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The rate of convergence on fractional power dissipative operator on some sobolev type spaces

CAO Zhen-bin   WANG Meng

Abstract. In [3], Chen, Deng, Ding and Fan proved that the fractional power dissipative operator is bounded on Lebesgue spaces $L^p(\mathbb{R}^n)$, Hardy spaces $H^p(\mathbb{R}^n)$ and general mixed norm spaces, which implies almost everywhere convergence of such operator. In this paper, we study the rate of convergence on fractional power dissipative operator on some sobolev type spaces.

§1 Introduction

We consider the fractional power dissipative equation

$$\begin{cases} u_t + (-\Delta)^\alpha u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 2$, $\alpha > 0$, $f : \mathbb{R}^n \to \mathbb{C}$ is the given initial data, and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. This interesting equation instantiates the heat equation if $\alpha = 1$ and the Poisson's equation if $\alpha = 1/2$. The solution of the above equation can be written by

$$u(x,t) = e^{-t(-\Delta)^\alpha} f(x) = (e^{-t|\xi|^{2\alpha}} \widehat{f})^\vee (x),$$

where $\widehat{f}$ is the Fourier transform of f, and $f^\vee$ is the inverse Fourier transform of f. We can also write $u(x,t)$ as a convolution operator:

$$u(x,t) = K_t^\alpha * f(x),$$

where

$$K_t^\alpha(x) = \frac{1}{t^{\frac{n}{2\alpha}}} K^\alpha\left(\frac{x}{t^{\frac{1}{2\alpha}}}\right),$$

and

$$K^\alpha(x) = \int_{\mathbb{R}^n} e^{-|\xi|^{2\alpha}} e^{2\pi i x \cdot \xi} d\xi.$$

In [10], Miao, Yuan and Zhang proved that $K^\alpha(x)$ satisfies that

$$|K^\alpha(x)| \lesssim (1+|x|)^{-(n+2\alpha)},$$

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for all $\alpha > 0$, which immediately implies $e^{-t(-\Delta)^\alpha} f$ is bounded on Lebesgue spaces $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$. Also, by the multiplier theorem of Calderón and Torchinsky [1], $e^{-t(-\Delta)^\alpha} f$ is bounded on Hardy spaces $H^p(\mathbb{R}^n)$ for all $0 < p < \infty$. We can see [3] on the boundedness of $e^{-t(-\Delta)^\alpha} f$ on other spaces. Also we can see [6,8-11,15] on the fractional power dissipative equations with different potentials.

For $\lambda > 0$, let $I_{-\lambda}$ denote the Riesz potential, i.e.

$$I_{-\lambda}(f) = (|\xi|^\lambda \widehat{f}(\xi))^\vee.$$

Let $H^p(\mathbb{R}^n)$ denote the Hardy space, which is defined by

$$\|f\|_{H^p(\mathbb{R}^n)} = \left\| \sup_{t>0} |(P_t * f)(x)| \right\|_{L^p(\mathbb{R}^n)},$$

where P is the Poisson kernel. A basic result is that $L^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$ are equivalent when $p > 1$. We further introduce the atomic decomposition: if $g \in H^p(\mathbb{R}^n)$, then

$$g = \sum_j c_j a_j, \quad \|g\|^p_{H^p(\mathbb{R}^n)} \sim \sum_j |c_j|^p < \infty,$$

where each a_j is a $(p, 2)$-atom. Here a is a $(p, 2)$-atom if there exists a cube Q such that
(1) a is supported in Q,
(2) $\|a\|_{L^2} \le |Q|^{1/2-1/p}$,
(3) $\int x^\gamma a(x)dx = 0$ for all γ with $|\gamma| \le [n/p - n]$,
where $[x]$ means the largest integer that is not more than x. In the following argument, we mainly discuss the Sobolev type spaces $I_{-\lambda}(H^p)(\mathbb{R}^n)$. $I_{-\lambda}(H^p)(\mathbb{R}^n)$ is defined as the space of all f satisfying $I_{-\lambda} f \in H^p(\mathbb{R}^n)$. For such function f, we define

$$\|f\|_{I_{-\lambda}(H^p)(\mathbb{R}^n)} = \|I_{-\lambda}(f)\|_{H^p(\mathbb{R}^n)}.$$

This space is the classical homogeneous Sobolev spaces for $p \ge 1$ and the Hardy-Sobolev spaces for $0 < p \le 1$ (see [7,14]).

