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Fractal interpolation: a sequential approach

N. Vijender[1]   M. A. Navascués[2]

Abstract. Fractal interpolation is a modern technique to fit and analyze scientific data. We develop a new class of fractal interpolation functions which converge to a data generating (original) function for any choice of the scaling factors. Consequently, our method offers an alternative to the existing fractal interpolation functions (FIFs). We construct a sequence of α-FIFs using a suitable sequence of iterated function systems (IFSs). Without imposing any condition on the scaling vector, we establish constrained interpolation by using fractal functions. In particular, the constrained interpolation discussed herein includes a method to obtain fractal functions that preserve positivity inherent in the given data. The existence of $\mathcal{C}^r$-α-FIFs is investigated. We identify suitable conditions on the associated scaling factors so that α-FIFs preserve r-convexity in addition to the $\mathcal{C}^r$-smoothness of original function.

§1 Introduction

Interpolation, which deals with the construction of a function in continuum from its availability in a finite set of points, has been cultivated for many decades. In view of its increasing relevance in this age of ever-increasing digitization, it is quite natural that the subject of interpolation is receiving more and more attention. Consequently, various types of interpolation schemes are being developed. The classical interpolation techniques fit an elementary function to the given data in order to render a connected visualization of a sample. Such elementary functions often imbue the visualization with a degree of smoothness that may not be consistent with the nature of a prescribed data set. Fractals and fractal interpolation functions have been applied to prevent such inappropriate smoothing [1, 2]. Utilizing the iterated function system (IFS) theory [3], Barnsley [4] proposed the concept of a fractal interpolation function such that it is the attractor of a specific IFS. In general, FIFs are fixed points of the Read-Bajraktarević operator, which are defined on suitable function spaces. Using fractal interpolation methodology, it is possible to construct interpolants with integer or non-integer dimensions. Many researchers have contributed to the theory of fractal functions by constructing different kinds of FIFs, including spline FIFs [5–7] and hidden variable FIFs [8,9]. The concept of FIF can be used

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to associate a family of functions to a given function f defined on a real compact interval I (for instance, see [4, 10–19]). An element of this family is denoted by f^α and Navascués [15] named it as α-fractal function associated with f. This function f^α contains a set of real parameters, namely, scaling factors. The rigidity properties for fractal functions were studied in [20, 21].

For every $n \in \mathbb{N}$, let $b_n : \mathcal{C}(I) \to \mathcal{C}(I)$ be bounded and nonidentity linear operator such that the following properties are satisfied for every $f \in \mathcal{C}(I)$.

$$b_n \neq f, b_n(f)(x_1) = f(x_1), b_n(f)(x_N) = f(x_N), \text{ and } ||b_n(f) - f||_\infty \to 0 \text{ as } n \to \infty. \quad (1)$$

There exist many operators which satisfy the above properties, for instance, Bernstein operator [22]. Using an interpolant $f \in \mathcal{C}(I)$ of a data set $\{(x_i, z_i) : i = 1, 2, \ldots, N\}$ and the functions $b_n(f), n \in \mathbb{N}$, we construct a sequence of α-FIFs f_n^α, $n \in \mathbb{N}$. Convergence of the α-FIFs f_n^α, $n \in \mathbb{N}$ towards f follows from the convergence of the sequence $\{b_n(f)\}_{n=1}^\infty$ towards f. Owing to this reason, the α-FIFs f_n^α, $n \in \mathbb{N}$ converge to the data generating function whenever the interpolant f converges uniformly to a given data generating function. Thus, the α-FIFs f_n^α, $n \in \mathbb{N}$ converge to the data generating function for all the scaling factors whereas the existing fractal interpolants converge to data generating function when the magnitude of the scaling factors goes to zero.

Finding an interpolating curve that lies completely above or below a prefixed curve is called constrained interpolation. There are practical situations wherein interpolating curves that lie completely above or below a prefixed curve at certain intervals, for instance, a polygonal (piecewise linear function) or a quadratic spline are sought-after. Many researchers have studied constrained fractal interpolants [7,23–30]. The existing constrained fractal interpolants converge to the data generating function if the magnitude of the scaling factors goes to zero. In this paper, we investigate the constrained interpolation problem by the proposed fractal functions without any condition on the scaling factors. Imposing the suitable conditions on the operators $b_n, n \in \mathbb{N}$, we study $\mathcal{C}^r$-α-FIFs. Further, we investigate the conditions on the scaling factors which ensure the r-convexity of $\mathcal{C}^r$-α-FIFs whenever a given data generating function possesses $\mathcal{C}^r$-continuity and r-convexity.

1.1 IFS and attractor

The following notation and terminologies will be used throughout the article. The set of real numbers will be denoted by $\mathbb{R}$, whilst the set of natural numbers by $\mathbb{N}$. For a fixed $N \in \mathbb{N}$, we shall write $\mathbb{N}_N$ for the set of first N natural numbers. Given real numbers x_1 and x_N with $x_1 < x_N$, let $I = [x_1, x_N]$. We define $\mathcal{C}^r(I)$ to be the space of all real-valued functions on I that are r-times differentiable with continuous r-th derivative. Let $(\mathcal{X}, d)$ be a complete metric space, and $\mathcal{H}(\mathcal{X})$ be the set of all nonempty compact subsets of $\mathcal{X}$. Then $\mathcal{H}(\mathcal{X})$ is a complete metric space with respect to the Hausdorff metric h_d, where h_d is defined as $\mathsf{h}_d(A, B) = \max\{d(A, B), d(B, A)\}$, and

$$d(A, B) = \max_{x \in A} \min_{y \in B} d(x, y).$$

Let $\vartheta_i : \mathcal{X} \to \mathcal{X}$ be continuous functions for $i \in \mathbb{N}_{N-1}$. The set $\mathcal{I} = \{\mathcal{X}; \vartheta_i, i \in \mathbb{N}_{N-1}\}$ is called an IFS. An IFS $\mathcal{I}$ is called *hyperbolic* if

$$\frac{d(\vartheta_i(x), \vartheta_i(y))}{d(x,y)} \le |c_i| < 1, \forall\ x \ne y \in \mathcal{X}.$$

