Appl. Math. J. Chinese Univ.
2021, 36(3): 354-369

Laguerre reproducing kernel method in Hilbert spaces for unsteady stagnation point flow over a stretching/shrinking sheet

M. R. Foroutan A. S. Gholizadeh Sh. Najafzadeh R. H. Haghi

Abstract. This paper investigates the nonlinear boundary value problem resulting from the exact reduction of the Navier-Stokes equations for unsteady magnetohydrodynamic boundary layer flow over the stretching/shrinking permeable sheet submerged in a moving fluid. To solve this equation, a numerical method is proposed based on a Laguerre functions with reproducing kernel Hilbert space method. Using the operational matrices of derivative, we reduced the problem to a set of algebraic equations. We also compare this work with some other numerical results and present a solution that proves to be highly accurate.

§1 Introduction

The magnetohydrodynamic flow and incompressible fluid over a stretching/shrinking sheet has attracted the attention of many researchers recently in view of its applications in multiple engineering problems such as magnetohydrodynamic generators, manufacturing processes of polymer, nuclear reactors, glass fiber production and geothermal energy extractions. Crane [10] was the first who studied the two-dimensional steady flow of an incompressible viscous fluid caused by a linearly stretching plate and obtained an exact solution in closed analytical form. Numerous studies have been conducted afterward to explore various aspects of the flow over a steady stretching sheet [1, 13, 30].

Recently, Mohamed et al. [24] studied the stagnation point flow over a stretching sheet and Hayat et al. [25] investigated the flow of a second grade fluid over a stretching surface with Newtonian heating. Most scientific problems like two-dimensional viscous flow between some expanding or contracting walls with various permeability rates and other fluids in mechanic are essentially nonlinear. There are many theoretical and experimental methods used to solve these

Received: 2019-02-19. Revised: 2019-09-11.
MR Subject Classification: 34B15, 76D03, 65M70.
Keywords: nonlinear boundary value problem, Laguerre reproducing kernel method, operational matrix of derivative, existence and nonexistence of solutions, approximate solution.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-021-3761-2.

equations. One of these methods that has been recently used in solving variable differential equations is reproducing kernel method.

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, machine learning and image precessing [2, 3, 23, 28, 29, 33]. Recently, based on the reproducing kernel theory, Cui and Geng [11, 17–19] have made much effort to solve some special boundary value problems. Furthermore, using the reproducing kernel method, some authors have proposed solutions to some two-point boundary value problems [12, 22]. For instance, Inc and Akgol have found some approximate solution for MHD squeezing fluid flow in [20]. Jiang and Lin used reproducing kernel theory to obtain approximate solutions of time-fractional telegraph equations in [21]. Some kinds of the second-order multi-point boundary value problems have been solved numerically by using the reproducing kernel method in [27]. Besides, Niu et al. presented convergence rate of the reproducing kernel method for solving boundary value problems in [34]. Abu Arqub applied reproducing kernel theory for the solutions of several integral, differential and fractional-order operators alongside with their theories [4–8]. Furthermore, Niu et al. investigated the approximate solutions to three point boundary value problems for parabolic equations by reproducing kernel space in [14, 26].

In this paper, we use reproducing kernel Hilbert space method to study the unsteady magnetohydrodynamic boundary layer flow over the stretching/shrinking permeable sheet submerged in a moving fluid. This problem has been solved by reproducing kernel Hilbert space method and it has been compared with some other numerical results. The reproducing kernel Hilbert space method, which accurately computes the series solution, is of great interest to applied sciences.

The advantages of the utilized Laguerre reproducing kernel method lie in the following powerful points; firstly, it can produce good globally smooth numerical solutions, and with ability to solve many differential equations with various boundary conditions; secondly, the numerical solutions and their derivatives are converge uniformly to the exact solutions and their derivatives, respectively; thirdly, the method is mesh-free, easily implemented, without discretization of the variables, effected by computation round off errors, with no necessity of large computer memory and time; fourthly, Laguerre reproducing kernel method reduce nonlinear problems to those of solving a system of algebraic equations thus greatly simplifying the problem [2].

The remaining sections of this paper are organized as follows: In section 2, the unsteady stagnation point flow over a stretching/shrinking sheet is addressed and transformed into a nonlinear ordinary differential equation. In section 3, it is demonstrated that Laguerre functions with the given properties can be applied to approximation third order nonlinear differential equations. According to our method, a brief introduction of the reproducing kernel Hilbert spaces is represented in section 4. In section 5, the existence of solutions for nonlinear ordinary differential introduced in section 5 is established. In section 6, we use Laguerre functions to solve this nonlinear equation and the operational matrix for these functions is derived. We also compare the proposed solutions with some well-known results and the comparisons show that

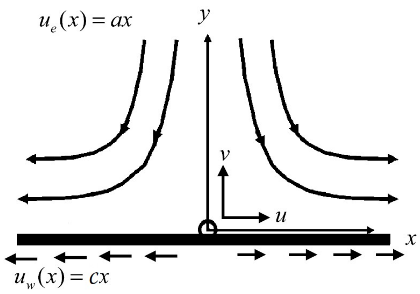

Figure 1. Physical model and coordinate system.

the present solutions are accurate.

