Appl. Math. J. Chinese Univ.
2021, 36(3): 390-402

On a new fractional-order Logistic model with feedback control

Manh Tuan Hoang[1] A. M. Nagy[2,3,*]

Abstract. In this paper, we formulate and analyze a new fractional-order Logistic model with feedback control, which is different from a recognized mathematical model proposed in our very recent work. Asymptotic stability of the proposed model and its numerical solutions are studied rigorously. By using the Lyapunov direct method for fractional dynamical systems and a suitable Lyapunov function, we show that a unique positive equilibrium point of the new model is asymptotically stable. As an important consequence of this, we obtain a new mathematical model in which the feedback control variables only change the position of the unique positive equilibrium point of the original model but retain its asymptotic stability. Furthermore, we construct unconditionally positive nonstandard finite difference (NSFD) schemes for the proposed model using the Mickens' methodology. It is worth noting that the constructed NSFD schemes not only preserve the positivity but also provide reliable numerical solutions that correctly reflect the dynamics of the new fractional-order model. Finally, we report some numerical examples to support and illustrate the theoretical results. The results indicate that there is a good agreement between the theoretical results and numerical ones.

§1 Introduction

Very recently, we have proposed and analyzed a fractional-order Logistic model with feedback control of the form [23]

$$\begin{aligned} {}_0^C D_t^q x &= a^q x(1 - b^q x) - c^q x u, \\ {}_0^C D_t^q u &= -d^q u + e^q x, \end{aligned} \tag{1}$$

where $q \in (0,1)$ and ${}_0^c D_t^q y$ stands for the Caputo fractional derivative of the function $y = y(t)$, and the parameters in the model are all positive. The model (1) is an extended version of a

Received: 2019-07-04. Revised: 2020-03-01.
MR Subject Classification: 26A33, 34A08, 37M05, 37N35, 93D30.
Keywords: fractional-order Logistic model, feedback control, Lyapunov functions, uniform asymptotic stability, nonstandard finite difference schemes.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-021-3851-1.
*Corresponding author.

system of ordinary differential equations of the form

$$\begin{aligned} x' &= x\left(1 - \frac{x}{K} - u\right), \\ u' &= -\eta u + ax. \end{aligned} \tag{2}$$

The model (2) was first introduced and analyzed by Fan and Wang in [21]. In [21], a new method has been proposed to establish the global asymptotic stability of the unique positive equilibrium of the model (2). Note that the model (2) without feedback control variables is the classical Logistics differential equation.

Mathematical analyses in [23] show that the model (1) always possesses a unique positive equilibrium point $E^* = (x^*, u^*)$ defined by

$$x^* := \frac{a^q d^q}{a^q b^q d^q + c^q e^q}, \qquad u^* := \frac{e^q}{d^q} x^*. \tag{3}$$

By using the Lyapunov stability theorem with the support of an appropriate Lyapunov function, it was proved that the unique positive equilibrium point of the model (1) is uniformly asymptotically stable [23]. Furthermore, nonstandard finite difference (NSFD) schemes were constructed to solve the model (1), and advantages and efficiency of NSFD schemes over standard finite difference schemes were also shown.

It is important to mention that the model (1) without the feedback control variables becomes the fractional-order Logistic differential equation:

$${}_0^C D_t^q x = a^q x(1 - b^q x). \tag{4}$$

The equation (4) has many useful applications in real-world situations, especially in biology, ecology, and physics. In the previous work [19], the existence, uniqueness, qualitative properties of solutions and numerical solutions for the equation (4) were investigated. Recently, an exact solution to the fractional-order Logistic equation (4) has been provided in [38].

It is easy to check that the equation (4) always has a unique positive equilibrium point given by $x_e := 1/b^q$. By a Lyapunov function defined by

$$V(x) = x - x_e - x_e \ln \frac{x}{x_e},$$

we can conclude that the equilibrium point x_e of the equation (4) is uniformly asymptotically stable.

The asymptotic stability of the models (1) and (4) imply that the feedback control variables do not affect the stability of the equation (4) but they change the position of the unique positive equilibrium point. This fact completely agrees with some well-known results on ordinary differential equation models with feedback controls [6–8, 21, 34]. Hence, the feedback controls variables have an important role in controlling the stability of the original model (4), especially they can allow us to change the value of the positive equilibrium point of (4) as we expect. Actually, the feedback control variables are really effective in the case x_e is not the desirable one (or affordable) and a smaller value of x_e is required. In this case, the feedback control variables can alter the model (4) structurally so as to make the population stabilize at a value lower than x_e. A similar comment can be found in [22]. Therefore, it is safe to say that the design of feedback control models for differential equation models is very important but not a simple work. Some similar ideas also have been explored in epidemic and opinion spreading (see [35,36]).