For any $A \in \mathcal{H}(\mathcal{X})$, we define the set valued Hutchinson map V on $\mathcal{H}(\mathcal{X})$ as

$$V(A) = \bigcup_{i \in \mathbb{N}_{N-1}} \vartheta_i(A).$$

If IFS $\mathcal{I}$ is *hyperbolic,* then it is easy to verify that V is a contraction map on $\mathcal{H}(\mathcal{X})$ with the contractive factor $c = \max\{|c_i| : i \in \mathbb{N}_{N-1}\}$. Then by the Banach Fixed Point Theorem, V has a unique fixed point (say G) and for any starting set A in $\mathcal{H}(\mathcal{X})$ with $V(A) = V^{\circ 1}(A)$, $V^{\circ m}(A) = V \circ V^{\circ m-1}(A)$ for $m \ge 2$,

$$\lim_{m \to \infty} V^{\circ m}(A) = G.$$

The set $G \in \mathcal{H}(\mathcal{X})$ is called the attractor or the deterministic fractal of the IFS $\mathcal{I}$.

§2 Construction of α-FIFs

Let the interpolation data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$, $x_1 < x_2 < \cdots < x_{N-1} < x_N$, be obtained from the original function $\Phi \in \mathcal{C}(I)$. Let $f \in \mathcal{C}(I)$ be a classical interpolant that interpolates the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$. Let K be a suitable compact subset of $\mathbb{R}$ such that $z_i \in K$, $i \in \mathbb{N}_N$. Let $L_i : I \to I_i$, $i \in \mathbb{N}_{N-1}$ be contractive homeomorphisms defined by $L_i(x) = a_i x + b_i$ such that

$$L_i(x_1) = x_i, L_i(x_N) = x_{i+1}. \tag{2}$$

Let $F_{n,i} : \mathcal{D} \times K \to K$, $i \in \mathbb{N}_{N-1}$ be continuous functions defined by

$$F_{n,i}(x, z) = \alpha_i z + f(L_i(x)) - \alpha_i b_n(f)(x), \tag{3}$$

where α_i is a real parameter satisfying $|\alpha_i| < 1$ and such that $b_n(f) \ne f, b_n(f)(x_1) = f(x_1)$, and $b_n(f)(x_N) = f(x_N)$, and $b_n(f) \to f$ as $n \to \infty$ uniformly. One can verify from (3) that

$$F_{n,i}(x_1, z_1) = z_i, F_{n,i}(x_N, z_N) = z_{i+1}, i \in \mathbb{N}_{N-1}, n \in \mathbb{N}. \tag{4}$$

Let $\mathcal{G} = \big\{g \in \mathcal{C}(I)\ |g(x_1) = z_1 \text{ and } g(x_N) = z_N\big\}$. Then $\mathcal{G}$ is a complete metric space with respect to the uniform metric ρ defined by

$$\rho(g, h) = \max\{|g(x) - h(x)| : x \in I\}\ \forall\ g, h \in \mathcal{G}.$$

Define the Read-Bajraktarević operator T_n on $(\mathcal{G}, \rho)$ as

$$T_n g(x, y) = F_{n,i}\big(L_i^{-1}(x), g(L_i^{-1}(x))\big),\ x \in I_i,\ i \in \mathbb{N}_{N-1}. \tag{5}$$

We prove first in the sequel that T_n maps $\mathcal{G}$ into itself. For $g \in \mathcal{G}$,

$$T_n g(x_1) = F_{n,1}\big(L_1^{-1}(x_1), g(L_1^{-1}(x_1))\big) = F_{n,1}(x_1, g(x_1)) = z_1,$$

$$T_n g(x_N) = F_{n,N-1}\big(L_{N-1}^{-1}(x_N), g(L_{N-1}^{-1}(x_N))\big) = F_{n,N-1}(x_N, g(x_N)) = z_N.$$

Using the properties of L_i, $F_{n,i}$, and (2)-(4), it is easy to verify that $T_n g$ is continuous on the intervals I_i, $i \in \mathbb{N}_{N-1}$, and at each of the points $x_2, \ldots, x_{N-1}$. Also,

$$\rho(T_n g, T_n h) \le |\alpha|_\infty \rho(g, h),$$

where $|\alpha|_\infty = \max\{|\alpha_i| : i \in \mathbb{N}_{N-1}\} < 1$. Hence, T_n is a contraction map on the complete metric space $(\mathcal{G}, \rho)$. Therefore, by the Banach fixed point theorem, T_n possesses a unique fixed point (say) f_n^α on $\mathcal{G}$, i.e., $(T_n f_n^\alpha)(x) = f_n^\alpha(x)$ for all $x \in I$. According to (5), the function f_n^α satisfies the following functional equation:

$$f_n^\alpha(x) = \alpha_i f_n^\alpha(L_i^{-1}(x)) + f(x) - \alpha_i b_n(f)(L_i^{-1}(x)),\ x \in I_i, i \in \mathbb{N}_{N-1}. \tag{6}$$