§2 Mathematical Formulation

We consider the unsteady magnetohydrodynamic stagnation-point flow of an incompressible viscous and incompressible fluid towards a stretching vertical sheet placed in the plane $y = 0$. The flow being confined to $y > 0$. The flow is generated by the stretching/ shrinking effect along the x-axis. This flow is divided into two streamlets leaving in positive and negative x-axis respectively. It is assumed that the external velocity $u_e(x)$ and the stretching velocity $u_w(x)$ are of the forms $u_e(x) = ax$ and $u_w(x) = cx$, where $a > 0$ is the straining constant and c being constant with $c > 0$ and $c < 0$ corresponding to stretching and shrinking cases, respectively. The physical model and coordinate system of this problem are shown in Figure 1. A uniform magnetic field of strength B_0 is assumed to be applied in the positive y-direction normal to the plate. Under the assumption of small magnetic Reynolds numbers, the boundary layer equations which govern the unsteady magnetohydrodynamic boundary layer flow take the following forms:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma {B_0}^2}{\rho}(u_e - u), \tag{2}$$

The boundary conditions of equations (1) and (2) are

$$u = cu_w(x) + L\frac{\partial u}{\partial y}v = u_w(x), \ at \ \ y = 0, \tag{3}$$

$$u \longrightarrow u_e(x), \ \ as \ \ y \longrightarrow \infty, \tag{4}$$

where u and v are velocity components in the x and y direction, respectively, $(\nu = \frac{\mu}{\rho})$ is Kinematic fluid viscosity, ρ is the fluid density, μ is the coefficient of fluid viscosity, σ is the electrical conductivity. and L is the velocity slip length. We now introduce the following similarity variables (see Foroutan et al. [15]):

$$\eta = (\frac{u_e}{vx})^{\frac{1}{2}}y, \ \ \psi = (vxu_e)^{\frac{1}{2}}f(\eta), \tag{5}$$

where ψ is the stream function defined as $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$, which identically satisfies Equation (1). Thus, we have

$$u = axf'(\eta), \;\; v = -(a\nu)^{\frac{1}{2}} f(\eta). \tag{6}$$

Substituting (5) and (6) into equation (2) gives the nonlinear differential ordinary equation:

$$f'''(\eta) + f(\eta)f''(\eta) - \big(f'(\eta)\big)^2 - M\Big(f'(\eta) - 1\Big) + 1 = 0, \tag{7}$$

where $M = \frac{\sigma B_0^2}{\rho a}$ is the magnetic parameter. The boundary conditions (3) and (4) can also be reduced to

$$f(0) = 0, \;\; f'(0) = \lambda + \delta f''(0), \;\; \lim_{\eta\to\infty} f'(\eta) = 1, \tag{8}$$

where $\lambda = \frac{c}{a}$ is the velocity ratio parameter and $\delta = L(\frac{a}{\nu})^{\frac{1}{2}}$ is the slip parameter.

§3 Laguerre function

The well-known Laguerre polynomials are orthogonal in the interval $(0, +\infty)$ in terms of weight function $w_\alpha(x) = x^\alpha e^{-x}$ and can be calculated via the following recurrence formula:

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = -x + \alpha + 1,$$

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n + \alpha + 1 - x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \quad n = 1, 2, \cdots .$$

Also, these polynomials satisfy conditions set by Strum-Liouville equation,

$$x(L_n^{(\alpha)}(x))'' + (\alpha + 1 - x)(L_n^{(\alpha)}(x))' + nL_n^{(\alpha)}(x) = 0, \quad n \in \mathbb{N}.$$

The Laguerre function is defined by

$$G_n^{(\alpha)}(x) = e^{\frac{-x}{2}} L_n^{(\alpha)}(x), \quad \alpha > -1, \quad x \in \mathbb{R}^+ := (0, +\infty). \tag{9}$$

Also, the Laguerre functions are well-behaved with the decay property:

$$|G_n^{(\alpha)}(x)| \to 0, \quad \text{as} \quad x \to +\infty,$$

Furthermore, the Laguerre functions are orthogonal with respect to the weight function $\chi_\alpha(x) = x^\alpha$, i.e.,

$$\int_0^{+\infty} G_n^{(\alpha)}(x)G_m^{(\alpha)}\chi_\alpha(x)dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\delta_{mn}. \tag{10}$$

In what follows, we restrict ourselves to the most commonly used case,

$$G_n(x) := G_n^{(0)}(x) = e^{\frac{-x}{2}} L_n(x), \quad x \in \mathbb{R}^+, \tag{11}$$

which satisfies the three-term recurrence relation:

$$G_0(x) = e^{\frac{-x}{2}}, \quad G_1(x) = (1-x)e^{\frac{-x}{2}}, \tag{12}$$

$$(n+1)G_{n+1}(x) = (2n + 1 - x)G_n(x) - nG_{n-1}(x). \tag{13}$$

Also, the Laguerre functions satisfy the following recurrence relations:

$$G_n'(x) = -\sum_{k=0}^{n-1} G_k(x) - \frac{1}{2}G_n(x), \tag{14}$$

$$\frac{1}{2}(G_n(x) - G_{n+1}(x)) = G_n'(x) - G_{n+1}'(x), \tag{15}$$

$$xG_n'(x) = n(G_n(x) - G_{n-1}(x)) - \frac{1}{2}xG_n(x), \tag{16}$$

and these functions are uniformly bounded. In particular, we have

$$|G_n(x)| \leq 1, \quad \forall x \in [0, +\infty).$$

The Rodrigues formula for Laguerre functions takes the form:

$$G_n(x) = \frac{e^{-x}}{2n!}\frac{d^n}{dx^n}\{x^n e^{-x}\}. \tag{17}$$

Furthermore, the the Laguerre functions satisfy conditions set by Strum-Liouville equation,

$$xG_n''(x) + G_n'(x) + (1 + n - \frac{1}{2}x)G_n(x) = 0. \tag{18}$$

3.1 Zeroes of Laguerre functions

We are going to review some properties of the zeroes of Laguerre functions. we refer for instance to Funaro [16] for proofs and further results. For any $N \geq 1$ we denote the N zeroes of G_n by $x_n,\ 0 \leq n \leq N$. A good approximation of the zeroes of Laguerre functions is obtained by the following procedure. For any $N \geq 1$, we first find the roots of the equation

$$z_n - sinz_n = 2\pi\frac{N - n + \frac{3}{4}}{2N + 3}, \qquad 0 \leq n \leq N.$$

Then, we set

$$\widehat{z_n} := (cos\frac{1}{2}z_n)^2, \qquad 0 \leq n \leq N,$$

and define for $0 \leq n \leq N$,

$$y_n = 2(2N + 3)\widehat{z_n} - \frac{1}{6(2N + 3)}(\frac{5}{4(1 - \widehat{z_n})^2} - \frac{1}{1 - \widehat{z_n}} - 1),$$

which provides a good approximation of x_n. Let us permute the value of $y_n,\ 0 \leq n \leq N$, in such a way they are in increasing order. After third rearrangement, we denote the new sequence by $\widehat{y_n},\ 0 \leq n \leq N$. Finally, we have $x_n \approx \widehat{y_n},\ 0 \leq n \leq N$.

