Appl. Math. J. Chinese Univ.
2021, 36(3): 420-442

Fractional sum and fractional difference on non-uniform lattices and analogue of Euler and Cauchy Beta formulas

CHENG Jin-fa

Abstract. As is well known, the definitions of fractional sum and fractional difference of $f(z)$ on non-uniform lattices $x(z) = c_1 z^2 + c_2 z + c_3$ or $x(z) = c_1 q^z + c_2 q^{-z} + c_3$ are more difficult and complicated. In this article, for the first time we propose the definitions of the fractional sum and fractional difference on non-uniform lattices by two different ways. The analogue of Euler's Beta formula, Cauchy' Beta formula on non-uniform lattices are established, and some fundamental theorems of fractional calculas, the solution of the generalized Abel equation on non-uniform lattices are obtained etc.

§1 Introduction

The definitions of non-uniform lattices date back to the approximation of the following differential equation of hypergeometric type:

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0, \tag{1}$$

where $\sigma(z)$ and $\tau(z)$ are polynomials of degrees at most two and one, respectively, and λ is a constant. Its solutions are some types of special functions of mathematical physics, such as the classical orthogonal polynomials, the hypergeometric and cylindrical functions, see G. E. Andrews, R. Askey, R. Roy [5, 6]. A. F. Nikiforov, V. B. Uvarov and S. K. Suslov [22, 23] generalized Eq. (1) to a difference equation of hypergeometric type case and studied the Nikiforov-Uvarov-Suslov difference equation on a lattice $x(s)$ with variable step size $\nabla x(s) = x(s) - x(s-1)$ as

$$\widetilde{\sigma}[x(s)]\frac{\Delta}{\Delta x(s-1/2)}\left[\frac{\nabla y(s)}{\nabla x(s)}\right] + \frac{1}{2}\widetilde{\tau}[x(s)]\left[\frac{\Delta y(s)}{\Delta x(s)} + \frac{\nabla y(s)}{\nabla x(s)}\right] + \lambda y(s) = 0, \tag{2}$$

where $\widetilde{\sigma}(x)$ and $\widetilde{\tau}(x)$ are polynomials of degrees at most two and one in $x(s)$, respectively, λ is a constant, $\Delta y(s) = y(s+1) - y(s)$, $\nabla y(s) = y(s) - y(s-1)$, and $x(s)$ is a lattice function that

Received: 2020-01-07. Revised: 2020-04-03.
MR Subject Classification: 33C45, 33D45, 26A33, 34K37.
Keywords: difference equation of hypergeometric type, non-uniform lattice, fractional sum, fractional difference, special functions, Euler's Beta formula, Cauchy' Beta formula.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-021-4013-1.
Supported by the National Natural Science Foundation Fujian province of China(2016J01032).

satisfies

$$\frac{x(s+1)+x(s)}{2}=\alpha x(s+\frac{1}{2})+\beta,\quad \alpha,\beta \text{ are constants,} \tag{3}$$

$$x^2(s+1)+x^2(s) \text{ is a polynomial of degree at most two w.r.t. } x(s+\frac{1}{2}). \tag{4}$$

It should be pointed out that the difference equation (2) obtained as a result of approximating the differential equation (1) on a non-uniform lattice is of independent importance and arises in a number of other questions. Its solutions essentially generalized the solutions of the original differential equation and are of interest in their own right [13–15, 21–25]. As it is known in ([22], P59), the general solutions $x(s)$ which satisfy the conditions in Eqs. (3) and (4) are

$$x(s)=\widetilde{c}_1 s^2+\widetilde{c}_2 s+\widetilde{c}_3; \tag{5}$$

or

$$x(s)=c_1q^s+c_2q^{-s}+c_3, q\neq 1 \tag{6}$$

Definition 1. *([22, 23]) Two kinds of lattice functions $x(s)$ are called* non-uniform lattices *which have the form (5) and (6), where $c_i,\widetilde{c}_i$ are arbitrary constants and $c_1c_2\neq 0$, $\widetilde{c}_1\widetilde{c}_2\neq 0$. When $c_1=1, c_2=c_3=0$, or $\widetilde{c}_2=1,\widetilde{c}_1=\widetilde{c}_3=0$, these two kinds of lattice functions $x(s)$*

$$x(s)=s \tag{7}$$

$$x(s)=q^s, \tag{8}$$

are called uniform lattices.

Let $x(s)$ be a non-uniform lattice, where $s\in\mathbb{C}$. For any real γ, $x_\gamma(s)=x(s+\frac{\gamma}{2})$ is also a non-uniform lattice. Given a function $F(s)$, define the difference operator with respect to $x_\gamma(s)$ as

$$\nabla_\gamma F(s)=\frac{\nabla F(s)}{\nabla x_\gamma(s)},$$

and

$$\nabla_\gamma^k F(z)=\frac{\nabla}{\nabla x_\gamma(z)}\frac{\nabla}{\nabla x_{\gamma+1}(z)}\cdots\frac{\nabla F(z)}{\nabla x_{\gamma+k-1}(z)}.(k=1,2,...)$$

Althought the discrete fractional calculus on uniform lattice (7) and (8) are more current, but great development has been made in this field [1–3, 8–12, 17, 18, 20]. In the recent monographs, J. F. Cheng [10], C. Goodrich and A. Peterson [19] provided the comprehensive treatment of the discrete fractional calculus with up-to-date references, and the developments in the theory of fractional q-calculas had been well reported by M. H. Annaby and Z. S. Mansour [4].

But we should mention that, in the case of nonuniform lattices (5) or (6), even when $n\in N$, the fomula of $n-$order difference on non-uniform lattices is a remarkable job, since it is very complicated and difficult to be obtained. In fact, in [22], A. Nikiforov, V. Uvarov, S. Suslov obtained the formula of $n-th$ difference $\nabla_1^{(n)}[f(s)]$ as follows:

Definition 2. *([22]) Let $n\in N^+$, for nonuniform lattices (5) or (6), then*

$$\nabla_1^{(n)}[f(s)]=\sum_{k=0}^{n}\frac{([-n]_q)_k}{[k]_q!}\frac{[\Gamma(2s-k+c)]_q}{[\Gamma(2s-k+n+1+c)]_q}f(s-k)\nabla x_{n+1}(s-k),$$

where $[\Gamma(s)]_q$ is modified $q-$gamma function which is defined as

$$[\Gamma(s)]_q=q^{-(s-1)(s-2)/4}\Gamma_q(s),$$

and function $\Gamma_q(s)$ is called the $q-$gamma function; it is a generalization of Euler's gamma function $\Gamma(s)$. It is defined by

$$\Gamma_q(s)=\begin{cases}\frac{\Pi_{k=0}^{\infty}(1-q^{k+1})}{(1-q)^{s-1}\Pi_{k=0}^{\infty}(1-q^{s+k})}, & when\ |q|<1;\\ q^{-(s-1)(s-2)/2}\Gamma_{1/q}(s), & when\ |q|>1.\end{cases}\tag{9}$$

and

$$[\mu]_q=\begin{cases}\frac{q^{\frac{\mu}{2}}-q^{-\frac{\mu}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}, if\ x(s)=c_1q^s+c_2q^{-s}+c_3;\\ \mu, if\ x(s)=\widetilde{c}_1s^2+\widetilde{c}_2s+\widetilde{c}_3,\end{cases}\tag{10}$$

where

$$c=\begin{cases}\frac{\log\frac{c_2}{c_1}}{\log q}, & when\ x(s)=c_1q^s+c_2q^{-s}+c_3,\\ \frac{\widetilde{c}_2}{\widetilde{c}_1}, & when\ x(s)=\widetilde{c}_1s^2+\widetilde{c}_2s+\widetilde{c}_3.\end{cases}$$

Now there exist two important and challenging problems that need to be further discussed:

(1) Assume that $g(s)$ be a given function, $f(s)$ be an unknown function, which satisfies the following generalized difference equation on non-uniform lattices

$$\nabla_1^{(n)}[f(s)]=g(s).\tag{11}$$

How to solve generalized difference equation (11)?

(2) The definitions of $\alpha-$order fractional difference and $\alpha-$order fractional sum on non-uniform lattices are very difficult and interesting problems. They have not appeared since the monographs [22, 23] were published. Can we give reasonable definitions of fractional sum and difference on non-uniform lattices?

We believe that as the most general discrete fractional calculus on non-uniform lattices, they should have an independent meaning and lead to many interesting new theories about them, which may be an important extension and development of the discrete fractional calculus.

The purpose of this paper is to inquire into the feasibility of establishing discrete fractional calculus on nonuniform lattices. In this article, for the first time we propose the definitions of the fractional sum and fractional difference on non-uniform lattices. In order to keep this paper to a reasonable length, we have chosen to restrict ourselves to some fundamental theorems of discrete fractional calculus, such as the analogue of Euler Beta formula, Cauchy Beta formula on non-uniform lattices, and the solution of the generalized Abel equation on non-uniform lattices etc. The other important results such as Taylor formula and Leibnize formula on non-uniform lattices will be given in a future. The results we obtain here are essentially new and have not been found in other literature.