Also, it is easy to verify that

$$f_n^\alpha(x_i) = z_i \ \forall\ i \in \mathbb{N}_N, n \in \mathbb{N}. \tag{7}$$

Now, define the functions $\omega_{n,i} : I \times K \to I_i \times K$, $i \in \mathbb{N}_{N-1}$ as

$$\omega_{n,i}(x,z) = \big(L_i(x), F_{n,i}(x,z)\big).$$

Let $\mathcal{X} := I \times K$ and consider the IFS $\mathcal{I}_n = \{\mathcal{X}; \omega_{n,i} : i \in \mathbb{N}_{N-1}\}$. Using [4], it follows that the above IFS $\mathcal{I}_n$ has unique attractor $G_n \in \mathcal{H}(I \times K)$, and G_n is graph of f_n^α. Hence, the above function f_n^α is called α-fractal interpolant function. Also, we conclude that there exists a sequence $\{f_n^\alpha(x)\}_{n=1}^\infty$ of α-FIFs in which every function interpolates the data set $\{(x_i, z_i) : i \in \mathbb{N}_N\}$.

From (6), we have

$$\begin{aligned} &||f_n^\alpha - f||_\infty \le |\alpha|_\infty ||f_n^\alpha - b_n(f)||_\infty \le |\alpha|_\infty [||f_n^\alpha - f||_\infty + ||f - b_n(f)||_\infty], \\ &\Longrightarrow ||f_n^\alpha - f||_\infty \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} ||f - b_n(f)||_\infty. \end{aligned} \tag{8}$$

Further, it is easy to see that

$$||\Phi - f_n^\alpha||_\infty \le ||\Phi - f||_\infty + ||f - f_n^\alpha||_\infty \le ||\Phi - f||_\infty + \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} ||f - b_n(f)||_\infty. \tag{9}$$

The proof of the following theorem follows from (1) and (9).

Theorem 2.1. *Let $\Phi \in \mathcal{C}(I)$ be the original function providing the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$. Assume that $b_n(f) \to f$ as $n \to \infty$ uniformly for every $f \in \mathcal{C}(I)$. If $f \in \mathcal{C}(I)$ interpolates Φ with respect to the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$ such that*

$$||\Phi - f|| = O(h^r), r > 0, h = \max_{i \in \mathbb{N}_{N-1}} (x_{i+1} - x_i),$$

then the sequence $\{f_n^\alpha(x)\}_{n=1}^\infty$ of α-FIFs converges to Φ as $h \to 0$ and $n \to \infty$.

2.1 α-Affine FIFs

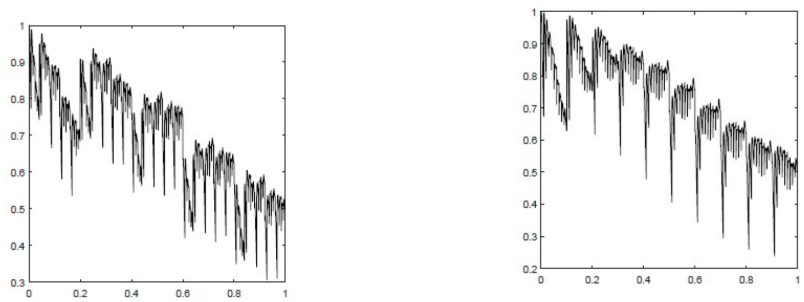
Let the data set $\{(x_i, z_i) : i \in \mathbb{N}_N\}$ be obtained from an unknown function $\Phi \in \mathcal{C}(I)$. It is easy to verify that the function

$$f(x) = \frac{z_{i+1} - z_i}{x_{i+1} - x_i} x + \frac{z_i x_{i+1} - z_{i+1} x_i}{x_{i+1} - x_i}, x \in I_i, i \in \mathbb{N}_{N-1}$$

interpolates the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$. Then, the corresponding fractal functions $f_n^\alpha, n \in \mathbb{N}$ are called the α-affine FIFs. From [31], it follows that $||\Phi - f||_\infty = O(h)$. Therefore, according to Theorem 2.1, α-affine FIFs $f_n^\alpha, n \in \mathbb{N}$, converge to the original function $\Phi \in \mathcal{C}(I)$ if $h \to 0$ and $n \to \infty$ whereas the existing affine FIFs [31] converge to the original function if $h \to 0$ and $|\alpha|_\infty \to 0$. To construct examples of α-affine FIFs, let us take Bernstein polynomial $B_n(f)(x)$ of f for $b_n(f)(x)$. That is, for all $x \in I = [0,1]$ and $n \in \mathbb{N}$,

$$b_n(f)(x) = B_n(f)(x) = \frac{1}{(x_N - x_1)^n} \sum_{k=0}^n \binom{n}{k} (x - x_1)^k (x_N - x)^{n-k} f\Big(x_1 + \frac{k(x_N - x_1)}{n}\Big).$$

Let $J_1 = \{(0,1),(0.2,0.9615),(0.4,0.8621),(0.6,0.7353),(0.8,0.6098),(1,0.5)\}$ and $J_2 = \{(0,1),$ $(0.1,0.9901),(0.2,0.9615),(0.3,0.9174),(0.4,0.8621),(0.5,0.8),(0.6,0.7353),(0.7,0.6711),(0.8,$ $0.6098),(0.9,0.5525),(1,0.5)\}$ be obtained from a unknown function Φ. The α-affine FIF f_3^α is generated in Figure 1(a) with respect to J_1 and $\alpha_i = 0.9$, $i \in \mathbb{N}_5$. Similarly, the α-affine FIF f_7^α is generated in Figure 1(b) with respect to J_2 and $\alpha_i = 0.9$, $i \in \mathbb{N}_{10}$. According to the Theorem 2.1, the α-affine FIF f_7^α provides a better approximation for the original function Φ than that obtained by f_3^α.