§4 Reproducing kernel function

We start with some basic definitions.which play a very important role in the study of reproducing kernel Hilbert spaces [1].

Definition 4.1. Let H be a Hilbert space of functions $f : X \to \mathbb{R}$ defined on a non-empty set X. A function $K : X \times X \to \mathbb{R}$ is called a reproducing kernel of H if it satisfies:

(i) $\forall x \in X, \quad K_x = K(x, 0) \in H.$

(ii) $\forall x \in X, \quad \forall f \in H, \quad < f, K_x >= f(x)$ (the reproducing property).

Also, a Hilbert space H is a reproducing Kernel Hilbert space if and only if it has a reproducing Kernel.

Theorem 4.2. *Let $\{G_n\}_{n=0}^{\infty}$ be a sequence of the generalized Laguerre functions. Then, for the orthonormal system $\{G_n\}_{n=0}^{\infty}$ and for*

$$K_n(x, y) \equiv \sum_{j=0}^{n} G_j(x)G_j(y), \quad (x, y \in \mathbb{R}^+), \tag{19}$$

we have

$$K_n(x,y) = (n+1)\frac{G_{n+1}(x)G_n(y) - G_n(x)G_{n+1}(y)}{y-x}, \quad if \quad x \neq y, \tag{20}$$

and

$$K_n(x,x) = (n+1)(G'_{n+1}(x)G_n(x) - G'_n(x)G_{n+1}(x)). \tag{21}$$

Proof. Let $x, y \in \mathbb{R}^+$. From (13), we have

$$G_{n+1}(x) = (\frac{2n+1}{n+1} - \frac{1}{n+1}x)G_n(x) - \frac{n}{n+1}G_{n-1}(x), \quad n \geq 1. \tag{22}$$

We first prove (20). By equation (22),

$$\begin{aligned} G_{j+1}(x)G_j(y) - G_j(x)G_{j+1}(y) &= [(\frac{2j+1}{j+1} - \frac{1}{j+1}x)G_j(x) - \frac{j}{j+1}G_{j-1}(x)]G_j(y) \\ &- G_j(x)[(\frac{2j+1}{j+1} - \frac{1}{j+1}y)G_j(y) - \frac{j}{j+1}G_{j-1}(y)]. \end{aligned}$$

Thus

$$(j+1)\frac{G_{j+1}(x)G_j(y) - G_j(x)G_{j+1}(y)}{y-x} - j\frac{G_j(x)G_{j-1}(y) - G_{j-1}(x)G_j(y)}{y-x} = G_j(x)G_j(y).$$

This relation also holds for $j = 0$ by defining $G_{-1} := 0$. Summing the above identities for $0 \leq j \leq n$ leads to (20). To prove (21), we observe that

$$\begin{aligned} \frac{G_{n+1}(x)G_n(y) - G_n(x)G_{n+1}(y)}{y-x} &= \frac{G_{n+1}(x) - G_{n+1}(y)}{y-x}G_n(y) \\ &- \frac{G_n(x) - G_n(y)}{y-x}G_{n+1}(y). \end{aligned}$$

Consequently, letting $y \to x$, we obtain (21) from (20) and the definition of the derivative. □

We denote the N+1 simple zero points of G_{N+1} on $(0, +\infty)$ by $\{x_n\}_{n=1}^{N+1}$. From equation (20), we have

$$K_N(x_m, x_n) = (N+1)\frac{G_{N+1}(x_m)G_N(x_n) - G_N(x_m)G_{N+1}(x_n)}{x_n - x_m} = 0.$$

Meanwhile, if $m = n$, from equation (21),

$$K_N(x_n, x_n) = (N+1)(G'_{N+1}(x_n)G_n(x_n) - G'_N(x_n)G_{N+1}(x_n)) = (N+1)G'_{N+1}(x_n)G_N(x_n).$$

Let $H_{K_N}(0, +\infty)$ be the (finite-dimensional) Hilbert space with the reproducing Kernel K_N. We consider the linear transform of $H_{K_N}(0, +\infty)$,

$$f(x_n) =< f, K_{x_n} >_{H_{K_N}(0,+\infty)} .$$

Therefore, if we set,

$$\chi_n = \|K_n\|^2_{H_{K_N}(0,+\infty)} = K(x_n, x_n) > 0,$$

we have

$$< K_{x_m}, K_{x_n} >_{H_{K_N}(0,+\infty)} = K(x_n, x_m) = \delta_{m,n}\chi_n.$$

Since $\{K_{x_n}\}_{n=0}^{N+1}$ are linearly independent, the linear mapping is isometric and we have the identity

$$\|f\|_{H_{K_N}(0,+\infty)} = (\sum_{n=0}^{K} \frac{|f(x)|^2}{\chi_n})^{\frac{1}{2}}.$$

Then, by sampling theorem, we have

$$\begin{aligned} f(x) &= < f, K_x >_{H_{K_N}(0,+\infty)} \\ &= \sum_{n=0}^{N} f(x_n) \frac{\overline{K(x_n, x)}}{\chi_n} \\ &= \sum_{n=0}^{N} f(x_n) \frac{K(x, x_n)}{K(x_n, x_n)} \\ &= \sum_{n=0}^{N} f(x_n) \frac{G_{N+1}(x)}{G'_{N+1}(x_n)(x - x_n)}. \end{aligned} \tag{23}$$

§5 Existence for f(η)

In this section, we will prove the existence and nonexistence of solutions to the problem (7) with conditions (8), and will also prove some qualitative results regarding the behavior of this solution.