In some contexts of real-world applications, a simple but important question is whether we can design other feedback control models that differ from the model (1) but possess the same features. If this question is resolved, we will have more options for models with feedback controls. This result is significant in both theory and applications. Motivates by the above question, in this paper we propose a new fractional-order Logistic model with feedback control which is different from the model (1). More precisely, the following fractional differential model is proposed

$$\begin{aligned} {}_0^C D_t^q x &= a^q x(1 - b^q x) - c^q xu := \hat{f}(x,u), \\ {}_0^C D_t^q u &= u(-d^q u + e^q x) := \hat{g}(x,u), \end{aligned} \tag{5}$$

where all the parameters in this model are positive. Although the second equations of the models (1) and (5) are not the same, their positive equilibrium points are identical. By the Lyapunov stability theorem for fractional dynamical systems and an appropriate Lyapunov function, we prove that the positive equilibrium point of the model (5) is also asymptotically stable. As a consequence of this result, we obtain a new feedback control model which is different from the model (1) but possesses the same properties and features.

Because it is very difficult to find the exact solution, our next objective is to consider reliable numerical methods for solving the model (5). For this purpose, we formulate nonstandard finite difference (NSFD) schemes preserving the positivity of the model (5). It should be emphasized that NSFD schemes were first introduced by Mickens to overcome serious limitations of standard finite difference schemes [28,29]. Nowadays, NSFD schemes have been widely used as one of the effective techniques for solving ordinary differential equations (see, for example, [1,9–16,24,25,28–30,39]), partial differential equations (see, for instance, [20,28–30,40]) and fractional differential equations (see, for example, [4,23,31]). Motivated by this, we construct unconditionally positive NSFD schemes for the model (5) using the Mickens' methodology. The result is that we obtain NSFD schemes preserving the positivity of the fractional-order model (5) for any finite step size. Importantly, the constructed NSFD schemes provide reliable numerical solutions that correctly reflect the dynamics of the model (5).

This paper is organized as follows. In Section 2 we recall some basic definitions of fractional derivatives and the Grunwald-Letnikov approximation. The asymptotic stability of the model (5) is established in Section 3. In Section 4, we propose NSFD schemes for the model (5) and report some numerical examples. Finally, some conclusions and remarks are given.

§2 Preliminaries

2.1 Caputo fractional derivative

We first recall the definition of the Caputo fractional derivative and some of its properties.

Definition 2.1. ([5]) *Suppose that $q > 0,\ t > 0,\ q, a, t \in \mathbb{R}$. The Caputo fractional derivative* *is given by*

$$ {}_a^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{f^{(n)}(\xi)}{(t-\xi)^{q+1-n}} d\xi, \qquad n-1 < q < n, n \in \mathbb{N}. $$

Remark 2.1. *The Caputo fractional derivative of order* $0 < q < 1$ *for a smooth function* $f = f(t)$ *is given by*

$$ {}_a^C D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_a^t \frac{1}{(t-\xi)^q} \frac{df(\xi)}{d\xi} d\xi. $$

Property 2.1. (Linearity property [18]). *Let* $f(t), g(t) : [a, b] \to \mathbb{R}$ *be such that* ${}_a^C D_t^q f(t)$ *and* ${}_a^C D_t^q g(t)$ *exist almost everywhere and let* $c_1, c_2 \in \mathbb{R}$. *Then,* ${}_a^C D_t^q (c_1 f(t) + c_2 g(t))$ *exists almost everywhere, and*

$$ {}_a^C D_t^q (c_1 f(t) + c_2 g(t)) = c_1 {}_a^C D_t^q f(t) + c_2 {}_a^C D_t^q g(t). $$

Property 2.2. (Caputo derivative of a constant [32]). *The fractional derivative for a constant function* $f(t) = c$ *is zero.*

Lemma 2.1. ([37, Lemma 3.1]). *Let* $x(t) \in \mathbb{R}^+$ *be a continuous and derivable function. Then, for any time instant* $t \geq t_0$

$$ {}_{t_0}^C D_t^q \Big[x(t) - x^* - x^* \ln \frac{x(t)}{x^*} \Big] \leq \Big(1 - \frac{x^*}{x(t)} \Big) {}_{t_0}^C D_t^q x(t). $$

2.2 Fractional dynamical systems

Consider the general type of fractional-order equations involving the Caputo derivative:

$$ {}_{t_0}^C D_t^q x(t) = f(t, x), \qquad q \in (0, 1), \tag{6} $$

subject to the initial condition $x_0 = x(t_0)$.