We have already known the fractional power dissipative operator and related maximal operator are bounded on $H^p(\mathbb{R}^n)$, which implies for each $f \in H^p(\mathbb{R}^n)$, $\lim_{t\to 0} e^{-t(-\Delta)^\alpha} f = f$ *a.e.* Then we ask: if f has more regularity, can we obtain better estimates? Our result is the following.

Theorem 1.1. *Let $\alpha > 0$, $p > 0$ and $0 \le \lambda < 2\alpha$. If $f \in I_{-\lambda}(H^p)(\mathbb{R}^n)$, then*

$$e^{-t(-\Delta)^\alpha} f(x) - f(x) = o(t^{\frac{\lambda}{2\alpha}}) \quad a.e. \text{ as } t \to 0.$$

Firstly we consider the case for $\lambda = 0$, by the method from [13], Theorem 1.1 can be reduced to

$$\left\| \sup_{t>0} \left| e^{-t(-\Delta)^\alpha} f - f \right| \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}.$$

This result is correct since the fractional power dissipative operator is bounded on $H^p(\mathbb{R}^n)$ and

$$\|f\|_{L^p(\mathbb{R}^n)} = \| \lim_{t\to 0} (e^{-t|x|^2} * f)\|_{L^p(\mathbb{R}^n)} \le \|f\|_{H^p(\mathbb{R}^n)}$$

holds for all $0 < p < \infty$.

From now on, let $0 < \lambda < 2\alpha$. We repeat the method of [13], then Theorem 1.1 can be reduced to the following result.

Theorem 1.2. *Let $\alpha > 0$, $p > 0$ and $0 < \lambda < 2\alpha$. If $f \in I_{-\lambda}(H^p)(\mathbb{R}^n)$, then*

$$\left\|\sup_{t>0} t^{-\frac{\lambda}{2\alpha}} \left|e^{-t(-\Delta)^\alpha} f - f\right|\right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{I_{-\lambda}(H^p)(\mathbb{R}^n)}.$$

We will prove this theorem in section 2. We decompose the integral in Theorem 1.2 into two parts. As the argument in Chen [2], the estimate of kernel on each part is core of our proof, which is left at Lemma 2.2.

Throughout this paper, $A \lesssim B$ means that there exists a constant $C > 0$ independent of all essential variables such that $A \leq CB$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. The space of all infinitely differentiable functions on $\mathbb{R}^n$ is denoted by $\mathcal{C}^\infty(\mathbb{R}^n)$. The space of $\mathcal{C}^\infty$ functions with compact support is denoted by $\mathcal{C}_0^\infty(\mathbb{R}^n)$. The space of $\mathcal{C}^\infty$ functions with all derivatives rapidly decreasing is denoted by $\mathcal{S}(\mathbb{R}^n)$. Let $\beta = (\beta_1, ..., \beta_n)$ where $\beta_j \in \mathbb{Z}$ and $\beta_j \geq 0$ for $0 \leq j \leq n$, $\partial^\beta f$ means the derivative $\partial^{\beta_1} ... \partial^{\beta_n} f$.

§2 Proof of Theorem 1.2

There is a standard result of Stein [12] on $H^p(\mathbb{R}^n)$. We state it here.

Lemma 2.1. [12] *Let $0 < p \leq 1$. Suppose that a function ζ vanishes at ∞ and satisfies*

$$\left|\partial^\beta \zeta(x)\right| \lesssim (1 + |x|)^{-A},$$

for $|\beta| = [n(1/p - 1)] + 1$ and $A > n/p$. Then there is a constant $C > 0$, for any $f \in H^p(\mathbb{R}^n)$,

$$\left\|\sup_{R>0} |f * \zeta_R|\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)},$$

where $\zeta_R(x) = R^{-n}\zeta(x/R)$ for $R > 0$.