(a): f_3^α with J_1 and $\alpha_i = 0.9$, $i \in \mathbb{N}_5$.

(b): f_7^α with respect to J_2 with $\alpha_i = 0.9$, $i \in \mathbb{N}_{10}$.

Figure 1. α-affine FIFs.

2.2 Constrained fractal functions

Theorem 2.2. *Let $f \in \mathcal{C}(I)$ be an interpolant of the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$ such that $f(x) \geq \phi(x)$ for all $x \in I$, where ϕ is a prefixed curve satisfying $\phi(x_i) \leq z_i, i \in \mathbb{N}_N$. Then, for every scaling vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$ and $\epsilon > 0$, there exists an approximating sequence $\{g_{n,\alpha}\}_{n=1}^\infty$ of fractal functions such that $g_{n,\alpha}(x) \geq \phi(x)$ for all $x \in I$ and $n \geq N_0(\epsilon) \in \mathbb{N}$.*

Proof. Under the stated conditions, Theorem 2.1 ensures the existence of a sequence $\{f_n^\alpha\}_{n=1}^\infty$ of α-FIFs such that

$$||f_n^\alpha - f||_\infty < \frac{\epsilon}{2} \ \forall\, n \geq N_0(\epsilon) \in \mathbb{N}.$$

Define fractal functions

$$g_{n,\alpha}(x) = f_n^\alpha(x) + \frac{\epsilon}{2} \ \forall\, x \in I, n \in \mathbb{N}.$$

Next,

$$g_{n,\alpha}(x) = f_n^\alpha(x) + \frac{\epsilon}{2} = f(x) + f_n^\alpha(x) + \frac{\epsilon}{2} - f(x) \geq f(x) + \frac{\epsilon}{2} - ||f_n^\alpha - f||_\infty \geq f(x) \geq \phi(x) \ \forall x \in I.$$

Thus, we complete the proof. □

The next theorem is an immediate consequence of the previous theorem.

Theorem 2.3. *(Positivity preserving fractal interpolation) Let $f \in \mathcal{C}(I)$ be an interpolant of the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$, where $z_i \geq 0, i \in \mathbb{N}_N$. Then, for every scaling vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$ and $\epsilon > 0$, there exists an approximating sequence $\{g_{n,\alpha}\}_{n=1}^\infty$ of fractal functions such that $g_{n,\alpha}(x) \geq 0$ for all $x \in I$ and $n \geq N_0(\epsilon) \in \mathbb{N}$.*

Theorem 2.4. *Let $f \in \mathcal{C}(I)$ be an interpolant of the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$ such that $f(x) \leq \phi(x)$ for all $x \in I$, where ϕ is a prefixed curve satisfying $\phi(x_i) \geq z_i, i \in \mathbb{N}_N$. Then, for every scaling vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$ and $\epsilon > 0$, there exists an approximating sequence $\{h_{n,\alpha}\}_{n=1}^{\infty}$ of fractal functions such that $h_{n,\alpha}(x) \leq \phi(x)$ for all $x \in I$ and $n \geq N_0(\epsilon) \in \mathbb{N}$.*

Proof. Let $\epsilon > 0$. Under the stated conditions, Theorem 2.1 ensures the existence of a sequence $\{f_n^\alpha\}_{n=1}^{\infty}$ of α-FIFs such that

$$||f_n^\alpha - f||_\infty < \frac{\epsilon}{2} \;\forall\, n \geq N_0(\epsilon) \in \mathbb{N}.$$

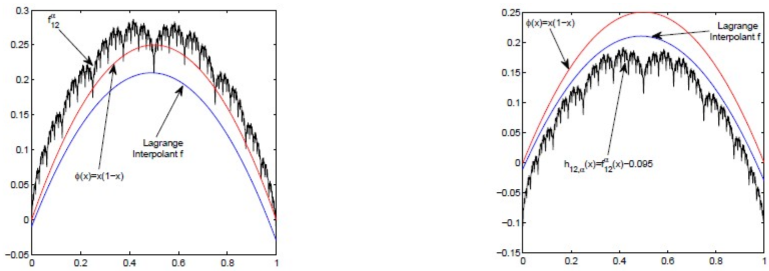
Define fractal functions

$$h_{n,\alpha}(x) = f_n^\alpha(x) - \frac{\epsilon}{2} \;\; \forall x \in I, n \in \mathbb{N}.$$

Next,

$$h_{n,\alpha}(x) = f_n^\alpha(x) - \frac{\epsilon}{2} = f(x) + f_n^\alpha(x) - \frac{\epsilon}{2} - f(x) \leq f(x) - \frac{\epsilon}{2} + ||f_n^\alpha - f||_\infty \leq f(x) \leq \phi(x) \;\; \forall x \in I.$$

Hence, the result follows. □



(a): The α-fractal function f_{12}^α with $\alpha_1 = \alpha_2 = 0.9$, $\phi(x) = x(1-x), x \in [0,1]$, and Lagrange interpolant f.