Definition 2.2. (See [26]). *The constant* x^* *is an equilibrium point of Caputo fractional dynamical system* (6) *if and only if* $f(t, x^*) = 0$.

The following theorem is an extension of the classical Lyapunov direct method for dynamical systems governed by ordinary differential equations.

Theorem 2.1. ([17, Theorem 3.1]) *Let* $x = 0$ *be an equilibrium point for* ${}^C D^q x(t) = f(t, x)$ *and* $\mathbb{D} \subset \mathbb{R}^n$ *be a domain containing* $x = 0$. *Let* $V(t, x) : [0, \infty] \times \mathbb{D} \to \mathbb{R}$ *be a continuously differentiable function such that*

$$ W_1(x) \leq V(t, x) \leq W_2(x), $$
$$ {}^C D^q V(t, x) \leq -W_3(x), $$

$\forall t \geq 0$, $\forall x \in \mathbb{D}$, $0 < q < 1$ *where* $W_1(x), W_2(x)$ *and* $W_3(x)$ *are continuous positive definite functions on* Ω. *Then* $x = 0$ *is uniformly asymptotically stable.*

2.3 The Grunwald-Letnikov approximation

In what follows, the Grunwald-Letnikov approximation for the Caputo derivative [4, 33] is provided. First, we present a direct definition of the fractional derivative ${}_0^C D_t^q x(t)$ based on finite differences of an equidistant grid in $[0,t]$ (see [4,33]). Suppose the function $D_t^q x(\tau)$ satisfies some smoothness conditions in every finite interval $(0,t)$. We use a grid given by

$$0=\tau_0<\tau_1<\ldots<\tau_{n+1}=t=(n+1)h, \qquad \tau_{n+1}-\tau_n=h,$$

and the classical notation of finite differences

$$\frac{1}{h^q}\Delta_h^q x(t)=\frac{1}{h^q}\left(x(\tau_{n+1})-\sum_{\nu}^{n+1}c_\nu^q x(\tau_{n+1-\nu})\right),$$

where,

$$c_\nu^q=(-1)^{\nu-1}\binom{q}{\nu},$$

and

$$\binom{q}{\nu}:=\frac{q(q-1)(q-2)\ldots(q-\nu+1)}{\nu!},$$

is the usual notation for the binomial coefficients [32, p. 43]. It is important to note that the binomial coefficients c_ν^q satisfy (see [33])

$$0<c_{\nu+1}^q<c_\nu^q<\ldots<c_1^q=q<1,$$

and in practice, the binomial coefficients c_ν^q can be recursively defined by

$$c_\nu^1=q,$$
$$c_\nu^q=\left(1-\frac{q+1}{\nu}\right)c_{\nu-1}^q, \qquad \nu>1.$$

Then, the Grunwald-Letnikov definition reads [32]

$$D_t^q x(t)=\lim_{t\to 0}\frac{1}{h^q}\Delta_h^q x(t).$$

Applying the Grunwald-Letnikov definition to the equation

$${}_0^C D_t^q x(t)=f(t,x(t)), \qquad x(t_0)=x_0,$$

we obtain the explicit and implicit Grunwald-Letnikov method for an equidistant grid as follows (see [4,33])

$$x_{n+1}-\sum_{\nu=1}^{n+1}c_\nu^q x_{n+1-\nu}-r_{n+1}^q x_0=h^q f(t_n,x_n)$$

and

$$x_{n+1}-\sum_{\nu=1}^{n+1}c_\nu^q x_{n+1-\nu}-r_{n+1}^q x_0=h^q f(t_{n+1},x_{n+1}),$$

where

$$r_{n+1}^q=h^q r_0^q(\tau_{n+1})=\gamma_{0,-1}^q(n+1)^{-q}$$

and

$$\gamma_{\mu,k}^q=\frac{\Gamma(\mu q+1)}{\Gamma(kq+1)}, \qquad \mu,k\in\mathbb{N}_0\cup\{-1\}.$$

Note that the coefficients r_{n+1}^q satisfy (see [33])

$$r_{n+1}^q<r_n^q<\ldots<r_1^q=\frac{1}{\Gamma(1-q)}$$

We refer the readers to [33] for other properties of the coefficients c_ν^q and r_ν^q and the Grunwald-Letnikov method for fractional differential equations.