Lemma 5.1. *Let $M \geq 0$ and $f \in C^3([0, +\infty))$ be a solution of the problem* (7) *with boundary value conditions* (8)*. Then the following assertions hold.*
(i) If η_0 be a minimum for f', then $f'(\eta_0) \geq 1$ or $f'(\eta_0) \leqslant -M - 1$.
(ii) If η_0 be a maximum for f', then $-M - 1 \leqslant f'(\eta_0) \leqslant 1$.

Proof. (i) Let f a solution of (7) and η_0 be a minimum of f'. so $f''(\eta_0) = 0$ and by equation (7), we have

$$f'''(\eta_0) + 1 - {f'}^2(\eta_0) + M(1 - f'(\eta_0) = 0.$$

Setting

$$m(x) = 1 - x^2 + M(1 - x).$$

Since $f''(\eta_0) \geq 0$, so $m(f'(\eta_0) \leqslant 0$ and the result follow.
(ii) We proceeding the same way as in the pervious Assertion, but with condition $m(f'(\eta_0)) \geq 0$. □

Lemma 5.2. *Let $M \geq 0, f''(0) > 0$ and $f \in C^3([0, +\infty))$ be a solution of the problem* (7) *with boundary value conditions* (8)*. Then $f''(\eta) > 0$ and $f'(0) < f'(\eta)$ for $\eta \in (0, +\infty)$*

Proof. Let

$$\eta_0 = sup\{\eta \in [0, +\infty) : \quad f''(\eta) > 0\}.$$

Since $f''(0) > 0$ and f'' is continuous on $(0, +\infty)$, $\quad \eta_0 > 0$. We prove that $\eta_0 = +\infty$. In fact, if

$$f''(\eta) < 0, \quad \text{on } (\eta_0, +\infty),$$

then f' is strictly decreasing on $(\eta_0, +\infty)$. Hence, for $\eta \in (\eta_0, +\infty)$, we have

$$1 = f'(\infty) < f'(\eta) < f'(\eta_0),$$

a condition. Because, it follows that $f'(\eta)$ has a positive maximum in $0 < \eta < +\infty$, which contradicts with (ii) in lemma (24). Equation $f'(0) < f'(\eta)$ follow from $f''(\eta) > 0$ for $\eta \in (0, +\infty)$. □

Lemma 5.3. *If* $\delta > 0$, $M > 0$, $\lambda > -1$ *and* $f \in C^3[0, +\infty)$ *is a solution of* (7)-(8), *then* f'' *is strictly decreasing on* $[0, +\infty)$ *and* $\lim_{\eta \to +\infty} f''(\eta) = 0$.

Proof. By $f'(+\infty) = 1$, We know

$$\lim_{\eta \to +\infty} inf f''(\eta) = 0, \tag{24}$$

and the exists η_m such that, for $\eta_m \to +\infty$,

$$f''(\eta_m) > 0, \quad and \quad f''(\eta_m) \to 0. \tag{25}$$

We prove that $f''(\eta)$ is strictly decreasing on $[0, +\infty)$. If there exist $0 \leq \eta_{m_1} \leq \eta_{m_2}$ such that $f''(\eta_{m_1}) < f''(\eta_{m_2})$. Then by equation (25), there exists $\eta_{m_3} \in (\eta_{m_2}, +\infty)$ such that $f''(\eta_{m_3}) < f''(\eta_{m_2})$. Let $\eta^*[\eta_{m_1}, \eta_{m_3}]$ such that

$$f''(\eta^*) = max\{f''(\eta) : \quad \eta \in [\eta_{m_1}, \eta_{m_2}]\} > 0. \tag{26}$$

This implies $f'''(\eta^*) = 0$ and $f^{(4)}(\eta^*) \leqslant 0$. Furthermore, by equation (7), we get

$$f^{(4)}(\eta) = f'(\eta)f''(\eta) - f(\eta)f'''(\eta) + Mf''(\eta). \tag{27}$$

By $M \geq 0$ and lemma (25), we have $f^{(4)} > 0$, then we have a contradiction. Hence, f'' is decreasing on $(0, +\infty)$. If there exist $0 \leqslant \eta_{m_1} < \eta_{m_2}$ such that $f''(\eta_{m_1}) = f''(\eta_{m_2})$, then

$$f''(\eta) = \text{constant}, \quad \text{on } (\eta_{m_1}, \eta_{m_2}),$$

and

$$f'''(\eta) = 0, \quad \text{on } (\eta_{m_1}, \eta_{m_2}).$$

By equation (27), we see

$$f^{(4)} = f'(\eta)f''(\eta) + Mf''(\eta) > 0, \quad on \quad (\eta_{m_1}, \eta_{m_2}),$$

which is a contradiction. Therefore $f''(\eta)$ is strictly increasing on $(0, +\infty)$, and then $\lim_{\eta \to +\infty} f''(\eta)$ exists. By (24), we obtain $\lim_{\eta \to +\infty} f''(\eta) = 0$. □

§6 Solving equation

In this section, we want to solve the nonlinear equation (7) with boundary conditions (8) through Laguerre reproducing kernel functions. Equation $\lim_{\eta \to \infty} f'(\eta) = 1$ implies the asymptotic behaviour $f(\eta) \to \eta + \alpha$, as $\eta \to \infty$, where α is an unknown constant. Therefore the application of Laguerre polynomials directly to the variable $f(\eta)$ is not possible and one must introduce a new unknown function by the following change of variables

$$f(\eta) = F(\eta) + \eta(1 - e^{-\frac{\eta}{2}}). \tag{28}$$

Thus from equation (28), we conclude that $f(0) = F(0)$, $f'(0) = F'(0)$, $f''(0) = 1 + F''(0)$ and $\lim_{\eta \to \infty} f'(\eta) = 1 + \lim_{\eta \to \infty} F'(\eta)$, where

$$\begin{aligned} f'(\eta) =& F'(\eta) + 1 - (1 - \frac{1}{2}\eta)e^{-\frac{\eta}{2}}, \\ f''(\eta) =& F''(\eta) + (1 - \frac{1}{4}\eta)e^{-\frac{\eta}{2}}, \\ f'''(\eta) =& F'''(\eta) - \frac{1}{8}(6 - \eta)e^{-\frac{\eta}{2}}. \end{aligned}$$