Write

$$\begin{aligned}
& t^{-\frac{\lambda}{2\alpha}} \left|e^{-t(-\Delta)^\alpha} f - f\right| \\
&= t^{-\frac{\lambda}{2\alpha}} \left|\int_{\mathbb{R}^n} (e^{-t|\xi|^{2\alpha}} - 1)\widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi\right| \\
&= t^{-\frac{\lambda}{2\alpha}} \left|\int_{\mathbb{R}^n} (e^{-|t^{\frac{1}{2\alpha}}\xi|^{2\alpha}} - 1)\frac{1}{|\xi|^\lambda}\widehat{f}(\xi)|\xi|^\lambda e^{2\pi i \xi \cdot x} d\xi\right| \\
&= \left|\int_{\mathbb{R}^n} (e^{-|t^{\frac{1}{2\alpha}}\xi|^{2\alpha}} - 1)\frac{1}{|t^{\frac{1}{2\alpha}}\xi|^\lambda}\widehat{f}(\xi)|\xi|^\lambda e^{2\pi i \xi \cdot x} d\xi\right| \\
&= \left|I_{-\lambda}(f) * (g^{\alpha,\lambda})_t\right|,
\end{aligned}$$

where

$$g^{\alpha,\lambda}(x) = \int_{\mathbb{R}^n} (e^{-|\xi|^{2\alpha}} - 1)\frac{1}{|\xi|^\lambda} e^{2\pi i \xi \cdot x} d\xi.$$

To prove Theorem 1.2, it suffices to show

$$\left\|\sup_{t>0} \left|I_{-\lambda}(f) * (g^{\alpha,\lambda})_t\right|\right\|_{L^p} \lesssim \|I_{-\lambda}(f)\|_{H^p}.$$

Set $h = I_{-\lambda}(f)$, the above inequality is reduced to

$$\left\|\sup_{t>0} |h * (g^{\alpha,\lambda})_t|\right\|_{L^p} \lesssim \|h\|_{H^p}.$$

Let $\phi_0, \phi_\infty \in \mathcal{C}^\infty(\mathbb{R}^n)$ be radial functions satisfying the following conditions:

$$\phi_0(\xi) = \begin{cases} 1 & |\xi| \le 1/2, \\ 0 & |\xi| \ge 1; \end{cases} \qquad \phi_\infty(\xi) = \begin{cases} 0 & |\xi| \le 1/2, \\ 1 & |\xi| > 1; \end{cases}$$

and $\phi_0 + \phi_\infty = 1$. Then we divide the integral $g^{\alpha,\lambda}$ to $g_0^{\alpha,\lambda}$ and $g_\infty^{\alpha,\lambda}$:

$$g_0^{\alpha,\lambda}(x) = \int_{\mathbb{R}^n} \phi_0(\xi)(e^{-|\xi|^{2\alpha}} - 1)\frac{1}{|\xi|^\lambda} e^{2\pi i\xi\cdot x} d\xi,$$

and

$$g_\infty^{\alpha,\lambda}(x) = \int_{\mathbb{R}^n} \phi_\infty(\xi)(e^{-|\xi|^{2\alpha}} - 1)\frac{1}{|\xi|^\lambda} e^{2\pi i\xi\cdot x} d\xi.$$

Firstly we consider $g_0^{\alpha,\lambda}$. Via the similar method of Lemma 2.1 in [4] (we also can see [5]), one concludes: for $0 < \lambda < 2\alpha$, all $\beta = (\beta_1, ..., \beta_n)$ with $\beta_j \ge 0$ for $0 \le j \le n$,

$$|\partial^\beta g_0^{\alpha,\lambda}(x)| \lesssim \frac{1}{(1+|x|)^{n+|\beta|+2\alpha-\lambda}}. \tag{1}$$

When $p > 1$, by (1),

$$|g_0^{\alpha,\lambda}(x)| \lesssim \frac{1}{(1+|x|)^{n+2\alpha-\lambda}},$$

which is controlled by one integrable radially decreasing function due to $0 < \lambda < 2\alpha$. Therefore

$$\left\|\sup_{t>0} |h * (g_0^{\alpha,\lambda})_t|\right\|_{L^p} \lesssim \|M(h)\|_{L^p} \lesssim \|h\|_{L^p} \sim \|h\|_{H^p}.$$