§3 Stability analysis of the model (5)

In this section, we establish the uniform asymptotic stability of the model (5) by using the uniform asymptotic stability theorem (Theorem 2.1). First, by using [27, Theorem 3.1] and [27, Remark 3.2], we obtain the existence and uniqueness of solutions of the model (5). Next, by making a similar argument as in [3, Theorem 1], we conclude that the set $\mathbb{R}_+^2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0\}$ is a positively invariant set of the model (5), i.e, if $x(0) \geq 0$ and $u(0) \geq 0$, then $x(t) \geq 0$ and $u(t) \geq 0$ for all $t > 0$.

It is easy to verify that the model (5) always has a unique positive equilibrium point $\hat{E}^* = (\hat{x}^*, \hat{u}^*)$ defined by $\hat{x}^* = x^*$ and $\hat{u}^* = u^*$, where x^* and u^* are given by (3). The following theorem can be considered as a main result of the paper.

Theorem 3.1. *The positive equilibrium point $\hat{E}^*$ of the model* (5) *is uniformly asymptotically stable with respect to the interior of* $\mathbb{R}_+^2$.

Proof. Since $\hat{E}^*$ is the positive equilibrium point of the model (5), we have

$$\begin{aligned} a^q\big(1 - b^q\hat{x}^*\big) - c^q\hat{u}^* &= 0, \\ -d^q\hat{u}^* + e^q\hat{x}^* &= 0. \end{aligned} \tag{7}$$

Thanks to (7), we can rewrite the model (5) in the form

$${}_0^C D_t^q x(t) = x\Big[-a^q b^q(x - \hat{x}^*) - c^q(u - \hat{u}^*)\Big], \quad {}_0^C D_t^q u(t) = u\Big[-d^q(u - \hat{u}^*) + e^q(x - \hat{x}^*)\Big]. \tag{8}$$

We now consider a Lyapunov function $V(x,u)$ defined by

$$V(x,u) = V_1(x) + V_2(u), \tag{9}$$

where

$$V_1(x) = \frac{e^q}{c^q}\left(x - \hat{x}^* - \hat{x}^* \ln\frac{x}{\hat{x}^*}\right), \qquad V_2(u) = u - \hat{u}^* - \hat{u}^* \ln\frac{u}{\hat{u}^*}.$$

Using Properties 2.1, 2.2 and Lemma 2.1, we have

$${}_0^C D_t^q V(x,u) \leq \frac{e^q}{c^q}\frac{x - \hat{x}^*}{x}\left({}_0^C D_t^q x\right) + \frac{u - \hat{u}^*}{u}\left({}_0^C D_t^q u\right). \tag{10}$$

Combining (8) and (10), we get

$$\begin{aligned} {}_0^C D_t^q V(x,u) &\leq \frac{e^q}{c^q}\frac{x - \hat{x}^*}{x}x\Big[-a^q b^q(x - \hat{x}^*) - c^q(u - \hat{u}^*)\Big] + \frac{u - \hat{u}^*}{u}u\Big[-d^q(u - \hat{u}^*) + e^q(x - \hat{x}^*)\Big] \\ &= -\frac{e^q}{c^q}a^q b^q(x - \hat{x}^*)^2 - d^q(u - \hat{u}^*)^2. \end{aligned}$$

Consequently, by Theorem 2.1, the unique positive equilibrium point $(\hat{x}^*, \hat{u}^*)$ is uniformly asymptotically stable in the interior of $\mathbb{R}_+^2$. The proof is complete. □

Remark 3.1. *The Lyapunov function $V(x,u)$ defined by* (9) *is different from the Lyapunov function that was used in [23]. Also, we can conclude the global asymptotic stability of the model* (5) *thanks to the Lyapunov function* (9).

§4 Numerical simulations

In this section, we report some numerical simulations to support and illustrate the theoretical results. For this purpose, positive NSFD schemes for the model (5) are constructed.