§2 Integer Sum and Fractional Sum on Non-uniform Lattices

Let $x(s)$ be a non-uniform lattice, where $s\in\mathbb{C}$. Let $\nabla_\gamma F(s)=f(s)$, then

$$F(s)-F(s-1)=f(s)\left[x_\gamma(s)-x_\gamma(s-1)\right].$$

Choose $z,a\in\mathbb{C}$, and $z-a\in N$. Summing from $s=a+1$ to z, we have

$$F(z)-F(a)=\sum_{s=a+1}^{z}f(s)\nabla x_\gamma(s).$$

Thus, we define

$$\int_{a+1}^{z} f(s)d_{\nabla}x_{\gamma}(s) = \sum_{s=a+1}^{z} f(s)\nabla x_{\gamma}(s).$$

It is easy to verify that

Proposition 3. *Given two function $F(z), f(z)$ with complex variable $z, a \in C$, and $z-a \in N$, we have*

$(1)\nabla_{\gamma}\left[\int_{a+1}^{z} f(s)d_{\nabla}x_{\gamma}(s)\right] = f(z),$

$(2)\int_{a+1}^{z} \nabla_{\gamma}F(s)d_{\nabla}x_{\gamma}(s) = F(z) - F(a).$

A generalized power $[x(s)-x(z)]^{(n)}$ on nonuniform lattice is given by

$$[x(s)-x(z)]^{(n)} = \prod_{k=0}^{n-1}[x(s)-x(z-k)], (n \in N^{+}),$$

and a more formal definition and further properties of the generalized powers $[x_{\nu}(s)-x_{\nu}(z)]^{(\alpha)}$ on nonuniform lattice are very important, which are defined as follows:

Definition 4. *(See [7, 25]) Let $\alpha \in C$, the generalized powers $[x_{\nu}(s)-x_{\nu}(z)]^{(\alpha)}$ are defined by*

$$[x_{\nu}(s)-x_{\nu}(z)]^{(\alpha)} = \tag{12}$$

$$\begin{cases} \frac{\Gamma(s-z+\alpha)}{\Gamma(s-z)}, & \text{if } x(s)=s, \\ \widetilde{c_1}^{\alpha}\frac{\Gamma(s-z+\alpha)\Gamma(s+z+\nu+c+1)}{\Gamma(s-z)\Gamma(s+z+\nu-\alpha+c+1)}, & \text{if } x(s)=\widetilde{c}_1 s^2+\widetilde{c}_2 s+\widetilde{c}_3, \\ (q-1)^{\alpha}q^{\alpha(\nu-\alpha+1)/2}\frac{\Gamma_q(s-z+\alpha)}{\Gamma_q(s-z)}, & \text{if } x(s)=q^s, \\ [c_1(1-q)^2]^{\alpha}q^{-\alpha(s+\frac{\nu}{2})}\frac{\Gamma_q(s-z+\alpha)\Gamma_q(s+z+\nu+c+1)}{\Gamma_q(s-z)\Gamma_q(s+z+\nu-\alpha+c+1)}, & \text{if } x(s)=c_1q^s+c_2q^{-s}+c_3. \end{cases} \tag{13}$$

Proposition 5. *[7, 25]. For $x(s)=c_1q^s+c_2q^{-s}+c_3$ or $x(s)=\widetilde{c}_1 s^2+\widetilde{c}_2 s+\widetilde{c}_3$, the generalized power $[x_{\nu}(s)-x_{\nu}(z)]^{(\alpha)}$ satisfy the following properties:*

$$[x_{\nu}(s)-x_{\nu}(z)][x_{\nu}(s)-x_{\nu}(z-1)]^{(\mu)} = [x_{\nu}(s)-x_{\nu}(z)]^{(\mu)}[x_{\nu}(s)-x_{\nu}(z-\mu)] \tag{14}$$

$$= [x_{\nu}(s)-x_{\nu}(z)]^{(\mu+1)}; \tag{15}$$

$$[x_{\nu-1}(s+1)-x_{\nu-1}(z)]^{(\mu)}[x_{\nu-\mu}(s)-x_{\nu-\mu}(z)]$$

$$= [x_{\nu-\mu}(s+\mu)-x_{\nu-\mu}(z)][x_{\nu-1}(s)-x_{\nu-1}(z)]^{(\mu)} = [x_{\nu}(s)-x_{\nu}(z)]^{(\mu+1)}; \tag{16}$$

$$\frac{\Delta_z}{\Delta x_{\nu-\mu+1}(z)}[x_{\nu}(s)-x_{\nu}(z)]^{(\mu)} = -\frac{\nabla_s}{\nabla x_{\nu+1}(s)}[x_{\nu+1}(s)-x_{\nu+1}(z)]^{(\mu)} \tag{17}$$

$$= -[\mu]_q[x_{\nu}(s)-x_{\nu}(z)]^{(\mu-1)}; \tag{18}$$

$$\frac{\nabla_z}{\nabla x_{\nu-\mu+1}(z)}\left\{\frac{1}{[x_{\nu}(s)-x_{\nu}(z)]^{(\mu)}}\right\} = -\frac{\Delta_s}{\Delta x_{\nu-1}(s)}\left\{\frac{1}{[x_{\nu-1}(s)-x_{\nu-1}(z)]^{(\mu)}}\right\} \tag{19}$$

$$= \frac{[\mu]_q}{[x_{\nu}(s)-x_{\nu}(z)]^{(\mu+1)}}. \tag{20}$$

where $[\mu]_q$ is defined as (10).

Now let us first define the integer sum on non-uniform lattices $x_\gamma(s)$ in detail, which is very helpful for us to define fractional sum on non-uniform lattices $x_\gamma(s)$.

For $\gamma \in R$, the 1-th order sum of $f(z)$ over $\{a+1, a+2, ..., z\}$ on non-uniform lattices $x_\gamma(s)$ is defined by

$$y_1(z) = \nabla_\gamma^{-1} f(z) = \int_{a+1}^{z} f(s) d_\nabla x_\gamma(s), \tag{21}$$

then by **Proposion** 3, we have

$$\nabla_\gamma^1 \nabla_\gamma^{-1} f(z) = \frac{\nabla y_1(z)}{\nabla x_\gamma(z)} = f(z), \tag{22}$$

and 2-th order sum of $f(z)$ over $\{a+1, a+2, ..., z\}$ on non-uniform lattices $x_\gamma(s)$ is defined by

$$\begin{aligned} y_2(z) = \nabla_\gamma^{-2} f(z) = \nabla_{\gamma+1}^{-1}[\nabla_\gamma^{-1} f(z)] &= \int_{a+1}^{z} y_1(s) d_\nabla x_{\gamma+1}(s) \\ &= \int_{a+1}^{z} d_\nabla x_{\gamma+1}(s) \int_{a+1}^{s} f(t) d_\nabla x_\gamma(t) \\ &= \int_{a+1}^{z} f(t) d_\nabla x_\gamma(t) \int_{t}^{z} d_\nabla x_{\gamma+1}(s) \\ &= \int_{a+1}^{z} [x_{\gamma+1}(z) - x_{\gamma+1}(t-1)] f(s) d_\nabla x_\gamma(s). \end{aligned} \tag{23}$$

Meanwhile, we have

$$\nabla_{\gamma+1}^1 \nabla_{\gamma+1}^{-1} y_1(z) = \frac{\nabla y_2(z)}{\nabla x_{\gamma+1}(z)} = y_1(z),$$

$$\nabla_\gamma^2 \nabla_\gamma^{-2} f(z) = \frac{\nabla}{\nabla x_\gamma(z)} \frac{\nabla y_2(z)}{\nabla x_{\gamma+1}(z)} = \frac{\nabla y_1(z)}{\nabla x_\gamma(z)} = f(z), \tag{24}$$

More generalaly, by the induction, we can define the k-th order sum of $f(z)$ over $\{a+1, a+2, ..., z\}$ on non-uniform lattices $x_\gamma(s)$ as

$$\begin{aligned} y_k(z) = \nabla_\gamma^{-k} f(z) = \nabla_{\gamma+k-1}^{-1}[\nabla_\gamma^{-(k-1)} f(z)] &= \int_{a+1}^{z} y_{k-1}(s) d_\nabla x_{\gamma+k-1}(s) \\ = \frac{1}{[\Gamma(k)]_q} \int_{a+1}^{z} [x_{\gamma+k-1}(z) - x_{\gamma+k-1}(t-1)]^{(k-1)} f(t) d_\nabla x_\gamma(t), (k = 1, 2, ...) \end{aligned} \tag{25}$$

And then we have

$$\nabla_\gamma^k \nabla_\gamma^{-k} f(z) = \frac{\nabla}{\nabla x_\gamma(z)} \frac{\nabla}{\nabla x_{\gamma+1}(z)} ... \frac{\nabla y_k(z)}{\nabla x_{\gamma+k-1}(z)} = f(z).(k = 1, 2, ...) \tag{26}$$

It is noted that the right hand side of (25) is still meanful when $k \in C$, so we can give the definition of fractional sum of $f(z)$ on non-uniform lattices $x_\gamma(s)$ as follows

Definition 6. *(Fractional sum on non-uniform lattices) For any* $\operatorname{Re}\alpha \in R^+$, *the* α*-th order sum of* $f(z)$ *over* $\{a+1, a+2, ..., z\}$ *on non-uniform lattices (5) and (6) is defined by*

$$\nabla_\gamma^{-\alpha} f(z) = \frac{1}{[\Gamma(\alpha)]_q} \int_{a+1}^{z} [x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(t-1)]^{(\alpha-1)} f(s) d_\nabla x_\gamma(s), \tag{27}$$

where

$$[\Gamma(\alpha)]_q = \begin{cases} q^{-(s-1)(s-2)} \Gamma_q(\alpha), \text{if } x(s) = c_1 q^s + c_2 q^{-s} + c_3; \\ \Gamma(\alpha), \text{if } x(s) = \widetilde{c}_1 s^2 + \widetilde{c}_2 s + \widetilde{c}_3, \end{cases}$$

which satisfy the following

$$[\Gamma(\alpha+1)]_q = [\alpha]_q[\Gamma(\alpha)]_q.$$

§3 The Analogue of Euler Beta Formula on Non-uniform Lattices

Euler Beta formula is well known as

$$\int_0^1 (1-t)^{\alpha-1}t^{\beta-1}dt = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, (\mathrm{Re}\,\alpha>0, \mathrm{Re}\,\beta>0)$$

or

$$\int_a^z \frac{(z-t)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-a)^{\beta-1}}{\Gamma(\beta)}dt = \frac{(z-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}.(\mathrm{Re}\,\alpha>0, \mathrm{Re}\,\beta>0)$$

In this section, we obtain the analogue Euler Beta formula on non-uniform lattices, which is very crucial for us to propose several new definitions in this manuscript, and is also of independent importance.

Theorem 7. *(Euler Beta formula on non-uniform lattices) For any $\alpha,\beta\in C$, then for non-uniform lattices $x(s)$, we have*

$$\int_{a+1}^z \frac{[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}}{[\Gamma(\beta)]_q}\frac{[x(t)-x(a)]^{(\alpha)}}{[\Gamma(\alpha+1)]_q}d_\nabla x_1(t)$$
$$= \frac{[x_\beta(z)-x_\beta(a)]^{(\alpha+\beta)}}{[\Gamma(\alpha+\beta+1)]_q}. \tag{28}$$

The proof of **Theorem** 7 should use some lemmas.

