - x_\beta(t)],
\]
and
\[
[x(t) - x(a)]^{(\alpha)} = [x_{-1}(t) - x_{-1}(a)]^{(\alpha - 1)}[x_{1 - \alpha}(t + \alpha - 1) - x_{1 - \alpha}(a)],
\]
these reduce to
\[
\sigma(t) \rho(t) = \frac{1}{[x_\beta(z) - x_\beta(t - 1)]^{(\beta)}} \frac{1}{[x_{-1}(t) - x_{-1}(a)]^{(\alpha - 1)}}.
\]
Making use of
\[
\Delta_t[f(t)g(t)] = g(t + 1)\Delta_t[f(t)] + f(t)\Delta_t[g(t)],
\]
where
\[
f(t) = \frac{1}{[x_{-1}(t) - x_{-1}(a)]^{(\alpha - 1)}}, \quad g(t) = \frac{1}{[x_\beta(z) - x_\beta(t - 1)]^{(\beta)}},
\]
and
\[
\frac{\Delta_t}{\Delta x_{-1}(t)} \frac{1}{[x_{-1}(t) - x_{-1}(a)]^{(\alpha - 1)}} = \frac{[1 - \alpha]q}{[x(t) - x(a)]^{(\alpha)}},
\]
\[
\frac{\Delta_t}{\Delta x_{-1}(t)} \left[ \frac{1}{[x_{\beta}(z) - x_{\beta}(t)][\beta]} \right] = \frac{\nabla_t}{\nabla x_1(t)} \left[ \frac{1}{[x_{\beta}(z) - x_{\beta}(t)][\beta]} \right] = \left[ [x_{\beta}(z) - x_{\beta}(t)][\beta] \right].
\]

then, we have
\[
\frac{\Delta_t}{\Delta x_{-1}(t)} \left\{ \sigma(t)\rho(t) \right\} = \frac{1}{[x_{\beta}(z) - x_{\beta}(t)][\beta]} \left[ x(t) - x(a) \right]^{[\alpha]}
+ \frac{1}{[x_{-1}(t) - x_{-1}(a)][\alpha - 1]} \left[ x_{\beta}(z) - x_{\beta}(t) \right]^{[\beta + 1]}
= \left\{ [1 - \alpha]_q [x_{\beta}(z) - x_{\beta}(t - \beta)] + [\beta]_q [x_{1-\alpha}(t + \alpha - 1) - x_{1-\alpha}(a)] \right\}
\times \frac{1}{[x(t) - x(a)]^{[\alpha]}} \left[ x_{\beta}(z) - x_{\beta}(t) \right]^{[\beta + 1]}
= \tau(t)\rho(t),
\]
where
\[
\tau(t) = [1 - \alpha]_q [x_{\beta}(z) - x_{\beta}(t - \beta)] + [\beta]_q [x_{1-\alpha}(t + \alpha - 1) - x_{1-\alpha}(a)],
\]
this is due to
\[
[x_{\beta}(z) - x_{\beta}(t)][\beta + 1] = [x_{\beta}(z) - x_{\beta}(t)][\beta][x_{\beta}(z) - x_{\beta}(t - \beta)].
\]
From Proposition 5 one has
\[
\frac{\Delta_t}{\Delta x_{-1}(t)} \left\{ \sigma(t)\rho(t) \right\} = \left\{ [1 - \alpha]_q [x_{\beta}(z) - x_{\beta}(a + 1 - \alpha - \beta)] + [\alpha + \beta - 1]_q [x(t) - x(a + 1 - \alpha)] \right\}
+ \frac{1}{[x_{\beta}(z) - x_{\beta}(t)][\beta + 1]} \left[ x(t) - x(a) \right]^{[\alpha]},
\]
or
\[
\Delta_t \left\{ \sigma(t)\rho(t) \right\} = \left\{ [1 - \alpha]_q [x_{\beta}(z) - x_{\beta}(a + 1 - \alpha - \beta)] + [\alpha + \beta - 1]_q [x(t) - x(a + 1 - \alpha)] \right\}
+ \frac{1}{[x_{\beta}(z) - x_{\beta}(t)][\beta + 1]} \left[ x(t) - x(a) \right]^{[\alpha]} \Delta x_{-1}(t).
\]
(82)