(b): The fractal function $h_{12,\alpha}$, $\phi(x) = x(1-x), x \in [0,1]$, and Lagrange interpolant f.

Figure 2. Constrained fractal functions.

Example: Let us illustrate the results in Theorem 2.4 using numerical examples. For this purpose, let $\phi(x) = x(1-x)$, $x \in [0,1]$ be the prefixed curve. Consider the data set $\{(0, -0.01), (0.5, 0.21), (1, -0.03)\}$. Note that the prescribed data set lies below the prefixed curve. Let f be the Lagrange interpolant of the prescribed data and f is evaluated as

$$f(x) = -0.02(x-1)\big(x - \frac{1}{2}\big) - 0.84x(x-1) - 0.06x\big(x - \frac{1}{2}\big), x \in [0,1].$$

The functions ϕ and f are plotted in Figures 2(a)-(b). To construct the α-fractal function f_n^α of f, let us take $b_n(f)(x) = B_n(f)(x)$ for all $x \in [0,1]$, where $B_n(f)(x)$ is the Bernstein polynomial of f. Let us fix $\epsilon = 0.19$ and $\alpha_1 = \alpha_2 = 0.9$. The α-fractal function f_{12}^α of f is generated in Figure 2(a). Also, it is calculated that $\|f_{12}^\alpha - f\|_\infty = 0.0872 < \frac{\epsilon}{2}$. From Figure 2(a), we can easily notice that f_{12}^α does not satisfy the condition $f_{12}^\alpha(x) \leq \phi(x)$ for all $x \in [0,1]$. Next, we have generated the fractal function $h_{12,\alpha}(x) = f_{12}^\alpha(x) - \frac{\epsilon}{2}$ in Figure 2(b). From Figure 2(b), it can be observed that $h_{12,\alpha}(x) < \phi(x)$ for all $x \in [0,1]$. Further, it calculated that $\|h_{12,\alpha} - f\|_\infty = 0.095 < \epsilon$. Hence, Theorem 2.4 is numerically verified.

§3 $\mathcal{C}^r$-α-FIFs

Theorem 3.1. *Let $f \in \mathcal{C}^r(I)$, $r \in \mathbb{N}$ be an interpolant of the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$, $x_1 < x_2 < \cdots < x_N$. For every $n \in \mathbb{N}$, let $b_n : \mathcal{C}^r(I) \to \mathcal{C}^r(I)$ be a bounded and linear operator such that the following properties are satisfied for every $\psi \in \mathcal{C}^r(I)$. For all $k \in \mathbb{N}_r \cup \{0\}$,*

$$b_n^{(k)}(\psi)(x_1) = \psi^{(k)}(x_1), b_n^{(k)}(\psi)(x_N) = \psi^{(k)}(x_N), \text{ and } ||b_n(\psi) - \psi||_{\mathcal{C}^r} \to 0 \text{ as } n \to \infty. \tag{10}$$

Suppose that $L_i : I \to I_i$, $i \in \mathbb{N}_{N-1}$ are affine maps $L_i(x) = a_i x + b_i$ satisfying $L_i(x_1) = x_i$ and $L_i(x_N) = x_{i+1}$, and $F_{n,i}(x, y) = \alpha_i y + f(L_i(x)) - \alpha_i b_n(f)(x)$, $i \in \mathbb{N}_{N-1}$, where α_i is scaling factor satisfying $|\alpha_i| < a_i^r$. Then, for every $n \in \mathbb{N}$ and scaling vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$, the IFS $\mathcal{I}_n^r = \{I \times K; (L_i(x), F_{n,i}(x, y)), i \in \mathbb{N}_{N-1}\}$ determines a r-times continuously differentiable α-FIF f_n^α. Further, for every scaling vector α, the sequence $\{\mathcal{I}_n^r\}_{n=1}^\infty$ of IFSs determines a sequence $\{f_n^\alpha\}_{n=1}^\infty$ of r-times continuously differentiable α-FIFs that converges in the $\mathcal{C}^r$-norm to $f \in \mathcal{C}^r(I)$.

Proof. Let $\mathcal{C}_f^r(I) = \Big\{g \in \mathcal{C}^r(I) \ : \ g^{(k)}(x_1) = f^{(k)}(x_1), g^{(k)}(x_N) = f^{(k)}(x_N), k \in \mathbb{N}_r \cup \{0\}\Big\}$. Then $\mathcal{C}_f^r(I)$ is a complete metric space with respect to the metric induced by the $\mathcal{C}_r$-norm. Let $T_n^\alpha : \mathcal{C}_f^r(I) \to \mathcal{C}_f^r(I)$ be the Read-Bajraktarević (RB) operator defined by

$$(T_n^\alpha g)x = \alpha_i g(L_i^{-1}(x)) + f(x) - \alpha_i b_n(f)(L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_{N-1}, \tag{11}$$

Using (10), and the continuity of $b_n^{(k)}(f)(x)$, $k \in \mathbb{N}_r \cup \{0\}$, it follows at once that RB operator $T_n^\alpha g$ is continuous on each subinterval I_i. Using differentiation, we obtain

$$(T_n^\alpha g)^{(k)}(x) = f^{(k)}(x) + \frac{\alpha_i}{a_i^k}\Big[g^{(k)}(L_i^{-1}(x)) - b_n^{(k)}(f)(L_i^{-1}(x))\Big], x \in I_i, i \in \mathbb{N}_{N-1}.$$