4.1 Positive NSFD schemes

Let N_0 be a positive integer and $[0,\, t_{end}]$ be a finite interval. We denote by $h = \Delta t = t_{end}/N_0$ the step size of the discretization:

$$0 = t_0 < t_1 < \ldots < t_{N_0} = t_{end} = N_0 h,$$

and let $t_n = nh$, with $n \in \{0, 1, 2, \ldots, N_0\}$. Let x_n and u_n be approximations for $x(t_n)$ and $u(t_n)$, respectively. Based on the previous works [4, 23] and Mickens' view of utilizing NSFD schemes [28–30], we obtain the following family of NSFD schemes for the model (5)

$$\begin{aligned} x_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu} - r_{n+1}^q x_0 &= \big(\varphi(h)\big)^q a^q (x_n - b^q x_n x_{n+1}) - \big(\varphi(h)\big)^q c^q x_{n+1} u_n, \\ u_{n+1} - \sum_{\nu=1}^{n+1} c_\nu^q u_{n+1-\nu} - r_{n+1}^q u_0 &= \big(\varphi(h)\big)^q u_n \Big(- d^q u_{n+1} + e^q x_{n+1} \Big), \end{aligned} \tag{11}$$

where $\varphi(h) = h + \mathcal{O}(h^2)$ as $h \to 0$.

Theorem 4.1. *Let (x_0, u_0) be any initial data for the initial value problem (5) with $x_0, u_0 \geq 0$ and $\big\{(x_n, u_n)\big\}_{n>0}$ be the approximations generated by the NSFD scheme (11). Then $x_n \geq 0$ and $u_n \geq 0$ for all $n > 0$. In other words, the NSFD scheme (11) preserves the positivity of the solutions of the model (5) for all finite step sizes.*

Proof. The theorem is proved by mathematical induction. It is easy to transform (11) to the explicit form:

$$\begin{aligned} x_{n+1} &= \frac{\sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu} + r_{n+1}^q x_0 + \varphi^q a^q x_n}{1 + \varphi^q a^q b^q x_n + \varphi^q c^q u_n}, \\ u_{n+1} &= \frac{\sum_{\nu=1}^{n+1} c_\nu^q u_{n+1-\nu} + r_{n+1}^q u_0 + \varphi^q e^q x_{n+1} u_n}{1 + \varphi^q d^q u_n}. \end{aligned}$$

Clearly, if $x_0 \geq 0$ and $u_0 \geq 0$, then $x_n \geq 0$ and $u_n \geq 0$ for all $n > 0$. Thus, the proof is completed. □

As a simple consequence of Theorem 4.1, we have the following positive NSFD schemes for the model without feedback control variables (4):

$$x_{n+1} = \frac{\sum_{\nu=1}^{n+1} c_\nu^q x_{n+1-\nu} + r_{n+1}^q x_0 + \varphi^q a^q x_n}{1 + \varphi^q a^q b^q x_n}. \tag{12}$$

This scheme will be used in the next section.

4.2 Numerical examples

First, we verify the asymptotic stability of the model (5). For this purpose, we consider the model (5) with the parameter

$$a = 8, \quad b = \frac{1}{32}, \quad c = 8, \quad d = 4, \quad e = \frac{1}{8}, \quad q \in \big\{0.8,\, 0.98\big\}.$$

In this example, the model (5) has the positive equilibrium points $\hat{E}^*_{0.8} = (8.0,\, 0.5)$ and $\hat{E}^*_{0.98} = (14.92,\, 0.50)$ for $q = 0.8$ and $q = 0.98$, respectively. Numerical solutions obtained by the NSFD scheme (11) with $\varphi(h) = h$ and $h = 0.001$ are depicted in Figures 1 and 2. It is clear that the stability of the model is shown in the given figures.