Here we use that L^p is equivalent to H^p when $p > 1$. When $0 < p \le 1$, by (1),

$$|\partial^\beta g_0^{\alpha,\lambda}(x)| \lesssim \frac{1}{(1+|x|)^{n+|\beta|+2\alpha-\lambda}}.$$

Considering $n + |\beta| + 2\alpha - \lambda > \frac{n}{p}$ when $|\beta| = [n(\frac{1}{p} - 1)] + 1$, by Lemma 2.1,

$$\left\|\sup_{t>0} |h * (g_0^{\alpha,\lambda})_t|\right\|_{L^p} \lesssim \|h\|_{H^p}.$$

Next we consider $g_\infty^{\alpha,\lambda}$. We have the following estimate (proof can be seen in the following Lemma 2.2):

$$|\partial^\beta g_\infty^{\alpha,\lambda}(x)| \lesssim \frac{1}{|x|^{n+|\beta|-\lambda+L}} \tag{2}$$

holds for all $L \ge 0$ so that $n + |\beta| - \lambda + L \ge 0$. When $p > 1$, by (2), $g_\infty^{\alpha,\lambda}$ is controlled by one integrable radially decreasing function, then

$$\left\|\sup_{t>0} |h * (g_\infty^{\alpha,\lambda})_t|\right\|_{L^p} \lesssim \|M(h)\|_{L^p} \lesssim \|h\|_{L^p} \sim \|h\|_{H^p}.$$

When $0 < p \le 1$, by the atomic decomposition of h, $h = \sum_j \lambda_j a_j$, where each a_j is a $(p, 2)$-atom and $\|h\|_{H^p}^p \sim \sum_j |\lambda_j|^p$. The properties of atom can be seen in section 1. Then

$$\begin{aligned} &\left\|\sup_{t>0} |h * (g_\infty^{\alpha,\lambda})_t|\right\|_{L^p}^p \\ \le &\left\|\sum_j |\lambda_j| \sup_{t>0} |a_j * (g_\infty^{\alpha,\lambda})_t|\right\|_{L^p}^p \\ \le &\sum_j |\lambda_j|^p \left\|\sup_{t>0} |a_j * (g_\infty^{\alpha,\lambda})_t|\right\|_{L^p}^p. \end{aligned}$$

So if for each atom a, we have

$$\left\|\sup_{t>0}|a*(g_\infty^{\alpha,\lambda})_t|\right\|_{L^p}\lesssim 1, \tag{3}$$

then

$$\left\|\sup_{t>0}|h*(g_\infty^{\alpha,\lambda})_t|\right\|_{L^p}\lesssim\left(\sum_j|\lambda_j|^p\right)^{1/p}\sim\|h\|_{H^p}.$$

We prove (3) in the last lemma. The proof of Theorem 1.2 is completed as all cases have been proved.

□

Finally, we give the proofs of (2) and (3).

Lemma 2.2. *For $\lambda>0$, $\beta=(\beta_1,...,\beta_n)$ with $\beta_j\ge 0$ for $0\le j\le n$, we have*

$$|\partial^\beta g_\infty^{\alpha,\lambda}(x)|\lesssim\frac{1}{|x|^{n+|\beta|-\lambda+L}}$$

holds for all $L\ge 0$ so that $n+|\beta|-\lambda+L\ge 0$.

Proof. Write

$$g_\infty^{\alpha,\lambda}(x)=\int_{\mathbb{R}^n}\frac{e^{-|\xi|^{2\alpha}}}{|\xi|^\lambda}\phi_\infty(\xi)e^{2\pi i\xi\cdot x}d\xi-\int_{\mathbb{R}^n}\frac{1}{|\xi|^\lambda}\phi_\infty(\xi)e^{2\pi i\xi\cdot x}d\xi.$$

We only estimate the second term that is denoted by $K^\lambda(x)$, then the first term can be treated by the same way.