Lemma 8. *For any α,β,we have*

$$[\alpha+\beta]_q x(t) - [\alpha]_q x_{-\beta}(t) - [\beta]_q x_\alpha(t) = const. \tag{29}$$

Proof. If we set $x(t)=\widetilde{c_1}t^2+\widetilde{c_2}t+\widetilde{c_3}$, then the left hand side of Eq.(29) is

$$LHS = \widetilde{c_1}[(\alpha+\beta)t^2 - \alpha(t-\frac{\beta}{2})^2 - \beta(t+\frac{\alpha}{2})^2]$$
$$+\widetilde{c_2}[(\alpha+\beta)t - \alpha(t-\frac{\beta}{2}) - \beta(t+\frac{\alpha}{2})] \tag{30}$$
$$= -\frac{\alpha\beta}{4}(\alpha+\beta)\widetilde{c_1} = const. \tag{31}$$

If we set $x(t)=c_1q^t+c_2q^{-t}+c_3$, then the left hand side of Eq.(29) is

$$LHS = c_1[\frac{q^{\frac{\alpha+\beta}{2}}-q^{-\frac{\alpha+\beta}{2}}}{q^{\frac12}-q^{-\frac12}}q^t - \frac{q^{\frac{\alpha}{2}}-q^{-\frac{\alpha}{2}}}{q^{\frac12}-q^{-\frac12}}q^{t-\frac{\beta}{2}} - \frac{q^{\frac{\beta}{2}}-q^{-\frac{\beta}{2}}}{q^{\frac12}-q^{-\frac12}}q^{t+\frac{\alpha}{2}}]$$
$$+c_2[\frac{q^{\frac{\alpha+\beta}{2}}-q^{-\frac{\alpha+\beta}{2}}}{q^{\frac12}-q^{-\frac12}}q^{-t} - \frac{q^{\frac{\alpha}{2}}-q^{-\frac{\alpha}{2}}}{q^{\frac12}-q^{-\frac12}}q^{-t+\frac{\beta}{2}} - \frac{q^{\frac{\beta}{2}}-q^{-\frac{\beta}{2}}}{q^{\frac12}-q^{-\frac12}}q^{-t-\frac{\alpha}{2}}] \tag{32}$$
$$=0. \tag{33}$$

□

Lemma 9. *For any α,β,we have*

$$\begin{aligned}&[\alpha+1]_q[x_\beta(z)-x_\beta(t-\beta)]-[\beta]_q[x_{1-\alpha}(t+\alpha)-x_{1-\alpha}(a)]\\&=[\alpha+1]_q[x_\beta(z)-x_\beta(a-\alpha-\beta)]\\&-[\alpha+\beta+1]_q[x(t)-x(a-\alpha)].\end{aligned}\tag{34}$$

Proof. (34) is equivalent to

$$\begin{aligned}&[\alpha+\beta+1]_qx(t)-[\alpha+1]_qx_\beta(t-\beta)-[\beta]_qx_{1-\alpha}(t+\alpha)\\&=[\alpha+\beta+1]_qx(a-\alpha)-[\alpha+1]_qx_\beta(a-\alpha-\beta)-[\beta]_qx_{1-\alpha}(a).\end{aligned}\tag{35}$$

Set $\alpha+1=\widetilde{\alpha}$, we only need to prove that

$$\begin{aligned}&[\widetilde{\alpha}+\beta]_qx(t)-[\widetilde{\alpha}]_qx_\beta(t-\beta)-[\beta]_qx_{2-\widetilde{\alpha}}(t+\widetilde{\alpha}-1)\\&=[\widetilde{\alpha}+\beta]_qx(a-\widetilde{\alpha}+1)-[\widetilde{\alpha}]_qx_\beta(a-\widetilde{\alpha}+1-\beta)-[\beta]_qx_{2-\widetilde{\alpha}}(a).\end{aligned}\tag{36}$$

That is

$$\begin{aligned}&[\widetilde{\alpha}+\beta]_qx(t)-[\widetilde{\alpha}]_qx_{-\beta}(t)-[\beta]_qx_{\widetilde{\alpha}}(t)\\&=[\widetilde{\alpha}+\beta]_qx(a-\widetilde{\alpha}+1)-[\alpha]_qx_{-\beta}(a-\widetilde{\alpha}+1)-[\beta]_qx_{\widetilde{\alpha}}(a-\widetilde{\alpha}+1).\end{aligned}\tag{37}$$

By **Lemma** 8, Eq. (37) holds, and then Eq. (34) holds. □

Using **Proposition** 5 and **Lemma** 9, now it is time for us to prove **Theorem** 7.

Proof of Theorem 7: Set

$$\rho(t)=[x(t)-x(a)]^{(\alpha)}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)},\tag{38}$$

and

$$\sigma(t)=[x_{1-\alpha}(t+\alpha)-x_{1-\alpha}(a)][x_\beta(z)-x_\beta(t)].\tag{39}$$

By **Proposition** 5, since

$$[x_{1-\alpha}(t+\alpha)-x_{1-\alpha}(a)][x(t)-x(a)]^{(\alpha)}=[x_1(t)-x_1(a)]^{(\alpha+1)}\tag{40}$$

and

$$[x_\beta(z)-x_\beta(t)][x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}=[x_\beta(z)-x_\beta(t)]^{(\beta)}.\tag{41}$$

so that we obtain

$$\sigma(t)\rho(t)=[x_1(t)-x_1(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t)]^{(\beta)},\tag{42}$$

Making use of

$$\nabla_t[f(t)g(t)]=g(t-1)\Delta_t[f(t)]+f(t)\nabla_t[g(t)],$$

where

$$f(t)=[x_1(t)-x_1(a)]^{(\alpha+1)},g(t)=[x_\beta(z)-x_\beta(t)]^{(\beta)},$$

let's calculate the $\frac{\nabla_t[\sigma(t)\rho(t)]}{\nabla x_1(t)}$.

From **Proposition** 5, we have

$$\frac{\nabla_t}{\nabla x_1(t)}\{[x_1(t)-x_1(a)]^{(\alpha+1)}\}=[\alpha+1]_q[x(t)-x(a)]^{(\alpha)},$$

and

$$\begin{aligned}&\frac{\nabla_t}{\nabla x_1(t)}\{[x_\beta(z)-x_\beta(t)]^{(\beta)}\}\\&=\frac{\Delta_t}{\Delta x_1(t-1)}\{[x_\beta(z)-x_\beta(t-1)]^{(\beta)}\}\\&=-[\beta]_q[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}.\end{aligned}$$

These yield

$$\begin{aligned}&\frac{\nabla_t}{\nabla x_1(t)}\{[x_1(t)-x_1(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t)]^{(\beta)}\}\\&=[\alpha+1]_q[x(t)-x(a)]^{(\alpha)}[x_\beta(z)-x_\beta(t-1)]^{(\beta)}\\&-[\beta]_q[x_1(t)-x_1(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}\\&=\{[\alpha+1]_q[x_\beta(z)-x_\beta(t-\beta)]-[\beta]_q[x_{1-\alpha}(t+\alpha)-x_{1-\alpha}(a)]\}\rho(t)\\&\equiv\tau(t)\rho(t),\end{aligned}\tag{43}$$

where

$$\tau(t)=[\alpha+1]_q[x_\beta(z)-x_\beta(t-\beta)]-[\beta]_q[x_{1-\alpha}(t+\alpha)-x_{1-\alpha}(a)],\tag{44}$$

this is due to

$$[x_\beta(z)-x_\beta(t-1)]^{(\beta)}=[x_\beta(z)-x_\beta(t-\beta)][x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}.$$

Then from **Lemma** 9, it yields

$$\tau(t)=[\alpha+1]_q[x_\beta(z)-x_\beta(a-\alpha-\beta)]-[\alpha+\beta+1]_q[x(t)-x(a-\alpha)].\tag{45}$$

So that we get

$$\begin{aligned}&\frac{\nabla_t}{\nabla x_1(t)}\{[x_1(t)-x_1(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t)]^{(\beta)}\}\\&=\{[\alpha+1]_q[x_\beta(z)-x_\beta(a-\alpha-\beta)]\\&-[\alpha+\beta+1]_q[x(t)-x(a-\alpha)]\}\rho(t).\end{aligned}$$

Or

$$\begin{aligned}&\nabla_t\{[x_1(t)-x_1(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t)]^{(\beta)}\}\\&=\{[\alpha+1]_q[x_\beta(z)-x_\beta(a-\alpha-\beta)]\\&-[\alpha+\beta+1]_q[x(t)-x(a-\alpha)]\}\\&\cdot[x(t)-x(a)]^{(\alpha)}[y_\beta(z)-x_\beta(t-1)]^{(\beta-1)}\nabla x_1(t).\end{aligned}\tag{46}$$

Summing from $a+1$ to z, we have

$$\begin{aligned}&\sum_{t=a+1}^{z}\nabla_t\{[x_1(t)-x_1(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t)]^{(\beta)}\}\\&=\int_{a+1}^{z}\{[\alpha+1]_q[x_\beta(z)-x_\beta(a-\alpha-\beta)]\end{aligned}\tag{47}$$

$$- [\alpha+\beta+1]_q[x(t)-x(a-\alpha)]\}$$
$$\cdot [x(t)-x(a)]^{(\alpha)}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}d_\nabla x_1(t). \tag{48}$$

Set

$$I(\alpha)=\int_{a+1}^{z}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}[x(t)-x(a)]^{(\alpha)}d_\nabla x_1(t), \tag{49}$$

and

$$I(\alpha+1)=\int_{a+1}^{z}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}[x(t)-x(a)]^{(\alpha+1)}d_\nabla x_1(t). \tag{50}$$

Then from (48) and by the use of **Proposition** 5, one has

$$\begin{aligned}
&\sum_{t=a+1}^{z}\nabla_t\{[x_1(t)-x_1(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t)]^{(\beta)}\}\\
&=[\alpha+1]_q[x_\beta(z)-x_\beta(a-\alpha-\beta)]\int_{a+1}^{z}[x(t)-x(a)]^{(\alpha)}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}d_\nabla x_1(t)\\
&-[\alpha+\beta+1]_q\int_{a+1}^{z}[x(t)-x(a-\alpha)][x(t)-x(a)]^{(\alpha)}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}d_\nabla x_1(t)\\
&=[\alpha+1]_q[x_\beta(z)-x_\beta(a-\alpha-\beta)]\int_{a+1}^{z}[x(t)-x(a)]^{(\alpha)}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}d_\nabla x_1(t)\\
&-[\alpha+\beta+1]_q\int_{a+1}^{z}[x(t)-x(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t-1)]^{(\beta-1)}d_\nabla x_1(t)\\
&=[\alpha+1]_q[x_\beta(z)-x_\beta(a-\alpha-\beta)I(\alpha)-[\alpha+\beta+1]_qI(\alpha+1).
\end{aligned}$$