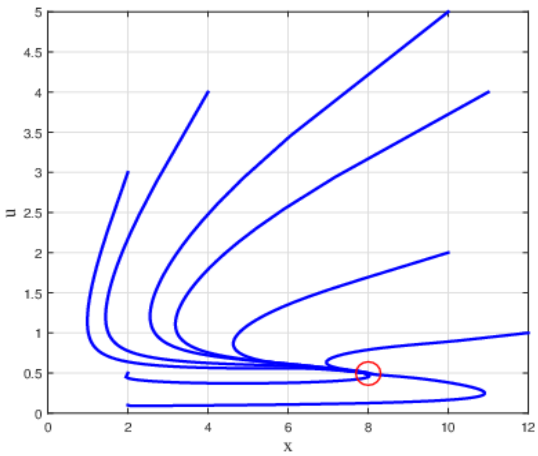

Figure 1. The numerical solutions obtained by the NSFD scheme (11) for $t \in [0, 50]$ and $q = 0.8$. The red circle indicates the equilibrium point $E^*_{0.8}$ and the blue curves indicate the phase planes of the model (5).

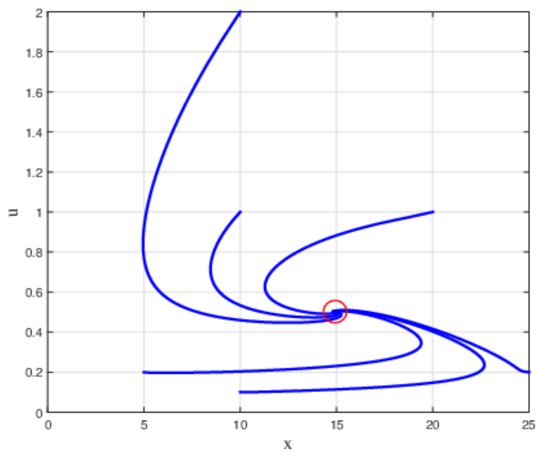

Figure 2. The numerical solutions obtained by NSFD scheme (11) for $q = 0.98$ and $t \in [0, 50]$. The red circle indicates the equilibrium point $E^*_{0.98}$ and the blue curves indicate the orbits of the model (5).

We now consider the model without feedback control (4). The model has the unique equilibrium point $x_e = 1/b^q$ and it is asymptotically stable. Numerical solutions of the model (4) by the NSFD scheme (12) with $h = 0.001$ and $t \in [0, 15]$ are depicted in Figures 3 and 4. Clearly, the equilibrium x_e is asymptotically stable.

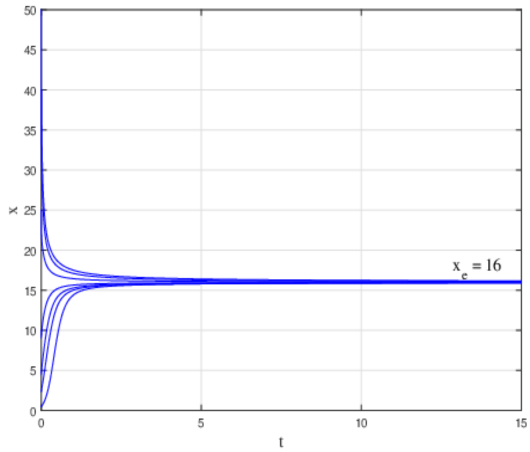

Figure 3. The numerical solutions generated by the NSFD scheme (12) when $q = 0.8$.

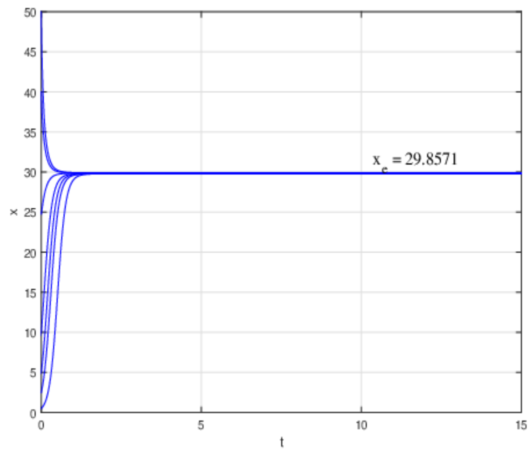

Figure 4. The numerical solutions obtained by the NSFD scheme (12) when $q = 0.98$.

From the above results, we observe that the feedback control variables do not change the stability but change the position of the positive equilibrium point. More clearly, $\hat{x}^* < x_e$. Consequently, the model (5) provides a new feedback control model which is different from the model (1).

§5 Conclusions and remarks

In this paper, a new fractional-order Logistic model with feedback control has been proposed and studied. The asymptotic stability of the new model has been investigated by a suitable Lyapunov function. The main result is that we obtain a new feedback control model that is different from the model (1) but possesses the same properties and features. In parallel, unconditionally positive NSFD schemes for the proposed model have been constructed and used for numerical simulations. The theoretical results are supported and illustrated by a set of numerical examples.