Next, using (2), one can verify for $k \in \mathbb{N}_r \cup \{0\}$ that

$$\lim_{x \to x_i^+}(T_n^\alpha g)^{(k)}(x) = f^{(k)}(x_i) = \lim_{x \to x_i^-}(T^\alpha g)^{(k)}(x), (T_n^\alpha g)^{(k)}(x_1) = f^{(k)}(x_1), (T_n^\alpha g)^{(k)}(x_N) = f^{(k)}(x_N).$$

Therefore, $T_n^\alpha g \in \mathcal{C}_f^r(I)$. For $g, h \in \mathcal{C}_f^r(I)$, we obtain

$$||(T_n^\alpha g)^{(k)} - (T_n^\alpha h)^{(k)}||_\infty \le \frac{1}{a_i^k}|\alpha_i|||(g - h)^{(k)}||_\infty, i \in \mathbb{N}_{N-1}.$$

The previous inequality implies

$$||(T_n^\alpha g) - (T_n^\alpha h)||_{\mathcal{C}^r} \le \max\Big\{\frac{1}{a_i^r}|\alpha_i| : i \in \mathbb{N}_{N-1}\Big\}||g - h||_{\mathcal{C}^r}.$$

The assumption on the scaling functions now ensure that T_n^α is a contraction map, and, hence by the Banach fixed point theorem, T_n^α has a unique fixed point f_n^α. Further, it follows that $(f_n^\alpha)^{(k)}$ obeys the functional equation:

$$(f_n^\alpha)^{(k)}(x) = f^{(k)}(x) + \frac{\alpha_i}{a_i^k}\Big[(f_n^\alpha)^{(k)}(L_i^{-1}(x)) - b_n^{(k)}(f)(L_i^{-1}(x))\Big], x \in I_i, i \in \mathbb{N}_{N-1}, \tag{12}$$

and $(f_n^\alpha)^{(k)}(x_i) = f^{(k)}(x_i)$ for all $k \in \mathbb{N}_r \cup \{0\}$, $i \in \mathbb{N}_N$. Also, we verify that

$$\begin{aligned} ||(f_n^\alpha)^{(k)} - f^{(k)}||_\infty &\le \frac{|\alpha|_\infty}{\Theta_k}||f^{(k)} - b_n^{(k)}(f)||_\infty, \Theta_k = \min_{i \in \mathbb{N}_{N-1}} a_i^k, k = 0, 1, \ldots, r, \\ &\le \frac{|\alpha|_\infty}{\Theta}||f^{(k)} - b_n^{(k)}(f)||_\infty, \Theta = \min_k \Theta_k. \end{aligned} \tag{13}$$

Hence, the previous inequality leads to

$$||f_n^\alpha - f||_{\mathcal{C}^r} \le \frac{|\alpha|_\infty}{\Theta}||f - b_n(f)||_{\mathcal{C}^r}. \tag{14}$$

From (12)-(14), one can observe the following:

- Each function of the sequence $\{(f_n^\alpha)^{(k)}\}_{n=1}^\infty$ interpolates $f^{(k)}$ at x_i, $i \in \mathbb{N}_N$ and $\lim_{n\to\infty}(f_n^\alpha)^{(k)} = f^{(k)}$ for $k \in \mathbb{N}_r \cup \{0\}$.
- Each function of the sequence $\{f_n^\alpha\}_{n=1}^\infty$ preserves the r-smoothness of f and $\lim_{n\to\infty} f_n^\alpha = f$.
- If $|\alpha|_\infty = 0$, then $(f_n^\alpha)^{(k)}(x) = f^{(k)}(x)$ for all $x \in I$ and $k \in \mathbb{N}_r \cup \{0\}$.
- Graph of $(f_n^\alpha)^{(k)}$ is the attractor of the IFS $\{I \times K; (L_i(x), F_{n,i}^k(x,y)), i \in \mathbb{N}_{N-1}\}$, where $F_{n,i}^k(x,y) = \frac{\alpha_i}{a_i^k}y + f^{(k)}(L_i(x)) - \frac{\alpha_i}{a_i^k}b_n^{(k)}(f)(x)$.

□

Theorem 3.2. *Let $\Phi \in \mathcal{C}^r(I)$ be the original continuous function providing the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$. Suppose that $f \in \mathcal{C}^r(I)$ interpolates Φ with respect to the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$. If scaling factors $\alpha_i, \in \mathbb{N}_{N-1}$ satisfy $|\alpha_i| < a_i^r$, then*

$$||\Phi^{(k)} - (f_n^\alpha)^{(k)}||_\infty \le ||\Phi^{(k)} - f^{(k)}||_\infty + \frac{|\alpha|_\infty}{\Theta}||f^{(k)} - b_n^{(k)}(f)||_\infty, k \in \mathbb{N}_r \cup \{0\}, \tag{15}$$

$$||\Phi - f_n^\alpha||_{\mathcal{C}^r} \le ||\Phi - f||_{\mathcal{C}^r} + \frac{|\alpha|_\infty}{\Theta}||f - b_n(f)||_{\mathcal{C}^r}. \tag{16}$$

Proof. Using (13) in the triangular inequality

$$||\Phi^{(k)} - (f_n^\alpha)^{(k)}||_\infty \le ||\Phi^{(k)} - f^{(k)}||_\infty + ||f^{(k)} - (f_n^\alpha)^{(k)}||_\infty.$$

we get (15). Since (15) is true for every $k \in \mathbb{N}_r \cup \{0\}$, we obtain (16). □