Since

$$\sum_{t=a+1}^{z}\nabla_t\{[x_1(t)-x_1(a)]^{(\alpha+1)}[x_\beta(z)-x_\beta(t)]^{(\beta)}\}=0, \tag{51}$$

therefore, we have prove that

$$\frac{I(\alpha+1)}{I(\alpha)}=\frac{[\alpha+1]_q}{[\alpha+\beta+1]_q}[x_\beta(z)-x_\beta(a-\alpha-\beta)]. \tag{52}$$

From (52), one has

$$\frac{I(\alpha+1)}{I(\alpha)}=\frac{\frac{[\Gamma(\alpha+2)]_q}{[\Gamma(\alpha+\beta+2)]_q}[x_\beta(z)-x_\beta(a)]^{(\alpha+\beta+1)}}{\frac{[\Gamma(\alpha+1)]_q}{[\Gamma(\alpha+\beta+1)]_q}[x_\beta(z)-x_\beta(a)]^{(\alpha+\beta)}}.$$

So that we can set

$$I(\alpha)=k\frac{[\Gamma(\alpha+1)]_q}{[\Gamma(\alpha+\beta+1)]_q}[x_\beta(z)-x_\beta(a)]^{(\alpha+\beta)}, \tag{53}$$

where k is undetermined.

Set $\alpha=0$, then

$$I(0)=k\frac{1}{[\Gamma(\beta+1)]_q}[x_\beta(z)-x_\beta(a)]^{(\beta)}, \tag{54}$$

From (49), one has

$$I(0) = \int_{a+1}^{z} [x_\beta(z) - x_\beta(t-1)]^{(\beta-1)} d_\nabla x_1(t)$$
$$= \frac{1}{[\beta]_q}[x_\beta(z) - x_\beta(a)]^{(\beta)}, \tag{55}$$

From (54) and (55), one gets

$$k = \frac{[\Gamma(\beta+1)]_q}{[\beta]_q} = [\Gamma(\beta)]_q.$$

Hence, we obtain that

$$I(\alpha) = \frac{[\Gamma(\beta)]_q[\Gamma(\alpha+1)]_q}{[\Gamma(\alpha+\beta+1)]_q}[x_\beta(z) - x_\beta(a)]^{(\alpha+\beta)}, \tag{56}$$

and the proof of **Theorem** 7 is completed.

§4 Generalized Abel Equation and Fractional Difference on Non-uniform Lattices

The definition of fractional difference of $f(z)$ on non-uniform lattices $x_\gamma(s)$ seems more difficult and complicated. Our idea is to start by solving the generalized Abel equation on non-uniform lattices. In detail, an important question is: Let $m-1 < \operatorname{Re}\alpha \leq m, f(z)$ over $\{a+1, a+2, ..., z\}$ be a given function, $g(z)$ over $\{a+1, a+2, ..., z\}$ be an unknown function, which satisfies the following generalized Abel equation

$$\nabla_\gamma^{-\alpha} g(z) = \int_{a+1}^{z} \frac{[x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(t-1)]^{(\alpha-1)}}{[\Gamma(\alpha)]_q} g(t) d_\nabla x_\gamma(t) = f(t). \tag{57}$$

How to solve generalized Abel equation (57)?

In order to solve equation (57), we should use the fundamental analogue of Euler Beta **Theorem** 7 on non-uniform lattices.

Theorem 10. *(Solution1 for Abel equation) Set functions $f(z)$ and $g(z)$ over $\{a+1, a+2, ..., z\}$ satisfy*

$$\nabla_\gamma^{-\alpha} g(z) = f(z), 0 < m-1 < \operatorname{Re}\alpha \leqslant m.$$

Then

$$g(z) = \nabla_\gamma^m \nabla_{\gamma+\alpha}^{-m+\alpha} f(z) \tag{58}$$

holds.

Proof. We only need to prove that

$$\nabla_\gamma^{-m} g(z) = \nabla_{\gamma+\alpha}^{-(m-\alpha)} f(z),$$

that is

$$\nabla_{\gamma+\alpha}^{-(m-\alpha)} f(z) = \nabla_{\gamma+\alpha}^{-(m-\alpha)} \nabla_\gamma^{-\alpha} g(z) = \nabla_\gamma^{-m} g(z).$$

In fact, by **Definition 6,** we have

$$\begin{aligned}\nabla_{\gamma+\alpha}^{-(m-\alpha)}f(z)&=\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(t-1)]^{(m-\alpha-1)}}{[\Gamma(m-\alpha)]_q}f(t)d_\nabla x_{\gamma+\alpha}(t)\\&=\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(t-1)]^{(m-\alpha-1)}}{[\Gamma(m-\alpha)]_q}d_\nabla x_{\gamma+\alpha}(t)\\&\cdot\int_{a+1}^{z}\frac{[x_{\gamma+\alpha-1}(t)-x_{\gamma+\alpha-1}(s-1)]^{(\alpha-1)}}{[\Gamma(\alpha)]_q}g(s)d_\nabla x_\gamma(s)\\&=\int_{a+1}^{z}g(s)\nabla x_\gamma(s)\int_{s}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(t-1)]^{(m-\alpha-1)}}{[\Gamma(m-\alpha)]_q}\\&\cdot\frac{[x_{\gamma+\alpha-1}(t)-x_{\gamma+\alpha-1}(s-1)]^{(\alpha-1)}}{[\Gamma(\alpha)]_q}d_\nabla x_{\gamma+\alpha}(t).\end{aligned}$$

In **Theorem 7,** replacing $a+1$ with $s;\alpha$ with $\alpha-1;\beta$ with $m-\alpha$, and replacing $x(t)$ with $x_{\nu+\alpha-1}(t)$, then $x_\beta(t)$ with $x_{\nu+m-1}(t)$, we can obtain the following equality

$$\begin{aligned}&\int_{s}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(t-1)]^{(m-\alpha-1)}}{[\Gamma(m-\alpha)]_q}\frac{[x_{\gamma+\alpha-1}(t)-x_{\gamma+\alpha-1}(s-1)]^{(\alpha-1)}}{[\Gamma(\alpha)]_q}d_\nabla x_{\gamma+\alpha}(t)\\&=\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s-1)]^{(-m-1)}}{[\Gamma(m)]_q},\end{aligned}$$

therefore, we have

$$\nabla_{\gamma+\alpha}^{-(m-\alpha)}f(z)=\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s-1)]^{(-m-1)}}{[\Gamma(m)]_q}g(s)d_\nabla x_\gamma(s)=\nabla_\gamma^{-m}g(z),$$

which yields

$$\nabla_\gamma^m\nabla_{\gamma+\alpha}^{-(m-\alpha)}f(z)=\nabla_\gamma^m\nabla_\gamma^{-m}g(z)=g(z).$$

□

Inspired by **Theorem** 10, This is natural that we give the α-th order $(0<m-1<\operatorname{Re}\alpha\le m)$ Riemann-Liouvile difference of $f(z)$ as follows:

Definition 11. *(Riemann-Liouvile fractional defference1) Let m be the smallest integer exceeding $\operatorname{Re}\alpha$, α-th order Riemann-Liouvile difference of $f(z)$ over $\{a+1,a+2,...,z\}$ on non-uniform lattices is defined by*

$$\nabla_\gamma^\alpha f(z)=\nabla_\gamma^m(\nabla_{\gamma+\alpha}^{\alpha-m}f(z)).\tag{59}$$

Formally, in **Definition 6**, if α is replaced by $-\alpha$, then the RHS of (27) become

$$\begin{aligned}&\int_{a+1}^{z}\frac{[x_{\gamma-\alpha-1}(z)-x_{\gamma-\alpha-1}(t-1)]^{(-\alpha-1)}}{[\Gamma(-\alpha)]_q}f(t)d_\nabla x_\gamma(t)\\&=\frac{\nabla}{\nabla x_{\gamma-\alpha}(t)}\left(\frac{\nabla}{\nabla x_{\gamma-\alpha+1}(t)}\cdots\frac{\nabla}{\nabla x_{\gamma-\alpha+n-1}(t)}\right)\\&\cdot\int_{a+1}^{z}\frac{[x_{\gamma+n-\alpha-1}(z)-x_{\gamma+n-\alpha-1}(t-1)]^{(n-\alpha-1)}}{[\Gamma(n-\alpha)]_q}f(t)d_\nabla x_\gamma(t) \qquad (60)\\&=\nabla_{\gamma-\alpha}^n\nabla_\gamma^{-n+\alpha}f(z)=\nabla_{\gamma-\alpha}^\alpha f(z). \qquad (61)\end{aligned}$$

From (61), we can also obtain α-th order difference of $f(z)$ as follows

Definition 12. *(Riemann-Liouvile fractional defference2) Let* $\mathrm{Re}\,\alpha > 0$, *α-th order Riemann-Liouvile difference of $f(z)$ over $\{a+1, a+2, ..., z\}$ on non-uniform lattices can be defined by*

$$\nabla^{\alpha}_{\gamma-\alpha} f(z) = \int_{a+1}^{z} \frac{[x_{\gamma-\alpha-1}(z) - x_{\gamma-\alpha-1}(t-1)]^{(-\alpha-1)}}{[\Gamma(-\alpha)]_q} f(t) d_{\nabla} x_{\gamma}(t). \tag{62}$$

Replacing $x_{\gamma-\alpha}(t)$ with $x_{\gamma}(t)$, Then

$$\nabla^{\alpha}_{\gamma} f(z) = \int_{a+1}^{z} \frac{[x_{\gamma-1}(z) - x_{\gamma-1}(t-1)]^{(-\alpha-1)}}{[\Gamma(-\alpha)]_{q[}} f(t) d_{\nabla} x_{\gamma+\alpha}(t), \tag{63}$$

where $\alpha \notin N$.