Therefore, the equation (7) changes to the following problem:

$$F'''(\eta)+F(\eta)F''(\eta)-(F'(\eta))^2+p(\eta)F''(\eta)+q(\eta)F'(\eta)+r(\eta)F(\eta)-s(\eta)=0, \tag{29}$$

where

$$\begin{aligned}p(\eta)=&\eta(1-e^{-\frac{\eta}{2}}),\\ q(\eta)=&-2-M+(2-\eta)e^{-\frac{\eta}{2}},\\ r(\eta)=&(1-\frac{1}{4}\eta)e^{-\frac{\eta}{2}},\\ s(\eta)=&-(M+2)(1-\frac{1}{2}\eta)e^{-\frac{\eta}{2}}+(1-\frac{1}{2}\eta)^2e^{-\eta}.\end{aligned}$$

Furthermore, the boundary conditions equation (8) changes to the following boundary conditions for the new unknown function F:

$$F(0)=0,\quad F'(0)=\lambda+\delta+\delta F''(0),\quad \lim_{\eta\to\infty}F'(\eta)=0. \tag{30}$$

If we assume that

$$\psi_n(\eta)=\begin{cases}\dfrac{G_{N+1}(\eta)}{G'_{N+1}(\eta_n)(\eta-\eta_n)}, & \eta\neq\eta_n,\\ 1, & \eta=\eta_n,\end{cases} \tag{31}$$

where $0\le n\le N$ and $\{\eta_n\}_{n=0}^N$ are zero points of $G_{N+1}(\eta)$. Of course, we have

$$\lim_{\eta\to\eta_n}\psi_n(\eta)=\lim_{\eta\to\eta_n}\frac{G'_{N+1}(\eta)}{G'_{N+1}(\eta_n)}=1,\quad 0\le n\le N.$$

So, the equation (23) changes to [15]

$$F(\eta)=\sum_{n=0}^{N}b_n\psi_n(\eta)=B^T\psi(\eta), \tag{32}$$

such that

$$\psi(\eta)=[\psi_0(\eta),\psi_1(\eta),\cdots,\psi_N(\eta)]^T,\quad B=[F(\eta_0),F(\eta_1),\cdots,F(\eta_N)]^T. \tag{33}$$

The derivation vector is defined by

$$\psi'(\eta)=\frac{d\psi(\eta)}{d\eta}=D\psi(\eta), \tag{34}$$

that D is a $(N+1)\times(N+1)$ matrix is called the derivative operator and the entries of that matrix estimated by

$$d_{nm}=\psi'_n(\eta_m).$$

Moreover, for any polynomial p of degree at most $N+1$, one gets:

$$p'(\eta_n)=\sum_{m=0}^{N}d_{nm}p'(\eta_m).$$

To find the entries of derivative operator, by equation (31), we have

$$[\frac{d}{d\eta}\psi_n](\eta)=\frac{G'_{N+1}(\eta)(\eta-\eta_n)-G_{N+1}(\eta)}{G'_{N+1}(\eta_n)(\eta-\eta_n)^2},\quad \eta\in(0,\infty),\quad \eta\neq\eta_n,\quad 0\le n\le N. \tag{35}$$

In the case $m=n$, from equation

$$G_{N+1}(\eta_n)=0,\quad n=0,1,\dots,N, \tag{36}$$

and equation (35), we deduce

$$\lim_{\eta \longrightarrow \eta_n} [\frac{d}{d\eta}\psi_n] = \lim_{\eta \longrightarrow \eta_n} \frac{G'_{N+1}(\eta)(\eta-\eta_n) - G_{N+1}(\eta)}{G'_{N+1}(\eta_n)(\eta-\eta_n)^2} = \frac{G''_{N+1}(\eta_n)}{2G'_{N+1}(\eta_n)}, \tag{37}$$

On the other hand, by equation (18), we have

$$G''_{N+1}(\eta_n) = -\frac{1}{\eta_n} G'_{N+1}(\eta_n), \quad n = 0, 1, \dots, N. \tag{38}$$

Substituting equation (38) into (37), we get

$$\lim_{\eta \longrightarrow \eta_n} [\frac{d}{d\eta}\psi_n] = -\frac{1}{2\eta_n}.$$

In the case $m \neq n$, by substituting equation (36) into (16), we have

$$G'_{N+1}(\eta_m) = -\frac{N+1}{\eta_m} G_N(\eta_m), \quad m = 0, 1, \dots, N. \tag{39}$$

Thus from (35), it follows that

$$[\frac{d}{d\eta}\psi_n](\eta_m) = \frac{\eta_n G_N(\eta_m)}{\eta_m G_N(\eta_n)(\eta_m - \eta_n)}.$$

Therefore

$$d_{nm} = \begin{cases} \dfrac{\eta_n G_N(\eta_m)}{\eta_m G_N(\eta_n)(\eta_m - \eta_n)}, & n \neq m, \\ -\dfrac{1}{2\eta_n}, & n = m. \end{cases} \tag{40}$$