Remark 3.1. *For $n \in \mathbb{N}$, let $\Delta_n = x_1 < x_{n,2} < x_{n,3} < \cdots < x_{n,N-1} < x_N$ be a partition of $I = [x_1, x_N]$ and $||\Delta_n||$ be the norm of the partition Δ_n. Let $I_{n,i} = [x_{n,i}, x_{n,i+1}]$ and*

$$H_{\Delta_n}^{r+1} = \{\phi \in \mathcal{C}^r(I) : \phi|_{I_{n,i}} \in \mathbb{P}_{2r+1}\},$$

where $\mathbb{P}_{2r+1}$ is the space consisting of all polynomials of degree at most $2r+1$. For $\psi \in \mathcal{C}^r(I)$, if $b_n(\psi) \in H_{\Delta_n}^{r+1}$ is $\mathcal{C}^r$-Hermite polynomial of ψ with respect to the partition Δ_n of I so that $||\Delta_n|| \to 0$ as $n \to \infty$, then (10) holds good, (for details, refer [32]).

Theorem 3.3. *Let $\Phi \in \mathcal{C}^q(I), q \ge 2r+2, r \in \mathbb{N}$ be the original continuous function providing the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$. Suppose that $f \in \mathcal{C}^q(I)$ interpolates Φ with respect to the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$. Let $b_n(f) \in H_{\Delta_n}^{r+1}$ be the Hermite spline of f with respect to the partition Δ_n of I so that $||\Delta_n|| \to 0$ as $n \to \infty$. If scaling factors $\alpha_i, \in \mathbb{N}_{N-1}$ satisfy $|\alpha_i| < a_i^r$, then, for $k \in \mathbb{N}_r \cup \{0\}$,*

$$||\Phi^{(k)} - (f_n^\alpha)^{(k)}||_\infty \le ||\Phi^{(k)} - f^{(k)}||_\infty + \frac{|\alpha|_\infty}{\Theta}\frac{||\Delta_n||^{2r+2-k}}{2^{2r+2-2k}k!(2r+2-2k)!}||f^{(2r+2)}||_\infty. \tag{17}$$

Proof. Under the sated conditions, from [14, 33], we get

$$||f^{(k)} - b_n^{(k)}(f)||_{\mathcal{C}^r} \le \frac{||\Delta_n||^{2r+2-k}}{2^{2r+2-2k}k!(2r+2-2k)!}||f^{(2r+2)}||_\infty, k \in \mathbb{N}_r \cup \{0\}. \tag{18}$$

Now, using the above inequality in (15), we get the desired result. □

Remark 3.2. *The partitions $\Delta_n, n \in \mathbb{N}$, used in the above are independent of the partition Δ used to define the α-FIFs.*

Theorem 3.4. *Let $\Phi \in \mathcal{C}^r(I)$ be the original function providing the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$ and $0 \le \Phi^{(r)}(x) \le \Gamma$ for all $x \in I$. Suppose that $f \in \mathcal{C}^r(I)$ interpolates Φ with respect to the data $\{(x_i, z_i) : i \in \mathbb{N}_N\}$ and $0 \le f^{(r)}(x) \le \Gamma$ for all $x \in I$. Then the corresponding FIFs $f_n^\alpha, n \in \mathbb{N}$ satisfy $0 \le (f_n^\alpha)^{(r)}(x) \le \Gamma$ for all $x \in I$ if the scaling factors $\alpha_1, \alpha_2, \dots, \alpha_{N-1}$ of f_n^α obey $|\alpha_i| < a_i^r$ and*

$$\max\left\{\frac{-a_i(K_n - M_i)}{M_n^*}, \frac{-a_i m_i}{K_n - m_n^*}\right\} \le \alpha_i \le \min\left\{\frac{a_i m_i}{M_n^*}, \frac{a_i(K_n - M_i)}{K_n - m_n^*}\right\}, \quad \textit{where} \tag{19}$$

$$m_i = \min_{x\in I} f^{(r)}(L_i(x)), M_i = \max_{x\in I} f^{(r)}(L_i(x)), m_n^* = \min_{x\in I} b_n^{(r)}(f)(x), M_n^* = \max_{x\in I} b_n^{(r)}(f)(x),$$

K_n is strictly greater than m_n^* and $||f^{(r)}||_\infty$ and the terms with zero denominator are avoided.

Proof. In the light of Theorem 3.1, it follows that the stated conditions on the scale factors ensure the $\mathcal{C}^r$-continuity of α-FIFs f_n^α, $n \in \mathbb{N}$. Further, from (12), it follows that $(f_n^\alpha)^{(r)}$, $n \in \mathbb{N}$, satisfy the functional equation

$$(f_n^\alpha)^{(r)}\big(L_i(x)\big) = f^{(r)}(L_i(x)) + \frac{\alpha_i}{a_i^r}\Big[(f_n^\alpha)^{(r)}(x) - b_n^{(r)}(f)(x)\Big], x \in I, i \in \mathbb{N}_{N-1}. \tag{20}$$

Thus, for each $n \in \mathbb{N}$, the functional equation (20) is a rule that computes the values of $(f_n^\alpha)^{(r)}$ at $(N-1)^{p+2}+1$ distinct points in I at $(p+1)$-th iteration using the values of $(f_n^\alpha)^{(r)}$ at $(N-1)^{p+1}+1$ distinct points in I at p-th iteration. It is easy to verify from (20) that

$$(f_n^\alpha)^{(r)}(x_i) = f^{(r)}(x_i) \ \forall\ i \in \mathbb{N}_N, n \in \mathbb{N}.$$

Therefore, to prove $0 \le (f_n^\alpha)^{(r)}(\tau) \le \Gamma$ for all $\tau \in I$, it is enough to show that $0 \le (f_n^\alpha)^{(r)}(x) \le \Gamma$, $x \in I$ implies that $0 \le (f_n^\alpha)^{(r)}\big(L_i(x)\big) \le \Gamma$, $x \in I$ and for all $i \in \mathbb{N}_{N-1}$.