The design of feedback control models that are different from the model (5) but possess similar properties and features should continue to be explored and studied. On the other hand, the models (1) and (5) should be studied in the context of other fractional derivatives, for instance, the Caputo-Fabrizio fractional derivative, the two-parameter fractional derivative in Caputo sense and the two-parameter fractional derivative in Riemann-Liouville sense, etc. This can help us figure out other properties and applications of fractional-order Logistic models with feedback controls. Especially, it is very interesting if applications of the model (5) in economics, biology, ecology, etc. are studied and analyzed.

In the near future, we will study the open problems mentioned above. Also, the construction of other numerical methods with high performance for solving the models (1) and (5) will be considered.

Acknowledgements

The authors would like to thank the editor and anonymous referees for useful and valuable comments that led to a great improvement of the paper.

References

[1] O Adekanye, T Washington. *Nonstandard finite difference scheme for a tacoma narrows bridge model*, Applied Mathematical Modelling, 2018, 62: 223-236.

[2] R Anguelov, J M -S Lubuma. *Nonstandard finite difference method by nonlocal approximations*, Mathematics and Computers in Simulation, 2003, 61(3-6): 465-475.

[3] R Almeida. *Analysis of a fractional SEIR model with treatment*, Applied Mathematics Letters, 2018, 84: 56-62.

[4] A J Arenas, G Gonzalez-Parra, B M Chen-Charpentier. *Construction of nonstandard finite difference schemes for the SI and SIR epidemic models of fractional order*, Mathematics and Computers in Simulation, 2016, 121: 48-63.

[5] M Caputo. *Linear models of dissipation whose Q is almost frequency independent-II*, Geophysical Journal International, 1967, 13(5): 529-539.

[6] L Chen, F Chen. *Global stability of a Leslie-Gower predator-prey model with feedback controls*, Applied Mathematics Letters, 2009, 22(9): 1330-1334.

[7] L Chen, J Sun. *Global stability of an SI epidemic model with feedback controls*, Applied Mathematics Letters, 2014, 28: 53-55.

[8] Q A Dang, M T Hoang, D Y Trejos, J C Valverde. *Feedback control variables to restrain the Babesiosis disease*, Mathematical Methods in the Applied Sciences, 2019, 42(18): 7517-7527.

[9] Q A Dang, M T Hoang. *Dynamically consistent discrete metapopulation model*, Journal of Difference Equations and Applications, 2016, 22(9): 1325-1349.

[10] Q A Dang, M T Hoang. *Lyapunov direct method for investigating stability of nonstandard finite difference schemes for metapopulation models*, Journal of Difference Equations and Applications, 2018, 24(1): 15-47.

[11] Q A Dang, M T Hoang. *Complete global stability of a metapopulation model and its dynamically consistent discrete models*, Qualitative Theory of Dynamical Systems, 2019, 18: 461-475.

[12] Q A Dang, M T Hoang. *Nonstandard finite difference schemes for a general predator-prey system*, Journal of Computational Science, 2019, 36: 101015.

[13] Q A Dang, M T Hoang. *Exact finite difference schemes for three-dimensional linear systems with constant coefficients*, Vietnam Journal of Mathematics, 2018, 46: 471-492.

[14] Q A Dang, M T Hoang, Q L Dang. *Nonstandard finite difference schemes for solving a modified epidemiological model for computer viruses**, Journal of Computer Science and Cybernetics, 2018, 32(2): 171-185.

[15] Q A Dang, M T Hoang. *Positivity and global stability preserving NSFD schemes for a mixing propagation model of computer viruses*, Journal of Computational and Applied Mathematics, 2020, 374: 112753.

[16] Q A Dang, M T Hoang. *Numerical dynamics of nonstandard finite difference schemes for a computer virus propagation model*, International Journal of Dynamics and Control, 2020, 8: 772-778.

[17] H Delavari, D Baleanu, J Sadati. *Stability analysis of Caputo fractional-order nonlinear systems revisited*, Nonlinear Dynamics, 2012, 67: 2433-2439.

[18] K Diethelm. *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer, 2010.

[19] A M A El-Sayed, A E M El-Mesiry, H A A El-Saka. *On the fractional-order logistic equation*, Applied Mathematics Letters, 2007, 20(7): 817-823.