§5 Caputo fractional Difference on Non-uniform Lattices

In this section, we give suitable definition of Caputo fractional difference on non-uniform lattices. By the use of $\nabla_{\nu}(f(s)g(s)) = f(s-1)\nabla_{\nu} g(s) + g(s)\nabla_{\nu} f(s)$, the following theorem can be verified straight forwardly.

Theorem 13. *(Sum by parts formula) Given two functions Let $f(s), g(s)$ with complex variable s, then*

$$\int_{a+1}^{z} g(s)\nabla_{\gamma} f(s) d_{\nabla} x_{\gamma}(s) = f(z)g(z) - f(a)g(a) - \int_{a+1}^{z} f(s-1)\nabla_{\gamma} g(s) d_{\nabla} x_{\gamma}(s),$$

where $z, a \in C$, and $z - a \in N$.

The idea of the definition of Caputo fractional difference on non-uniform lattices is also inspired by the the solution of generalized Abel equation (57). In section 4, we have obtained that the solution of the generalized Abel equation

$$\nabla^{-\alpha}_{\gamma} g(z) = f(z), 0 < m-1 < \alpha \leqslant m$$

is

$$g(z) = \nabla^{\alpha}_{\gamma} f(z) = \nabla^{m}_{\gamma} \nabla^{-m+\alpha}_{\gamma+\alpha} f(z). \tag{64}$$

Now we will give a new expression of (64) by parts formula. In fact, we have

$$\begin{aligned}\nabla^{\alpha}_{\gamma} f(z) &= \nabla^{m}_{\gamma} \nabla^{-m+\alpha}_{\gamma+\alpha} f(z) \\ &= \nabla^{m}_{\gamma} \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha-1)}}{[\Gamma(m-\alpha)]_q} f(s) d_{\nabla} x_{\gamma+\alpha}(s). \end{aligned} \tag{65}$$

In view of the identity

$$\begin{aligned}\frac{\nabla_{(s)}[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s)]^{(m-\alpha)}}{\nabla x_{\gamma+\alpha}(s)} &= \frac{\Delta_{(s)}[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha)}}{\Delta x_{\gamma+\alpha}(s-1)} \\ &= -[m-\alpha]_q [x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha-1)},\end{aligned}$$

then the expression

$$\int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha-1)}}{[\Gamma(m-\alpha)]_q} f(s) d_{\nabla} x_{\gamma+\alpha}(s)$$

can be written as

$$\int_{a+1}^{z} f(s)\nabla_{(s)}\{\frac{-[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s)]^{(m-\alpha)}}{[\Gamma(m-\alpha+1)]_q}\}d_\nabla s$$
$$=\int_{a+1}^{z} f(s)\nabla_{\gamma+\alpha-1}\{\frac{-[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s)]^{(m-\alpha)}}{[\Gamma(m-\alpha+1)]_q}\}d_\nabla x_{\gamma+\alpha-1}(s).$$

Summing by parts formula, we get

$$\int_{a+1}^{z} f(s)\nabla_{\gamma+\alpha-1}\{\frac{-[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s)]^{(m-\alpha)}}{[\Gamma(m-\alpha+1)]_q}\}d_\nabla x_{\gamma+\alpha-1}(s)$$
$$=f(a)\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(a)]^{(m-\alpha)}}{[\Gamma(m-\alpha+1)]_q}$$
$$+\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s-1)]^{(m-\alpha)}}{[\Gamma(m-\alpha+1)]_q}\nabla_{\gamma+\alpha-1}[f(s)]d_\nabla x_{\gamma+\alpha-1}(s).$$

Therefore, we lead to

$$\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s-1)]^{(m-\alpha-1)}}{[\Gamma(m-\alpha)]_q}f(s)d_\nabla x_{\gamma+\alpha}(s) \tag{66}$$
$$=f(a)\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(a)]^{(m-\alpha)}}{[\Gamma(m-\alpha+1)]_q}$$
$$+\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s-1)]^{(m-\alpha)}}{[\Gamma(m-\alpha+1)]_q}\nabla_{\gamma+\alpha-1}[f(s)]d_\nabla x_{\gamma+\alpha-1}(s).$$

By mathematical induction we can obtain

$$\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s-1)]^{(m-\alpha+k-1)}}{[\Gamma(m-\alpha+k)]_q}\nabla^k_{\gamma+\alpha-k}[f(s)]d_\nabla x_{\gamma+\alpha-k}(s)$$
$$=\nabla^k_{\gamma+\alpha-k}f(a)\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(a)]^{(m-\alpha+k)}}{[\Gamma(m-\alpha+k+1)]_q}+$$
$$+\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s-1)]^{(m-\alpha+k)}}{[\Gamma(m-\alpha+k+1)]_q}\nabla^{k+1}_{\gamma+\alpha-(k+1)}f(s)d_\nabla x_{\gamma+\alpha-(k+1)}(s). \tag{67}$$

$(k=0,1,...,m-1)$

Substituting (66) and (67) into (65), we get

$$\nabla^\alpha_\gamma f(z)=\nabla^m_\gamma\{f(a)\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(a)]^{(m-\alpha)}}{[\Gamma(m-\alpha+1)]_q}+$$
$$+\nabla_{\gamma+\alpha-1}f(a)\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(a)]^{(m-\alpha+1)}}{[\Gamma(m-\alpha+2)]_q}+$$
$$+\nabla^k_{\gamma+\alpha-k}f(a)\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(a)]^{[m-\alpha+k]}}{[\Gamma(m-\alpha+k+1)]_q}$$
$$+...+\nabla^{m-1}_{\gamma+\alpha-(m-1)}f(a)\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(a)]^{(2m-\alpha-1)}}{[\Gamma(2m-\alpha)]_q}+$$
$$+\int_{a+1}^{z}\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(s-1)]^{(2m-\alpha-1)}}{[\Gamma(2m-\alpha)]_q}\nabla^m_{\gamma+\alpha-m}f(s)d_\nabla x_{\gamma+\alpha-m}(s)\}$$

$$= \nabla_{\gamma}^{m}\{\sum_{k=0}^{m-1}\nabla_{\gamma+\alpha-k}^{k}f(a)\frac{[x_{\gamma+m-1}(z)-x_{\gamma+m-1}(a)]^{(m-\alpha+k)}}{[\Gamma(m-\alpha+k+1)]_q}+$$
$$+\nabla_{\gamma+\alpha-m}^{\alpha-2m}\nabla_{\gamma+\alpha-m}^{m}f(z)\}$$
$$=\sum_{k=0}^{m-1}\nabla_{\gamma+\alpha-k}^{k}f(a)\frac{[x_{\gamma-1}(z)-x_{\gamma-1}(a)]^{(-\alpha+k)}}{[\Gamma(-\alpha+k+1)]_q}+\nabla_{\gamma+\alpha-m}^{\alpha-m}\nabla_{\gamma+\alpha-m}^{m}f(z).$$

As a result, we have the following

Theorem 14. *(Solution2 for Abel equation) Set functions $f(z)$ and $g(z)$ over $\{a+1,a+2,...,z\}$ satisfy*

$$\nabla_{\gamma}^{-\alpha}g(z)=f(z), 0<m-1<\operatorname{Re}\alpha\leqslant m,$$

then

$$g(z)=\sum_{k=0}^{m-1}\nabla_{\gamma+\alpha-k}^{k}f(a)\frac{[x_{\gamma-1}(z)-x_{\gamma-1}(a)]^{(-\alpha+k)}}{[\Gamma(-\alpha+k+1)]_q}+\nabla_{\gamma+\alpha-m}^{\alpha-m}\nabla_{\gamma+\alpha-m}^{m}f(z) \tag{68}$$

holds.

Inspired by **Theorem** 14, this is also natural that we give the α-th order $(0<m<\operatorname{Re}\alpha\leq m-1)$ Caputo fractional difference of $f(z)$ as follows:

Definition 15. *(Caputo fractional difference)Let m be the smallest integer exceeding $\operatorname{Re}\alpha$, α-th order Caputo fractional difference of $f(z)$ over $\{a+1,a+2,...,z\}$ on non-uniform lattices is defined by*

$$^{C}\nabla_{\gamma}^{\alpha}f(z)=\nabla_{\gamma+\alpha-m}^{\alpha-m}\nabla_{\gamma+\alpha-m}^{m}f(z). \tag{69}$$

§6 Complex Variable Approach for Riemann-Liouville Fractional Difference On Non-uniform Lattices

In this section, we represent $k\in N^{+}$ order difference and $\alpha\in C$ order fractional difference on non-uniform lattices in terms of complex integration.

Theorem 16. *Let $n\in N$, Γ be a simple closed positively oriented contour. If $f(s)$ is analytic in simple connected domain D bounded by Γ and z is any nonzero point lies inside D, then*

$$\nabla_{\gamma-n+1}^{n}f(z)=\frac{[n]_q!}{2\pi i}\frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\oint_{\Gamma}\frac{f(s)\nabla x_{\gamma+1}(s)ds}{[x_{\gamma}(s)-x_{\gamma}(z)]^{(n+1)}}, \tag{70}$$

where Γ enclosed the simple poles $s=z,z-1,...,z-n$ in the complex plane.