Assume that $0 \le (f_n^\alpha)^{(r)}(x) \le \Gamma$, $x \in I$. Firstly, let us consider, $0 \le \alpha_i \le a_i^k$. Then, we obtain that

$$f^{(r)}\big(L_i(x)\big) - \frac{\alpha_i}{a_i^r} b_n^{(r)}(f)(x) \le (f_n^\alpha)^{(r)}(L_i(x)) \le f^{(r)}\big(L_i(x)\big) + \frac{\alpha_i}{a_i^r}\Big(\Gamma - b_n^{(r)}(f)(x)\Big). \tag{21}$$

Therefore, to prove that $0 \le (f_n^\alpha)^{(r)}(x) \le \Gamma$, it is enough to obtain the conditions on the scaling factors so that the following inequalities are satisfied.

$$0 \le f^{(r)}\big(L_i(x)\big) - \frac{\alpha_i}{a_i^r} b_n^{(r)}(f)(x) \le \Gamma\Big(1 - \frac{\alpha_i}{a_i^r}\Big). \tag{22}$$

Now, using the definition of m_i and M_n^*, it is easy to verify that first of the two inequalities in (22) is satisfied if

$$\alpha_i \le \frac{a_i^r m_i}{M_n^*}. \tag{23}$$

Similarly, observing $m_n^* \le b_n^{(r)}(f)(x_1) = f^{(r)}(x_1) \le \Gamma$ and using the definition of M_i and m_n^*, we ensure that second inequality in (22) is fulfilled if

$$\alpha_i \le \frac{a_i^r\Big(\Gamma - M_i\Big)}{\Gamma - m_n^*}. \tag{24}$$

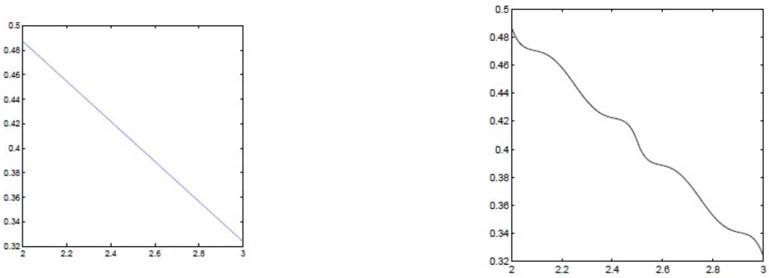
Next, let $-a_i^r \le \alpha_i \le 0$. Then it follows that

$$0 \le (f_n^\alpha)^{(r)}\big(L_i(x)\big) \le \Gamma$$

is satisfied if

$$\frac{-\alpha_i}{a_i^r}\Gamma \le f^{(r)}\big(L_i(x)\big) - \frac{\alpha_i}{a_i^r} b_n^{(r)}(f)(x) \le \Gamma. \tag{25}$$

Now, using the analysis which is similar to that used for obtaining (23)-(24), we obtain the conditions on the scaling factors which ensure (25). $\square$



(a): The function $f^{(1)}(x), x \in [2,3]$.

(b): The fractal function $(f_5^\alpha)^{(1)}(x), x \in [2,3]$.

Figure 3. $\mathcal{C}^1$-Constrained fractal function.

Example: Let us construct some numerical experiments to illustrate Theorem 3.4. For this purpose, let us consider the data $\{(2, 0.6931), (2.5, 0.9163), (3, 1.0986)\}$ obtained from the function $\Phi(x) = \ln(x), x \in [2,3]$. It is clear that $\Phi \in \mathcal{C}^1[2,3]$ and $0 < \Phi'(x) \le 0.5$ for all $x \in [2,3]$. The Lagrange interpolant f of $\Phi(x) = \ln(x), x \in [2,3]$ with respect to the above data set is given by $f(x) = -0.816x^2 + 0.8137x - 0.6076$. The graph of $f^{(1)}(x), x \in [2,3]$ is generated in Figure 3(a), and from this one can easily notice that $0 < f^{(1)}(x) \le 0.5$ for all $x \in [2,3]$. Next, by taking $b_5(f)$ as the fifth degree $\mathcal{C}^1$-Hermite interpolant (polynomial) of f and scaling factors $\alpha_1 = \alpha_2 = 0.001$ (these scaling factors are calculated according to Theorem 3.4), we have generated fractal function $(f_5^\alpha)^{(1)}$ in Figure 3(b). It is clear that $0 < (f_5^\alpha)^{(1)}(x) \le 0.5$ for all $x \in [2,3]$.

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[1]Department of Mathematics, Visvesvaraya National Institute of Technology Nagpur, Nagpur 440010, India.

Email: vijendernallapu@gmail.com

[2]Departamento de Matemática Aplicada, Escuela de Ingeniería y Arquitectura, Universidad de Zaragoza, Zaragoza 50018, Spain.

Email: manavas@unizar.es