Now using equations (32) and (33), we can write

$$F'(\eta) = B^T\psi'(\eta) = B^T D\psi(\eta), \qquad F''(\eta) = B^T D\psi'(\eta) = B^T D^2\psi(\eta), \tag{41}$$

and

$$F'''(\eta) = B^T D^3 \psi(\eta). \tag{42}$$

Also using equation (32) the functions $p(\eta)$, $q(\eta)$, $r(\eta)$ and $s(\eta)$ can be expanded as:

$$\begin{aligned} p(\eta) &\approx \sum_{n=0}^{N} p(\eta_n)\psi_n(\eta) = P^T\psi(\eta), \qquad q(\eta) \approx \sum_{n=0}^{N} q(\eta_n)\psi_n(\eta) = Q^T\psi(\eta), \\ r(\eta) &\approx \sum_{n=0}^{N} r(\eta_n)\psi_n(\eta) = R^T\psi(\eta), \qquad s(\eta) \approx \sum_{n=0}^{N} s(\eta_n)\psi_n(\eta) = S^T\psi(\eta), \end{aligned} \tag{43}$$

where

$$\begin{aligned} P &= [p(\eta_0), p(\eta_1), \cdots, p(\eta_N)]^T, \qquad Q = [q(\eta_0), q(\eta_1), \cdots, q(\eta_N)]^T, \\ R &= [r(\eta_0), r(\eta_1), \cdots, r(\eta_N)]^T, \qquad S = [s(\eta_0), s(\eta_1), \cdots, s(\eta_N)]^T. \end{aligned} \tag{44}$$

From expressions (32), (41), (42) and (43) we have

$$\begin{aligned} Res(\eta) = {} & B^T D^3\psi(\eta) + (B^T\psi(\eta))(B^T D^2\psi(\eta)) - (B^T D\psi(\eta))^2 + (P^T\psi(\eta))(B^T D^2\psi(\eta)) \\ & + (Q^T\psi(\eta))(B^T D\psi(\eta)) + (R^T\psi(\eta))(B^T\psi(\eta)) - S^T\psi(\eta) = 0. \end{aligned} \tag{45}$$

By collocating equation (45) in $N-1$ points η_j for $j = 3, 4, \dots, N$ we get

$$\begin{aligned} Res(\eta_j) = {} & B^T D^3\psi(\eta_j) + (B^T\psi(\eta_j))(B^T D^2\psi(\eta_j)) - (B^T D\psi(\eta_j))^2 \\ & + (P^T\psi(\eta_j))(B^T D^2\psi(\eta_j)) + (Q^T\psi(\eta_j))(B^T D\psi(\eta_j)) \\ & + (R^T\psi(\eta_j))(B^T\psi(\eta_j)) - S^T\psi(\eta_j) = 0. \end{aligned} \tag{46}$$

Using equation (32) and equation (41) in equation (30), we get

$$B^T\psi(0) = 0, \qquad B^T D\psi(0) - \delta B^T D^2\psi(0) = \lambda + \delta. \tag{47}$$

Table 1. Values of $f^{''}(0)$ for different values of λ and M with α= 0.

	$M = 0$		$M = 0.5$	$M = 1.0$	$M = 2.0$
λ	**Wang [31]**	**Present**	**Present**	**Present**	**Present**
5.0	-10.26475	-10.2647502	-10.4550325	-11.0072589	-12.9920640
2.0	-1.88731	-1.8873077	-1.9513821	-2.1326775	-2.7438711
1.0	0.0	0.0	0.0	0.0	0.0
0.5	0.71330	0.7132974	0.7553048	0.8696273	1.2260970
0.2	1.05113	1.0511301	1.1239352	1.3189875	1.9115629
0.0	1.232588	1.2325857	1.3294099	1.5853318	2.3466605

Considering $\lim_{\eta\to\infty} \frac{d}{d\eta}\psi(\eta) = 0$, it seems that the boundary condition $F^{'}(+\infty) = 0$ is automatically met. Equations (45) and (46) give an $N + 1$ system of nonlinear equation, which can be solve for $F(\eta_j),\ j = 0, 1, \cdots N$, using Newtons iterative method. So the unknown function of $F(\eta)$ can be found. In the process of computation, all the symbolic and numerical computations are performed by using Maple 16 Software Package.

An algorithm is a precisely defined sequence of steps for performing a specified task. The aim of the next algorithm is to implement a procedure to solve nonlinear boundary value problems in numeric form in terms of their grid nodes based on the use of Laguerre reproducing kernel method.

Algorithm 1 To approximate the solution of equations (7) and (8), we do the following steps:
Input: The numbers $\alpha, \delta, M \in \mathbb{R}$ and $N \in \mathbb{N}$.
Output: The approximation solution $f(\eta)$ for equations (7) and (8).
Step 1: Define the unknown function $F(\eta)$ by function $f(\eta)$ as in equation (28);
Step 2: Define the functions $G_n(x)$ for $n = 0, 1, \ldots, N$ by equation (11);
Step 3: Compute η_n the zeroes of Laguerre functions as in section 3.1;
Step 4: Construct the functions $\psi_n(\eta)$ for $n = 0, 1, \ldots, N$ by equation (31);
Step 5: Compute the square operational matrix D with the entries of d_{ij} as in equation (40);
Step 5: Make the unknown vector B and vector $\psi(\eta)$ by equation (33);
Step 6: Assign the vectors P, Q, R and S by equation (44);
Step 7: Solve the nonlinear algebraic equations system (46), (47) and compute the unknown vector B;
Step 8: Put $F(\eta) = B^T\psi(\eta)$;
Step 9: Use the transformation $f(\eta) = F(\eta) + \eta(1 - e^{-\frac{\eta}{2}})$.

To get a clear insight of the physical problem, numerical results are shown in Tables 1-3. Note that in (7), the second derivative at zero $f^{''}(0)$ is an important point of the function.

Table 2. Comparison of the values of skin friction coefficient $f^{''}(0)$ for $\delta = 0$, $M = 0$ and several values of λ.