Proof. Since the set of points $\{z-i, i=0,1,...,n\}$ lie inside D. Hence, from the genaralized Cauchy's integral formula, we obtain

$$f(z)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(s)x_{\gamma}'(s)ds}{[x_{\gamma}(s)-x_{\gamma}(z)],} \tag{71}$$

and it yields

$$f(z-1)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(s)x_{\gamma}'(s)ds}{[x_{\gamma}(s)-x_{\gamma}(z-1)].} \tag{72}$$

Substitutig with the value of $f(z)$ and $f(z-1)$ into $\frac{\nabla f(z)}{\nabla x_\gamma(z)}=\frac{f(z)-f(z-1)}{x_\gamma(z)-x_\gamma(z-1)}$, then we have

$$\frac{\nabla f(z)}{\nabla x_\gamma(z)}=\frac{1}{2\pi i}\oint_\Gamma\frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s)-x_\gamma(z)][x_\gamma(s)-x_\gamma(z-1)]}$$

$$=\frac{1}{2\pi i}\oint_\Gamma\frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s)-x_\gamma(z)]^{(2)}}.$$

Substitutig with the value of $\frac{\nabla f(z)}{\nabla x_\gamma(z)}$ and $\frac{\nabla f(z-1)}{\nabla x_\gamma(z-1)}$ into $\frac{\frac{\nabla f(z)}{\nabla x_\gamma(z)}-\frac{\nabla f(z-1)}{\nabla x_\gamma(z-1)}}{x_\gamma(z)-x_\gamma(z-2)}$, then we have

$$\frac{\frac{\nabla f(z)}{\nabla x_\gamma(z)}-\frac{\nabla f(z-1)}{\nabla x_\gamma(z-1)}}{x_\gamma(z)-x_\gamma(z-2)}=\frac{1}{2\pi i}\oint_\Gamma\frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s)-x_\gamma(z)]^{(3)}}.$$

In view of

$$x_\gamma(z)-x_\gamma(z-2)=[2]_q\nabla x_{\gamma-1}(z),$$

we obtain

$$\frac{\nabla}{\nabla x_{\gamma-1}(z)}\left(\frac{\nabla f(z)}{\nabla x_\gamma(z)}\right)=\frac{[2]_q}{2\pi i}\oint_\Gamma\frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s)-x_\gamma(z)]^{(3)}}.$$

More generalaly, by the induction, we can obtain

$$\frac{\nabla}{\nabla x_{\gamma-n+1}(z)}\left(\frac{\nabla}{\nabla x_{\gamma-n+2}(z)}\cdots\left(\frac{\nabla f(z)}{\nabla x_\gamma(z)}\right)\right)=\frac{[n]_q!}{2\pi i}\oint_\Gamma\frac{f(s)x'_\gamma(s)ds}{[x_\gamma(s)-x_\gamma(z)]^{(n+1)}},$$

where

$$[x_\gamma(s)-x_\gamma(z)]^{(n+1)}=\prod_{i=0}^{n}[x_\gamma(s)-x_\gamma(z-i)].$$

And last, by the use of identity

$$x'_\gamma(s)=\frac{\log q}{q^{\frac12}-q^{-\frac12}}\nabla x_{\gamma+1}(s),$$

we have

$$\nabla^n_{\gamma-n+1}f(z)=\frac{[n]_q!}{2\pi i}\frac{\log q}{q^{\frac12}-q^{-\frac12}}\oint_\Gamma\frac{f(s)\nabla x_{\gamma+1}(s)ds\xi}{[x_\gamma(s)-x_\gamma(z)]^{(n+1)}}. \tag{73}$$

□

Inspired by formula (73), so we can give the definition of fractional difference of $f(z)$ over $\{a+1,a+2,...,z\}$ on non-uniform lattices as follows

Definition 17. *(Complex fractional difference on non-uniform lattices) Let Γ be a simple closed positively oriented contour. If $f(s)$ is analytic in simple connected domain D bounded by Γ, assume that z is any nonzero point inside D, $a+1$ is a point inside D, and $z-a\in N$, then for any $\alpha\in R^+$, the α-th order fractional difference of $f(z)$ over $\{a+1,a+2,...,z\}$ on non-uniform lattices is defined by*

$$\nabla^\alpha_{\gamma-\alpha+1}f(z)=\frac{[\Gamma(\alpha+1)]_q}{2\pi i}\frac{\log q}{q^{\frac12}-q^{-\frac12}}\oint_\Gamma\frac{f(s)\nabla x_{\gamma+1}(s)ds}{[x_\gamma(s)-x_\gamma(z)]^{(\alpha+1)}}, \tag{74}$$

where Γ enclosed the simple poles $s=z,z-1,...,a+1$ in the complex plane.

We can calculate the integral (74) by Cauchy's residue theorem. In detail, we have

Theorem 18. *(Fractional difference on non-uniform lattices)Assume $z,a\in C, z-a\in N,\alpha\in R^+$.*

(1) Let $x(s)$ be quadratic lattices (5), then the α-th order fractional difference of $f(z)$ over $\{a+1, a+2, ..., z\}$ on non-uniform lattices can be rewritten by

$$\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)] = \sum_{k=0}^{z-(a+1)} f(z-k)\frac{\Gamma(2z-k+\gamma-\alpha)\nabla x_{\gamma+1}(z-k)}{\Gamma(2z+\gamma+1-k)}\frac{(-\alpha)_k}{k!}; \tag{75}$$

(2) Let $x(s)$ be quadratic lattices(6), then the α-th order fractional difference of $f(z)$ over $\{a+1, a+2, ..., z\}$ on non-uniform lattices can be rewritten by

$$\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)] = \sum_{k=0}^{z-(a+1)} f(z-k)\frac{[\Gamma(2z-k+\gamma-\alpha)]_q\nabla x_{\gamma+1}(z-k)}{[\Gamma(2z+\gamma+1-k)]_q}\frac{([-\alpha]_q)_k}{[k]_q!}. \tag{76}$$

Proof. From (74), in the case of the quadratic lattices (5), one has

$$\begin{aligned}\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)] &= \frac{\Gamma(\alpha+1)}{2\pi i}\oint_{\Gamma}\frac{f(s)\nabla x_{\gamma+1}(s)ds}{[x_\gamma(s)-x_\gamma(z)^{(\alpha+1)}}\\ &= \frac{\Gamma(\alpha+1)}{2\pi i}\oint_{\Gamma}\frac{f(s)\nabla x_{\gamma+1}(s)\Gamma(s-z)\Gamma(s+z+\gamma-\alpha)ds}{\Gamma(s-z+\alpha+1)\Gamma(s+z+\gamma+1)}.\end{aligned}$$

According to the assumption of **Definition** 17, $\Gamma(s-z)$ has simple poles at $s = z-k, k = 0, 1, 2, ..., z-(a+1)$. The residue of $\Gamma(s-z)$ at the point $s-z=-k$ is

$$\begin{aligned}&\lim_{s\to z-k}(s-z+k)\Gamma(s-z)\\ &= \lim_{s\to z-k}\frac{(s-z)(s-z+1)...(s-z+k-1)(s-z+k)\Gamma(s-z)}{(s-z)(s-z+1)...(s-z+k-1)}\\ &= \lim_{s\to z-k}\frac{\Gamma(s-z+k+1)}{(s-z)(s-z+1)...(s-z+k-1)}\\ &= \frac{1}{(-k)(-k+1)...(-1)} = \frac{(-1)^k}{k!}.\end{aligned}$$

Then by the use of Cauchy's residue theorem, we have

$$\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)] = \Gamma(\alpha+1)\sum_{k=0}^{z-(a+1)} f(z-k)\frac{\Gamma(2z-k+\gamma-\alpha)\nabla x_{\gamma+1}(z-k)}{\Gamma(\alpha+1-k)\Gamma(2z+\gamma+1-k)}\frac{(-1)^k}{k!}.$$

Since

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)} = \alpha(\alpha-1)...(\alpha-k+1),$$

and

$$\alpha(\alpha-1)...(\alpha-k+1)(-1)^k = (-\alpha)_k,$$

therefore, we get

$$\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)] = \sum_{k=0}^{z-(a+1)} f(z-k)\frac{\Gamma(2z-k+\gamma-\alpha)\nabla x_{\gamma+1}(z-k)}{\Gamma(2z+\gamma+1-k)}\frac{(-\alpha)_k}{k!}.$$

From (74), in the case of the quadratic lattices (6), we have

$$\begin{aligned}\nabla^{\alpha}_{\gamma-\alpha+1}f(z) &= \frac{[\Gamma(\alpha+1)]_q}{2\pi i}\frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\oint_{\Gamma}\frac{f(s)\nabla x_{\gamma+1}(s)ds}{[x_\gamma(s)-x_\gamma(z)]^{(\alpha+1)}}\\ &= \frac{[\Gamma(\alpha+1)]_q}{2\pi i}\frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\oint_{\Gamma}\frac{f(s)\nabla x_{\gamma+1}(s)[\Gamma(s-z)]_q[\Gamma(s+z+\gamma-\alpha)]_q ds}{[\Gamma(s-z+\alpha+1)]_q[\Gamma(s+z+\gamma+1)]_q}\end{aligned} \tag{77}$$

From the assumption of **Definition** 17, $[\Gamma(s-z)]_q$ has simple poles at $s=z-k, k=0,1,2,...,z-(a+1)$. The residue of $[\Gamma(s-z)]_q$ at the point $s-z=-k$ is

$$\begin{aligned}
&\lim_{s\to z-k}(s-z+k)[\Gamma(s-z)]_q\\
&=\lim_{s\to z-k}\frac{s-z+k}{[s-z+k]_q}[s-z+k]_q[\Gamma(s-z)]_q\\
&=\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{\log q}\lim_{s\to z-k}[s-z+k]_q[\Gamma(s-z)]_q\\
&=\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{\log q}\lim_{s\to z-k}\frac{[s-z]_q[s-z+1]_q...[s-z+k-1]_q[s-z+k]_q[\Gamma(s-z)]_q}{(s-z)(s-z+1)...(s-z+k-1)}\\
&=\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{\log q}\lim_{s\to z-k}\frac{[\Gamma(s-z+k+1)]_q}{[s-z]_q[s-z+1]_q...[s-z+k-1]_q}\\
&=\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{\log q}\frac{1}{[-k]_q[-k+1]_q...[-1]_q}=\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{\log q}\frac{(-1)^k}{[k]_q!}.
\end{aligned}$$

Then by the use of Cauchy's residue theorem, we have

$$\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)]=[\Gamma(\alpha+1)]_q\sum_{k=0}^{z-(a+1)}f(z-k)\frac{[\Gamma(2z-k+\gamma-\alpha)]_q\nabla x_{\gamma+1}(z-k)}{[\Gamma(\alpha+1-k)]_q[\Gamma(2z+\gamma+1-k)]_q}\frac{(-1)^k}{[k]_q!}.$$

Since

$$\frac{[\Gamma(\alpha+1)]_q}{[\Gamma(\alpha+1-k)]_q}=[\alpha]_q[\alpha-1]_q...[\alpha-k+1]_q,$$

and

$$[\alpha]_q[\alpha-1]_q...[\alpha-k+1](-1)^k=([-\alpha])_k,$$

therefore, we obtain that

$$\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)]=\sum_{k=0}^{z-(a+1)}f(z-k)\frac{[\Gamma(2z-k+\gamma-\alpha)]_q\nabla x_{\gamma+1}(z-k)}{[\Gamma(2z+\gamma+1-k)]_q}\frac{([-\alpha]_q)_k}{k!}.$$

□

So far, with respect to the definition of the R-L fractional difference on non-uniform lattices, we have given two kinds of definitions, such as **Definition** 11 or **Definition** 12 in section 4 and **Definition** 17 or **Theorem** 18 in section 6 through two different ideas and methods. Now let's compare **Definition** 12 in section 4 and **Theorem** 18 in section 6.