[20] M Ehrhardt, R E Mickens. *A nonstandard finite difference scheme for convection-diffusion equations having constant coefficients*, Applied Mathematics and Computation, 2013, 219(12): 6591-6604.

[21] Y H Fan, L L Wang. *Global asymptotical stability of a Logistic model with feedback control*, Nonlinear Analysis: Real World Applications, 2010, 11(4): 2686-2697.

[22] K Gopalsamy, Pei-Xuan Weng. *Feedback regulation of logistic growth*, International Journal of Mathematics and Mathematical Sciences, 1993, 16(1): 177-192.

[23] M T Hoang, A M Nagy. *Uniform asymptotic stability of a Logistic model with feedback control of fractional order and nonstandard finite difference schemes*, Chaos, Solitons & Fractals, 2019, 123: 24-34.

[24] M T Hoang, O F Egbelowo. *Nonstandard finite difference schemes for solving an SIS epidemic model with standard incidence*, Rendiconti del Circolo Matematico di Palermo Series 2, 2020, 69: 753769.

[25] M T Hoang, O F Egbelowo. *On the global asymptotic stability of a hepatitis B epidemic model and its solutions by nonstandard numerical schemes*, Boletín de la Sociedad Matemática Mexicana, 2020, 26: 1113-1134.

[26] Y Li, Q Y Chen, I Podlubny. *Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability*, Computers and Mathematics with Applications, 2010, 59(5): 1810-1821.

[27] W Lin. *Global existence theory and chaos control of fractional differential equations*, J Math Anal Appl, 2007, 332(1): 709-726.

[28] R E Mickens. *Applications of nonstandard finite difference schemes*, World Scientific, Singapore, 2000.

[29] R E Mickens. *Nonstandard finite difference schemes for differential equations*, Journal of Difference Equations and Applications, 2002, 8(9): 823-847.

[30] R E Mickens. *A nonstandard finite-difference scheme for the Lotka-Volterra system*, Applied Numerical Mathematics, 2003, 45(2-3): 309-314.

[31] A M Nagy. *Non-Standard finite difference schemes for solving variable-order fractional differential equations*, Differential Equations and Dynamical Systems, 2017, DOI: https://doi.org/10.1007/s12591-017-0378-2.

[32] I Podlubny. *Fractional differential equations, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego, CA, 1999.

[33] R Scherer, S Kalla, Y Tang, J Huang. *The Grunwald-Letnikov method for fractional differential equations*, Computers and Mathematics with Applications, 2011, 62(3): 902-917.

[34] Y Shang. *Global stability of disease-free equilibria in a two-group SI model with feedback control*, Nonlinear Analysis: Modelling and Control, 2015, 20(4): 501-508.

[35] Y Shang. *Degree distribution dynamics for disease spreading with individual awareness*, Journal of Systems Science and Complexity, 2015, 28(1): 96-104.

[36] Y Shang. *Lie algebraic discussion for affinity based information diffusion in social networks*, Open Physics, 2017, 15: 705-711

[37] C Vargas-De-Leon. *Volterra-type Lyapunov functions for fractional-order epidemic systems*, Communications in Nonlinear Science and Numerical Simulation, 2015, 24(1-3): 75-85.

[38] B J West. *Exact solution to fractional logistic equation*, Physica A: Statistical Mechanics and its Applications, 2015, 429: 103-108.

[39] D T Wood, H V Kojouharov, D T Dimitrov. *Universal approaches to approximate biological systems with nonstandard finite difference methods*, Mathematics and Computers in Simulation, 2017, 133(c): 337-350.

[40] Y Yang, J Zhou, X Ma, T Zhang. *Nonstandard finite difference scheme for a diffusive within-host virus dynamics model with both virus-to-cell and cell-to-cell transmissions*, Computers & Mathematics with Applications, 2016, 72(4): 1013-1020.

[1]Department of Mathematics, FPT University, Hoa Lac Hi-Tech Park, Km29 Thang Long Blvd, Hanoi, Viet Nam.

Email: tuanhm14@fe.edu.vn, hmtuan01121990@gmail.com

[2]Kuwait University, Faculty of Science, Department of Mathematics, Safat 13060, Kuwait.

[3]Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt.

Email: abdelhameed_nagy@yahoo.com, abdelhameed.nagy@fsc.bu.edu.eg