As to $K^\lambda(x)$, we decompose it by the partition of unity,

$$1=\eta(\xi)+\sum_{j=0}^\infty\delta(2^{-j}\xi),$$

where η and δ are all in $\mathcal{S}(\mathbb{R}^n)$, supp $\eta\subset\{x\mid|x|\le 1\}$, supp $\delta\subset\{x\mid 1/2\le|x|\le 2\}$, so supp $\delta(2^{-j}\cdot)\subset\{x\mid 2^{j-1}\le|x|\le 2^{j+1}\}$ for $j\ge 1$. Noting the support of ϕ_∞, we have

$$K^\lambda(x)=\sum_{j=1}^\infty K_j^\lambda(x),$$

where

$$K_j^\lambda(x)=\int_{\mathbb{R}^n}\frac{1}{|\xi|^\lambda}\delta(2^{-j}\xi)e^{2\pi i\xi\cdot x}d\xi.$$

We claim:

$$|\partial^\beta K_j^\lambda(x)|\lesssim|x|^{-M}2^{j(n+|\beta|-\lambda-M)} \tag{4}$$

holds for all $M\ge 0$.

In fact, write

$$(-2\pi ix)^\gamma\partial^\beta K_j^\lambda(x)=\int_{\mathbb{R}^n}\partial_\xi^\gamma\left[(2\pi i\xi)^\beta\frac{\delta(2^{-j}\xi)}{|\xi|^\lambda}\right]e^{2\pi i\xi\cdot x}d\xi.$$

Using the support of $\delta(2^{-j}\cdot)$, we obtain

$$|x^\gamma\partial^\beta K_j^\lambda(x)|\lesssim\int_{\mathbb{R}^n}\delta(2^{-j}\xi)|\xi|^{|\beta|-\lambda-|\gamma|}d\xi\lesssim 2^{j(n+|\beta|-\lambda-|\gamma|)}. \tag{5}$$

There is a basic fact: for $\gamma=(\gamma_1,...,\gamma_n)$,

$$|x|^{|\gamma|}\le C_{n,\gamma}\sum_{|\beta|=|\gamma|}|x^\beta|$$

holds for all $x \neq 0$. Since γ is arbitrary, we take the supremum over all γ with $|\gamma| = M$, then (5) implies (4).

Since $K^\lambda(x) = \sum_{j=1}^\infty K_j^\lambda(x)$, it suffices to estimate $\sum_{j=1}^\infty |\partial^\beta K_j^\lambda(x)|$. When $0 < |x| \le 1$, we divide it to two parts: the first part $2^j \le |x|^{-1}$, the second part $2^j > |x|^{-1}$. For the first term, we use (4) when $M = 0$ to get

$$\sum_{2^j \le |x|^{-1}} |\partial^\beta K_j^\lambda(x)| \le \sum_{2^j \le |x|^{-1}} 2^{j(n+|\beta|-\lambda)},$$

which is $O(|x|^{-n-|\beta|+\lambda})$ when $n + |\beta| - \lambda > 0$, and $O(\log(|x|^{-1}) + 1)$ when $n + |\beta| - \lambda \le 0$. In either case we get the bound

$$O(|x|^{-n-|\beta|+\lambda-L}),$$

with the restrictions that $|x| \le 1$, $L \ge 0$, and $n + |\beta| - \lambda + L \ge 0$. For the second term, we choose $M > n + |\beta| - \lambda$ in (4), then we get

$$O(|x|^{-M}) \sum_{2^j > |x|^{-1}} 2^{j(n+|\beta|-\lambda-M)} = O(|x|^{-n-|\beta|+\lambda}),$$

so the bound is $O(|x|^{-n-|\beta|+\lambda-L})$ if $L \ge 0$. When $|x| \ge 1$, we choose $M > n+|\beta|-\lambda+L$, then from (4) we get the bound $|x|^{-M}$, which is $O(|x|^{-n-|\beta|+\lambda-L})$ for every L. □

Lemma 2.3. *Suppose that a is a $(p, 2)$-atom, then*

$$\left\| \sup_{t>0} |a * (g_\infty^{\alpha,\lambda})_t| \right\|_{L^p} \lesssim 1.$$