Here follows a theorem connecting the R-L fractional difference (63) and the complex generalization of fractional difference (74) :

Theorem 19. *For any $\alpha\in R^+$, let Γ be a simple closed positively oriented contour. If $f(s)$ is analytic in simple connected domain D bounded by Γ, assume that z is any nonzero point inside D, $a+1$ is a point inside D, such that $z-a\in N$, then the complex generalization fractional integral (74) equals the R-L fractional defference (62) or (63):*

$$\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)]=\sum_{k=a+1}^{z}\frac{[x_{\gamma-\alpha}(z)-x_{\gamma-\alpha}(k-1)]^{(-\alpha-1)}}{[\Gamma(-\alpha)]_q}f(k)\nabla x_{\gamma+1}(k).$$

Proof. By **Theorem** 18, we have

$$\begin{aligned}\nabla^{\alpha}_{\gamma+1-\alpha}[f(z)] &= \sum_{k=0}^{z-(a+1)} \frac{([-\alpha]_q)_k}{[k]_q!} \frac{[\Gamma(2z-k+\gamma-\alpha)]_q}{[\Gamma(2z-k+\gamma+1)]_q} f(z-k)\nabla x_{\gamma+1}(z-k).\\ &= \sum_{k=0}^{z-(a+1)} \frac{[\Gamma(k-\alpha)]_q}{[\Gamma(-\alpha)]_q[\Gamma(k+1)]_q} \frac{[\Gamma(2z-k+\gamma-\alpha)]_q}{[\Gamma(2z-k+\gamma+1)]_q} f(z-k)\nabla x_{\gamma+1}(z-k)\\ &= \sum_{k=0}^{z-(a+1)} \frac{[x_{\gamma-\alpha}(z)-x_{\gamma-\alpha}(z-k-1)]^{(-\alpha-1)}}{[\Gamma(-\alpha)]_q} f(z-k)\nabla x_{\gamma+1}(z-k)\\ &= \sum_{k=a+1}^{z} \frac{[x_{\gamma-\alpha}(z)-x_{\gamma-\alpha}(k-1)]^{(-\alpha-1)}}{[\Gamma(-\alpha)]_q} f(k)\nabla x_{\gamma+1}(k).\end{aligned}$$

So that the two **Theorem** 12 and **Theorem** 18 are consistent. □

Set $\alpha=\gamma$ in **Theorem** 18, we obtain

Corollary 20. *Assume that conditions of **Definition** 17 and **Theorem** 18 hold, then*

$$\begin{aligned}\nabla^{\gamma}_{1}[f(z)] &= \frac{[\Gamma(\gamma+1)]_q}{2\pi i} \frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \oint_{\Gamma} \frac{f(s)\nabla x_{\gamma+1}(s)ds}{[x_{\gamma}(s)-x_{\gamma}(z)^{(\gamma+1)}}\\ &= \sum_{k=0}^{z-(a+1)} f(z-k)\frac{[\Gamma(2z+\mu-k)]_q\nabla x_{\gamma+1}(z-k)}{[\Gamma(2z+\gamma+\mu+1-k)]_q} \frac{([-\gamma]_q)_k}{[k]_q!}.\end{aligned}$$

where Γ enclosed the simple poles $s=z,z-1,...,a+1$ in the complex plane.

Remark 21. *When $\gamma=n\in N^{+}$,we have*

$$\begin{aligned}\nabla^{n}_{1}[f(z)] &= \frac{[\Gamma(n+1)]_q}{2\pi i} \frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \oint_{\Gamma} \frac{f(s)\nabla x_{\gamma+1}(s)ds}{[x_{n}(s)-x_{n}(z)^{(n+1)}}\\ &= \sum_{k=0}^{n} f(z-k)\frac{[\Gamma(2z+\mu-k)]_q\nabla x_{n+1}(z-k)}{[\Gamma(2z+n+\mu+1-k)]_q} \frac{([-n]_q)_k}{k!},\end{aligned} \tag{78}$$

where Γ enclosed the simple poles $s=z,z-1,...,z-n$ in the complex plane.

*This is consistent with **Definition** 2 proposed by Nikiforov. A, Uvarov. V, Suslov. S in [22]*

.

Finally, for complex integral of Riemann-Liouville fractional difference on non-uniform lattices, we can establish an analogue of Cauchy Beta formula on non-uniform lattices, which is also of independent importance:

Theorem 22. *(Cauchy Beta formula) Let $\alpha,\beta\in C$, and assume that*

$$\oint_{\Gamma} \Delta_t\{\frac{1}{[x_{\beta}(z)-x_{\beta}(t)]^{(\beta)}} \frac{1}{[x_{-1}(t)-x_{-1}(a)]^{(\alpha-1)}}\}dt=0,$$

then

$$\frac{1}{2\pi i}\frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \oint_{\Gamma} \frac{[\Gamma(\beta+1)]_q}{[x_{\beta}(z)-x_{\beta}(t)]^{(\beta+1)}} \frac{[\Gamma(\alpha)]_q\Delta y_{-1}(t)dt}{[x(t)-x(a)]^{(\alpha)}} = \frac{[\Gamma(\alpha+\beta)]_q}{[x_{\beta}(z)-x_{\beta}(a)]^{(\alpha+\beta)}},$$

where Γ be a simple closed positively oriented contour, a lies inside C.

In order to prove **Theorem** 22, we first give a lemma.

Lemma 23. *For any α, β, then we have*

$$[1-\alpha]_q[x_\beta(z)-x_\beta(t-\beta)]+[\beta]_q[x_{1-\alpha}(t+\alpha-1)-x_{1-\alpha}(a)]$$
$$=[1-\alpha]_q[x_\beta(z)-x_\beta(a+1-\alpha-\beta)]+[\alpha+\beta-1]_q[x(t)-x(a+1-\alpha)]. \tag{79}$$

Proof. (79) is equivalent to

$$[\alpha+\beta-1]_q x(t)+[1-\alpha]_q x_\beta(t-\beta)-[\beta]_q x_{1-\alpha}(t+\alpha-1)$$
$$=[\alpha+\beta-1]_q x(a+1-\alpha)+[1-\alpha]_q x_\beta(a+1-\alpha-\beta)-[\beta]_q x_{1-\alpha}(a). \tag{80}$$

Set $\alpha-1=\widetilde{\alpha}$, then (80) can be written as

$$[\widetilde{\alpha}+\beta]_q x(t)-[\widetilde{\alpha}]_q x_{-\beta}(t)-[\beta]_q x_{\widetilde{\alpha}}(t)$$
$$=[\widetilde{\alpha}+\beta]_q x(a-\widetilde{\alpha})-[\widetilde{\alpha}]_q x_{-\beta}(a-\widetilde{\alpha})-[\beta]_q x_{\widetilde{\alpha}}(a-\widetilde{\alpha}). \tag{81}$$

By the use of **Lemma** 9, then Eq. (81) holds, and then Eq. (79) holds. □

Proof of Theorem 22: Set

$$\rho(t)=\frac{1}{[x_\beta(z)-x_\beta(t)]^{(\beta+1)}}\frac{1}{[x(t)-x(a)]^{(\alpha)}},$$

and

$$\sigma(t)=[x_{\alpha-1}(t+\alpha-1)-x_{\alpha-1}(a)][x_\beta(z)-x_\beta(t)].$$

Since

$$[x_\beta(z)-x_\beta(t)]^{(\beta+1)}=[x_\beta(z)-x_\beta(t-1)]^{(\beta)}[x_\beta(z)-x_\beta(t)],$$

and

$$[x(t)-x(a)]^{(\alpha)}=[x_{-1}(t)-x_{-1}(a)]^{(\alpha-1)}[x_{1-\alpha}(t+\alpha-1)-x_{1-\alpha}(a)],$$

these reduce to

$$\sigma(t)\rho(t)=\frac{1}{[x_\beta(z)-x_\beta(t-1)]^{(\beta)}}\frac{1}{[x_{-1}(t)-x_{-1}(a)]^{(\alpha-1)}}.$$

Making use of

$$\Delta_t[f(t)g(t)]=g(t+1)\Delta_t[f(t)]+f(t)\Delta_t[g(t)],$$

where

$$f(t)=\frac{1}{[x_{-1}(t)-x_{-1}(a)]^{(\alpha-1)}},\, g(t)=\frac{1}{[x_\beta(z)-x_\beta(t-1)]^{(\beta)}},$$

and

$$\frac{\Delta_t}{\Delta x_{-1}(t)}\{\frac{1}{[x_{-1}(t)-x_{-1}(a)]^{(\alpha-1)}}\}=\frac{[1-\alpha]_q}{[x(t)-x(a)]^{(\alpha)}},$$

$$\begin{aligned}
&\frac{\Delta_t}{\Delta x_{-1}(t)}\{\frac{1}{[x_\beta(z)-x_\beta(t-1)]^{(\beta)}}\}\\
&=\frac{\nabla_t}{\nabla x_1(t)}\{\frac{1}{[x_\beta(z)-x_\beta(t)]^{(\beta)}}\}\\
&=\frac{[\beta]_q}{[x_\beta(z)-x_\beta(t)]^{(\beta+1)}}.
\end{aligned}$$

then, we have

$$\begin{aligned}
&\frac{\Delta_t}{\Delta x_{-1}(t)}\{\sigma(t)\rho(t)\}\\
&=\frac{1}{[x_\beta(z)-x_\beta(t)]^{(\beta)}}\frac{[1-\alpha]_q}{[x(t)-x(a)]^{(\alpha)}}+\\
&+\frac{1}{[x_{-1}(t)-x_{-1}(a)]^{(\alpha-1)}}\frac{[\beta]_q}{[x_\beta(z)-x_\beta(t)]^{(\beta+1)}}\\
&=\{[1-\alpha]_q[x_\beta(z)-x_\beta(t-\beta)]+[\beta]_q[x_{1-\alpha}(t+\alpha-1)-x_{1-\alpha}(a)]\}\\
&\times\frac{1}{[x(t)-x(a)]^{(\alpha)}}\frac{1}{[x_\beta(z)-x_\beta(t)]^{(\beta+1)}}\\
&=\tau(t)\rho(t),
\end{aligned}$$

where

$$\tau(t)=[1-\alpha]_q[x_\beta(z)-x_\beta(t-\beta)]+[\beta]_q[x_{1-\alpha}(t+\alpha-1)-x_{1-\alpha}(a)],$$

this is due to

$$[x_\beta(z)-x_\beta(t)]^{(\beta+1)}=[x_\beta(z)-x_\beta(t)]^{(\beta)}[x_\beta(z)-x_\beta(t-\beta)].$$