	Bhattacharyya [9]		**Present**	
λ	**First solution**	**Second solution**	**First solution**	**Second solution**
-0.25	1.40224051	——	1.4022407	——
-0.50	1.49566948	——	1.4956699	——
-0.75	1.48929834	——	1.4892927	——
-1.00	1.32881689	0.0	1.3288170	0.0
-1.15	1.08223164	0.11667340	1.0822315	0.1167031
-1.20	0.93247253	0.23364909	0.9324738	0.2336497
-1.24	——	——	0.7066019	0.4356713
-1.246	——	——	0.5843750	0.5542150
-1.2465	0.58429146	0.55428565	0.5842918	0.5542896
-1.24657	——	——	0.5745288	0.5542964
-1.246579	——	——	0.5692646	0.5550269

Here, we have to calculate to this equation. To validate the numerical results obtained in this study, they are compared with values obtained by Wang [31] as given in Table 1, for several values of M and λ. From this Table it is evident that the present results are found to be in good agreement with the results of Wang [31] in the cases of $M = 0$. According to Ref [9], the unsteady stagnation-point flow over a shrinking sheet has self-similar solution for certain values of velocity ratio parameter λ. For $\delta = 0, 0.1$, with $M = 0$ of the ranges λ where dual solutions exist are $-1.2465 \leq \lambda < -1$, $-1.31067 \leq \lambda < -1$, respectively with no similarity solution for $\lambda < -1.2465$, $\lambda < -1.31067$, respectively.

Table 3. Skin friction coefficient $f^{''}(0)$ with $\delta = 0.2$, $M = 0.2$. and several values of λ.

	Present	
λ	**First solution**	**Second solution**
-1.24	1.2446725	0.1976892
-1.22	1.2818786	0.1590704
-1.2	1.3144899	0.1258411
-1.0	1.4973281	0.0037855
1.0	0.0	——
3.0	-3.4237221	——
5.0	-7.7860408	——
10.0	-21.2162615	——

Table 2 shows a comparison of the values of $f^{''}(0)$ obtained in the present study and those obtained by Bhattacharyya [9], showing that these values are in excellent agreement. We observed that, for $M = 0$ and $\delta = 0$, it has dual solutions for $-1.246579 \leq \lambda \leq -1$, the solution

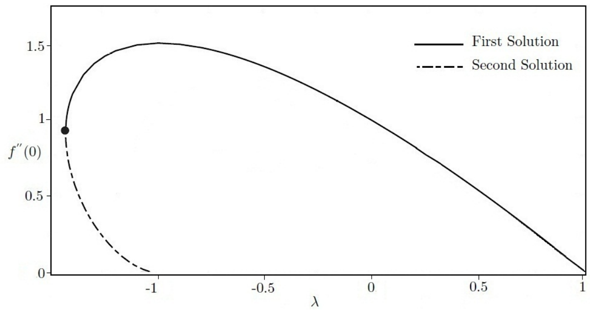

Figure 2. Skin friction coefficient $f^{''}(0)$ with $\lambda = -1.1$ and $M = 0$.

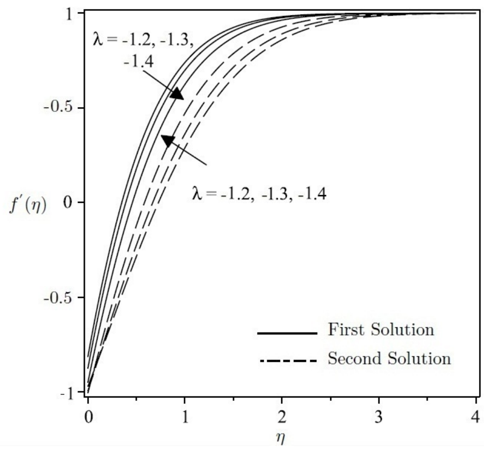

Figure 3. Shear stress profiles $f^{'}(\eta)$ with $\delta = 0.1,\ M = 0.1$ and several values of λ.

is unique for $\lambda > -1$ but there is no solution for $\lambda < -1.246579$.
The results for the stretching and shrinking case are given in Table 3. For the axisymmetric case, solutions seem to be unique for $\lambda > -1$.

Figure 2 shows the variation of the skin friction coefficient for $M = 0$ and $\lambda = -1.1$. As it is shown, there are dual solutions for the skin friction coefficient $f^{''}(0)$ for some values of unsteadiness parameter λ, that is, for $-1.424 < \lambda < -1$, but there is no solution for $\lambda < -1.424$, and a unique solution exists for $\lambda > -1$. The velocity, shear stress and temperature profiles for several values of λ are demonstrated in Figure3. In Figure3, the dual velocity profiles $f^{'}(\eta)$ confirm that the velocity decreases with increasing values of λ in first solution and conversely for second solution.

§7 Conclusion

In the present paper, a new computational method is proposed to solve a class of nonlinear differential equations driven by stagnation flow towards a shrinking and stretching sheet. The method is based on a new class of orthogonal polynomials, namely the Laguerre reproducing kernel method. We applied the Laguerre reproducing kernel method to singular third order three-point boundary value problems and obtained approximate solutions with a high degree of accuracy. The operational matrix of the derivative is derived and used in the implementation of the proposed method. These matrix play an important role in modelling of problems. The main advantage of the proposed method is the reduction of the problem to a simpler one, which consists of solving a system of nonlinear algebraic equations. In addition, the nonlinear ordinary differential equation is solved numerically through using an efficient and accurate numerical technique called the Laguerre reproducing kernel method, where the second derivative at zero is an important point of the function, so we have computed $f^{''}(0)$ and have compared it with other results. Moreover, the reliability and applicability of the proposed method are demonstrated by solving several numerical examples.

References

[1] M S Abel, P S Datti, N Mahesha. *Flow and heat transfer in a power-law fluid over a stretching sheet with variable thermal conductivity and nonuniform heat source*, Int J Heat Mass Transfer, 2009, 852(11-12): 2902-2913.

[2] O A Arqub. *Numerical algorithm for the solutions of fractional order systems of Dirichlet function types with comparative analysis*, Fundam Inform, 2019, 166: 111-137.

[3] O A Arqub. *Numerical solutions of systems of first-order, two-point BVPs based on the reproducing kernel algorithm*, Calcolo, 2018, 55: 1-28.

[4] O A Arqub, M Al-Smadi. *Atangana Baleanu fractional approach to the solutions of Bagley Torvik and Painleve equations in Hilbert space*, Chaos Solitons Fractals, 2018, 117: 161-167.