Proof: We assume that supp $a \subset Q$, where Q is a cube with the center of zero. Other cases are similar since this operator is a convolution operator (it commutes with translations). Then

$$\begin{aligned} &\left\| \sup_{t>0} |a * (g_\infty^{\alpha,\lambda})_t| \right\|_{L^p}^p \\ =& \int_{\mathbb{R}^n} \sup_{t>0} |a * (g_\infty^{\alpha,\lambda})_t|^p \\ =& \int_{2Q} \sup_{t>0} |a * (g_\infty^{\alpha,\lambda})_t|^p + \int_{(2Q)^c} \sup_{t>0} |a * (g_\infty^{\alpha,\lambda})_t|^p \\ =& I + II. \end{aligned}$$

For I, since $g_\infty^{\alpha,\lambda}$ is controlled by one integrable radially decreasing function, $T(f) = \sup_{t>0} |f * (g_\infty^{\alpha,\lambda})_t|$ is bounded on L^2. Then

$$\begin{aligned} I = \int_{2Q} |T(a)|^p &\le \left(\int_{2Q} 1\right)^{1-\frac{p}{2}} \left(\int_{2Q} T(a)^2\right)^{\frac{p}{2}} \\ &\lesssim |Q|^{1-\frac{p}{2}} \|a\|_{L^2}^p \le |Q|^{1-\frac{p}{2}} |Q|^{\frac{p}{2}-1} = 1. \end{aligned}$$

Next we start to estimate II. Set $N = [n(\frac{1}{p} - 1)]$. Considering the vanishing property of a, for some $\theta \in (0, 1)$ such that

$$\begin{aligned} &|a * (g_\infty^{\alpha,\lambda})_t| \\ =& \frac{1}{t^n} \left| \int_Q a(y) g_\infty^{\alpha,\lambda}(\frac{x-y}{t}) dy \right| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t^n}\left|\int_Q a(y)\left[g_\infty^{\alpha,\lambda}(\frac{x-y}{t}) - \sum_{|\beta|\le N}\partial^\beta g_\infty^{\alpha,\lambda}(\frac{x}{t})\frac{(-y/t)^\beta}{\beta!}\right]dy\right| \\
&= \frac{1}{t^n}\left|\int_Q a(y)\left[\sum_{|\beta|=N+1}\partial^\beta g_\infty^{\alpha,\lambda}(\frac{x-\theta y}{t})\frac{(-y/t)^\beta}{\beta!}\right]dy\right| \\
&\lesssim \frac{1}{t^n}\int_Q |a(y)||\frac{y}{t}|^{N+1}\frac{1}{|\frac{x}{t}|^{n+N-\lambda+L+1}}dy,
\end{aligned}
$$

where $|x-\theta y| \ge |x| - |y| \ge \frac{1}{2}|x|$ since $y \in Q$ and $x \in (2Q)^c$, and the last line is due to Lemma 2.2. We choose $L = \lambda > 0$, then

$$
|a * (g_\infty^{\alpha,\lambda})_t| \lesssim \frac{1}{|x|^{n+N+1}}\int_Q |a(y)||y|^{N+1}dy \lesssim \frac{|Q|^{1-\frac{1}{p}+\frac{N+1}{n}}}{|x|^{n+N+1}},
$$

hence

$$
\begin{aligned}
II &= \int_{(2Q)^c}\sup_{t>0}|a*(g_\infty^{\alpha,\lambda})_t|^p \\
&\lesssim \int_{(2Q)^c}\frac{|Q|^{p-1+\frac{p(N+1)}{n}}}{|x|^{p(n+N+1)}}dx \\
&\lesssim |Q|^{p-1+\frac{p(N+1)}{n}}\int_{(2Q)^c}\frac{1}{|x|^{p(n+N+1)}}dx.
\end{aligned}
$$

Since $p(n+N+1) > n$ when $N = [n(\frac{1}{p}-1)]$,

$$
II \lesssim |Q|^{p-1+\frac{p(N+1)}{n}}|Q|^{-p+1-\frac{p(N+1)}{n}} = 1.
$$

We finish the proof of this lemma.

□

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Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
Email: 11735002@zju.edu.cn, mathdreamcn@zju.edu.cn