Set
\[
I(\alpha) = \frac{1}{2\pi i} \log q \int_{\mathbb{R}} \frac{1}{q^{-\frac{x}{2}} q^{\frac{x}{2}} - x} \frac{\nabla y_1(t) dt}{[x_{\beta}(z) - x_{\beta}(t)][\beta + 1]} \left[ x(t) - x(a) \right]^{[\alpha]},
\]
(83)
and

\[
I(\alpha - 1) = \frac{1}{2\pi i} \log q \oint \frac{1}{x_\beta(z) - x_\beta(t)} \frac{\nabla y_1(t)}{|x(t) - x(a)|^{\alpha - 1}} dt.
\]

Since

\[
[x(t) - x(a)]^{(\alpha - 1)}[x(t) - x(a + 1 - \alpha)] = [x(t) - x(a)]^{(\alpha)},
\]

then

\[
I(\alpha - 1) = \frac{1}{2\pi i} \log q \oint \frac{1}{x_\beta(z) - x_\beta(t)} \frac{[x(t) - x(a + 1 - \alpha)]\nabla y_1(t)}{|x(t) - x(a)|^{\alpha}} dt.
\]

Integrating both sides of equation (82), then we have

\[
\oint \Delta t \{\sigma(t)\rho(t)\} dt = [1 - \alpha]_q [x_\beta(z) - x_\beta(a + 1 - \alpha - \beta)]I(\alpha) - [\alpha + \beta - 1]_q I(\alpha - 1).
\]

If

\[
\oint \Delta t \{\sigma(t)\rho(t)\} dt = 0,
\]

then we obtain that

\[
\frac{I(\alpha - 1)}{I(\alpha)} = \frac{[\alpha - 1]_q}{[\alpha + \beta - 1]_q} [y_\beta(z) - y_\beta(a + 1 - \alpha - \beta)].
\]

That is

\[
\frac{I(\alpha - 1)}{I(\alpha)} = \frac{\Gamma(\alpha + \beta - 1)_q}{\Gamma(\alpha + \beta)_q} \frac{[x_\beta(z) - x_\beta(a)]^{(\alpha - 1)}}{[x_\beta(z) - x_\beta(a)]^{(\alpha + \beta)}}.
\] (84)

From (84), we set

\[
I(\alpha) = k [\Gamma(\alpha + \beta)_q] \frac{1}{\Gamma(\alpha)_q} \frac{1}{[x_\beta(z) - x_\beta(a)]^{(\alpha + \beta)}},
\] (85)

where \(k\) is undetermined.

Set \(\alpha = 1\), one has

\[
I(1) = k [\Gamma(1 + \beta)_q] \frac{1}{[x_\beta(z) - x_\beta(a)]^{(1 + \beta)}},
\] (86)

and from (83) and generalized Cauchy residue theorem, one has

\[
I(1) = \frac{1}{2\pi i} \log q \oint \frac{1}{x_\beta(z) - x_\beta(t)} \frac{\nabla y_1(t)}{|x(t) - x(a)|^{(1)}} dt = \frac{1}{2\pi i} \oint \frac{1}{x_\beta(z) - x_\beta(t)} \frac{x'(t)}{|x(t) - x(a)|} dt = \frac{1}{[x_\beta(z) - x_\beta(a)]^{(1 + 1)}}.
\] (87)

From (86) and (87), we get

\[
k = \frac{1}{[\Gamma(1 + \beta)_q].
\]

Therefore, we obtain that
\[ I(\alpha) = \frac{[\Gamma(\alpha + \beta)]_q}{[\Gamma(\beta + 1)]_q[\Gamma(\alpha)]_q} \frac{1}{[x_\beta(z) - x_\beta(a)]^{(\alpha+\beta)}} \]

and Theorem 22 is completed.

References


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