[5] O A Arqub, B Maayah. *Fitted fractional reproducing kernel algorithm for the numerical solutions of ABC-fractional Volterra integro-differential equations*, Chaos Solitons Fractals, 2019, 126: 394-402.

[6] O A Arqub, B Maayah. *Modulation of reproducing kernel hilbert space method for numerical solutions of Riccati and Bernoulli equations in the Atangana Baleanu fractional sense*, Chaos Solitons Fractals, 2019, 125: 163-170.

[7] O A Arqub, B Maayah. *Numerical solutions of integrodifferential equations of fredholm operator type in the sense of the Atangana Baleanu fractional operator*, Chaos Solitons Fractals, 2018, 117: 117-124.

[8] O A Arqub, N Shawagfeh. *Application of reproducing kernel algorithm for solving Dirichlet time-fractional diffusion-Gordon types equations in porous media*, J Porous Media, 2019, 22(4): 411-434.

[9] K Bhattacharyya, S Mukhopadhyay, G C Layek. *Slip effects on boundary layer stagnation-point flow and heat transfer towards a shrinking sheet*, Int J Heat Mass Transf, 2011, 54: 308-313.

[10] L J Crane. *Flow past a stretching plate*, Z Angew Math Phys, 1970, 21: 645-647.

[11] M G Cui, F Z Geng. *A computational method for solving one-dimensional variable-coefficient Burgers equation*, Appl Math Comput, 2007, 188: 1389-1401.

[12] M G Cui, F Z Geng. *Solving singular two-point boundary problem in reproducing kernel space*, J Comput Appl Math, 2007, 205: 6-15.

[13] T Fang, S Yao, I Pop. *Slip MHD viscous flow over a stretching sheet an exact solution*, Comm Nonlinear Sci Numer Simulat, 2009, 14(11): 3731-3737.

[14] M R Foroutan, A Ebadian, R Asadi. *Reproducing kernel method in Hilbert spaces for solving the linear and nonlinear four-point boundary value problems*, Int J Comput Math, 2018, 95(10): 2128-2142.

[15] M R Foroutan, A Ebadian, Sh Najafzadeh. *Analysis of unsteady stagnation-point flow over a shrinking sheet and solving the equation with rational Chebyshev functions*, Math Meth Appl Sci, 2016, 40(7): 2610-2622.

[16] D Funaro. *Polynomial Approximation of Differential Equations*, Berlin, Springer-Verlag, 1992.

[17] F Z Geng. *Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method*, Appl Math Comput, 2009, 215: 2095-2102.

[18] F Z Geng, M G Cui. *Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space*, Appl Math Comput, 2007, 192: 389-398.

[19] F Z Geng, S P Qian, S Li. *A numerical method for sinqularly perturbed turning point problems with an interior layer*, J Comput Appl Math, 2013, 255: 97-105.

[20] M Inc, A Akgül. *Approximate solutions for MHD squeezing fluid flow by a novel method*, Bound Value Probl, 2014, 18: 1-17.

[21] W Jiang, Y Lin. *Representation of exact solution for the time-fractional telegraph equation in the reproducing Kernel space*, Commun Nonlinear Sci Numer Simul, 2011, 16(9): 3639-3645.

[22] M Kaleghi, E Babolian, S Abbasbandy. *Chebyshev reproducing kernel method: application to two-point boundary value problems*, Adv differential equ, 2017, 2017(26): 1-19.

[23] Y Z Lin, J Niu, M G Cui. *A numerical solution to nonlinear second order three-point boundary value problems in the reproducing kernel space*, Appl Math Comput, 2012, 218: 7362-7368.

[24] M K A Mohamed, M Z Salleh, R Nazar, A Ishak. *Stagnation point flow over a stretching sheet with Newtonian heating*, Sains Malays, 2012, 41(11): 1467-1473.

[25] T Hayat, Z Iqbal, M Mustafa. *Flow of a second grade fluid over a stretching surface with Newtonian heating*, J Mech, 2012, 28(1): 209-216.

[26] J Niu, Y Z Lin, M Cui. *Approximate solutions to three point boundary value problems with two-space integral condition for parabolic equations*, Abstr Appl Anal, 2012, 2012, Article ID: 414612, 9 pages, doi: 10.1155/2012/414612.

[27] J Niu, Y Z Lin, C P Zhang. *Approximate solution of nonlinear multi-point boundary value problem on the half-line*, Math Model Anal, 2012, 2: 190-202.

[28] J Niu, Y Z Lin, C P Zhang. *Numerical solution of nonlinear three-point boundary value problem on the positive half-line*, Math Methods Appl Sci, 2012, 35: 1601-1610.

[29] X Qing, J Niu. *An improved reproducing kernel method for Fredholm integro-differential type two-point boundary value problems*, Int J Comput Math, 2017, 95(5): 1015-1023.

[30] M Suail, N M A Nik Long, A Ishak. *Unsteady stagnation point flow and heat transfer over a stretching/shrinking sheet with prescribed surface heat flux*, Appl Math Comput Intel, 2012, 1: 1-11.

[31] C Y Wang. *Stagnation flow towards a shrinking sheet*, Int J Non-linear Mech, 2008, 43: 377-382.

[32] G C Yang, K Q Lan. *Nonexistence of the reversed flow solutions of the Falkner-Skan equations*, Nonlinear Anal, 2011, 74: 5327-5339.

[33] C P Zhang, J Niu, Y Z Lin. *Numerical solutions for the three-point boundary value problem of nonlinear fractional differential equations*, Abstr Appl Anal, 2012, 2012, Article ID: 360631, 16 pages, doi: 10.1155/2012/360631

[34] Z H Zhao, Y Z Lin, J Niu. *Convergence Order of the Reproducing Kernel Method for Solving Boundary Value Problems*, Math Model Anal, 2016, 21: 466-477.

Department of Mathematics, Payame Noor University, P O Box 19395-3697, Tehran, Iran.
Email: *foroutan_mohammadreza@yahoo.com*