From **Proposition** 5 one has

$$\begin{aligned}
&\frac{\Delta_t}{\Delta x_{-1}(t)}\{\sigma(t)\rho(t)\}\\
&=\{[1-\alpha]_q[x_\beta(z)-x_\beta(a+1-\alpha-\beta)]+[\alpha+\beta-1]_q[x(t)-x(a+1-\alpha)]\}\\
&\cdot\frac{1}{[x_\beta(z)-x_\beta(t)]^{(\beta+1)}}\frac{1}{[x(t)-x(a)]^{(\alpha)}},
\end{aligned}$$

or

$$\begin{aligned}
&\Delta_t\{\sigma(t)\rho(t)\}\\
&=\{[1-\alpha]_q[x_\beta(z)-x_\beta(a+1-\alpha-\beta)]+[\alpha+\beta-1]_q[x(t)-x(a+1-\alpha)]\}\\
&\cdot\frac{1}{[x_\beta(z)-x_\beta(t)]^{(\beta+1)}}\frac{1}{[x(t)-x(a)]^{(\alpha)}}\Delta x_{-1}(t).
\end{aligned}\tag{82}$$

Set

$$I(\alpha)=\frac{1}{2\pi i}\frac{\log q}{q^{\frac12}-q^{-\frac12}}\oint_\Gamma\frac{1}{[x_\beta(z)-x_\beta(t)]^{(\beta+1)}}\frac{\nabla y_1\,(t)\,dt}{[x(t)-x(a)]^{(\alpha)}},\tag{83}$$

and

$$I(\alpha-1)=\frac{1}{2\pi i}\frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\oint_{\Gamma}\frac{1}{[x_{\beta}(z)-x_{\beta}(t)]^{(\beta+1)}}\frac{\nabla y_1\,(t)\,dt}{[x(t)-x(a)]^{(\alpha-1)}}.$$

Since

$$[x(t)-x(a)]^{(\alpha-1)}[x(t)-x(a+1-\alpha)]=[x(t)-x(a)]^{(\alpha)},$$

then

$$I(\alpha-1)=\frac{1}{2\pi i}\frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\oint_{\Gamma}\frac{1}{[x_{\beta}(z)-x_{\beta}(t)]^{(\beta+1)}}\frac{[x(t)-x(a+1-\alpha)]\nabla x_1\,(t)\,dt}{[x(t)-x(a)]^{(\alpha)}}.$$

Integrating both sides of equation (82), then we have

$$\begin{aligned}\oint_{\Gamma}\Delta_t\{\sigma(t)\rho(t)\}dt=&[1-\alpha]_q[x_{\beta}(z)-x_{\beta}(a+1-\alpha-\beta)]I(\alpha)\\&-[\alpha+\beta-1]_qI(\alpha-1).\end{aligned}$$

If

$$\oint_{\Gamma}\Delta_t\{\sigma(t)\rho(t)\}dt=0,$$

then we obtain that

$$\frac{I(\alpha-1)}{I(\alpha)}=\frac{[\alpha-1]_q}{[\alpha+\beta-1]_q}[y_{\beta}(z)-y_{\beta}(a+1-\alpha-\beta)].$$

That is

$$\frac{I(\alpha-1)}{I(\alpha)}=\frac{\frac{[\Gamma(\alpha+\beta-1)]_q}{[\Gamma(\alpha-1)]_q}\frac{1}{[x_{\beta}(z)-x_{\beta}(a)]^{(\alpha+\beta-1)}}}{\frac{[\Gamma(\alpha+\beta)]_q}{[\Gamma(\alpha)]_q}\frac{1}{[x_{\beta}(z)-x_{\beta}(a)]^{(\alpha+\beta)}}}.\tag{84}$$

From (84), we set

$$I(\alpha)=k\frac{[\Gamma(\alpha+\beta)]_q}{[\Gamma(\alpha)]_q}\frac{1}{[x_{\beta}(z)-x_{\beta}(a)]^{(\alpha+\beta)}},\tag{85}$$

where k is undetermined.

Set $\alpha=1$, one has

$$I(1)=k[\Gamma(1+\beta)]_q\frac{1}{[x_{\beta}(z)-x_{\beta}(a)]^{(1+\beta)}},\tag{86}$$

and from (83) and generalized Cauchy residue theorem, one has

$$\begin{aligned}I(1)&=\frac{1}{2\pi i}\frac{\log q}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\oint_{\Gamma}\frac{1}{[x_{\beta}(z)-x_{\beta}(t)]^{(\beta+1)}}\frac{\nabla x_1\,(t)\,dt}{[x(t)-x(a)]^{(1)}}\\&=\frac{1}{2\pi i}\oint_{\Gamma}\frac{1}{[x_{\beta}(z)-x_{\beta}(t)]^{(\beta+1)}}\frac{x'\,(t)\,dt}{[x(t)-x(a)]}\\&=\frac{1}{[x_{\beta}(z)-x_{\beta}(a)]^{(\beta+1)}},\end{aligned}\tag{87}$$

From (86) and (87), we get

$$k=\frac{1}{[\Gamma(1+\beta)]_q}.$$

Therefore, we obtain that

$$I(\alpha)=\frac{[\Gamma(\alpha+\beta)]_q}{[\Gamma(\beta+1)]_q[\Gamma(\alpha)]_q}\frac{1}{[x_\beta(z)-x_\beta(a)]^{(\alpha+\beta)}},$$

and **Theorem** 22 is completed.

References

[1] R P Agarwal. *Certain fractional q-integral and q-derivative*, Proc Camb Phil Soc, 1969, 66: 365-370.

[2] W A Al-Salam. *Some fractional q-integral and q-derivatives*, Proc Edinb Math Soc v2, 1966/1967, 15: 135-140.

[3] G A Anastassiou. *Nabla discrete fractional calculus and nalba inequalities*, Mathematical and Computer Modelling, 2010, 51(5-6): 562-571.

[4] M H Annaby, Z S Mansour. *q-Fractional Calculus and Equations*, Springer, 2012.

[5] G E Andrews, R Askey, R Roy. *Special functions*, Encyclopedia of Mathematics and its Applications, 71, Cambridge University Press, Cambridge, 1999.

[6] R Askey, J A Wilson. *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem Amer Math Soc, 1985, 319.

[7] N M Atakishiyev, S K Suslov. *Difference hypergeometric functions*, In: Progress in Approximation Theory, Springer New York, 1992, pp: 1-35.

[8] F M Atici, P W Eloe. *Discrete fractional calculus with the nable operator*, Electronic Journal of Qualitative Theory of Differential Equations, Spec Ed I, 2009, 2009(3): 1-12.

[9] J Baoguo, L Erbe, A Peterson. *Two monotonicity results for nabla and delta fractional differences,* Arch Math (Basel), 2015, 104: 589-597.

[10] J F Cheng. *Theory of Fractional Difference Equations*, Xiamen University Press, Xiamen, 2011. (in Chinese)

[11] L K Jia, J F Cheng, Z S Feng. *A q-analogue of Kummer's equation*, Electron J Differential Equations, 2017, 2017(31): 1-20.

[12] J F Cheng, W Z Dai. *Higer-order fractional Green and Gauss formulas*, J Math Anal Appl, 2018, 462(1): 157-171.

[13] J F Cheng, L K Jia. *Hypergeometric Type Difference Equations on Nonuniform Lattices: Rodrigues Type Representation for the Second Kind Solution*, Acta Mathematics Scientia, 2019, 39A(4): 875-893.

[14] J F Cheng, L K Jia. *Generalizations of Rodrigues type formulas for hypergeometric difference equations on nonuniform lattices,* Journal of Difference Equations and Applications, 2020, 26(4): 435-457.

[15] J F Cheng. *On the Complex Difference Equation of Hypergeometric Type on Non-uniform Lattices*, Acta Mathematica Sinica, English Series, 2020, 36(5): 487-511.

[16] J F Cheng. *Hypergeometric Equations and Discrete Fractionl Calculus on Non-uniform Lattices*, Science Press, Beijing, 2021. (in Chinese)

[17] J B Diaz, T J Osler. *Differences of fractional order*, Math Comp, 1974, 28(125): 185-202.

[18] Rui A C Ferreira, Delfim F M Torres. *Fractional $h-$differences arising from the calculus of variations*, Appl Anal Discrete Math, 2011, 5: 110-121.

[19] C Goodrich, A C Peterson. *Discrete Fractional Calculus*, Springer International Publishing, Switzerland Springer, Switzerland, 2015.

[20] H L Gray, N F Zhang. *On a new definition of the fractional difference*, Mathematics of Computation, 1988, 50(182): 513-529.

[21] M E H Ismail, R Zhang. *Diagonalization of certain integral operators*, Advance in Math Soc, 1994, 109 (1): 1-33.

[22] A F Nikiforov, S K Suslov, V B Uvarov. *Classical orthogonal polynomials of a discrete variable*, Translated from the Russian, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1991.

[23] A F Nikiforov, V B Uvarov. *Special functions of mathematical physics: A unified introduction with applications*, Translated from the Russian by Ralph P Boas, Birkhauser Verlag, Basel, 1988.

[24] M Rahman, S K Suslov. *The Pearson equation and the Beta Integrals*, SIAM J Math Anal, 1994, 25(2): 646-693.

[25] S K Suslov. *On the theory of difference analogues of special functions of hypergeometric type*, Russian Math Surveys , 1989, 44(2): 227-278.

Department of Mathematics, Xiamen University, Xiamen 361005, China.
Email: jfcheng@xmu.edu.cn