Appl. Math. J. Chinese Univ.
2021, 36(3): 317-329

Numerical solution of stochastic Itô-Volterra integral equations based on Bernstein multi-scaling polynomials

A. R. Yaghoobnia M. Khodabin* R. Ezzati

Abstract. In this paper, first, Bernstein multi-scaling polynomials (BMSPs) and their properties are introduced. These polynomials are obtained by compressing Bernstein polynomials (BPs) under sub-intervals. Then, by using these polynomials, stochastic operational matrices of integration are generated. Moreover, by transforming the stochastic integral equation to a system of algebraic equations and solving this system using Newton's method, the approximate solution of the stochastic Itô-Volterra integral equation is obtained. To illustrate the efficiency and accuracy of the proposed method, some examples are presented and the results are compared with other methods.

§1 Introduction

Mathematical expression and modeling of many scientific and natural phenomena are carried out using differential equations. On the other hand, in many of these models, stochastic factors are involved that include stochastic processes that generate stochastic differential equations. These stochastic equations are seen in various branches of sciences, specially in engineering, economics, physics, biology and financial mathematic. Some of these stochastic equations have analytical solutions, but many of them cannot be solved analytically and it is necessary to obtain their approximate solutions using numerical methods. In this paper, numerical solution of a set of stochastic differential equations are discussed which are as follows:

$$dy(x) = \lambda_1 \ f(x, y(x))dx + \lambda_2 \ g(x, y(x))dB(x), \quad y(x_0) = y_0, \tag{1.1}$$

where λ_1 and λ_2 are parameters and $y(x)$, $f(x, y(x))$ and $g(x, y(x))$ for $x \in [0, 1)$ are stochastic processes defined on the some probability space $(\Omega, \mathscr{F}, P)$. Moreover, $y(x)$ is the unknown function and $B(x)$ is the Brownian motion. In [4][10][23][29] stochastic differential equations and their applications have introduced. Also, we can study about the stochastic equations in [12][27][30][32] and we can see the various methods for numerical solution of these equations

Received: 2018-12-24. Revised: 2019-12-30.
MR Subject Classification: 65C30, 60H20, 60Gxx, 45Dxx.
Keywords: Bernstein multi-scaling polynomial, stochastic operational matrix, stochastic Itô-Volterra integral equation, Brownian motion process.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-021-3694-9.
*Corresponding author.

in [2][7][9][24]. For functions that are based on Brownian motion, laws of Leibnitz-Newton's calculation do not hold, any more such functions are not differentiable at any point in the path. So, through the integration of the two sides of the (1.1), the following stochastic Itô integral equation is obtained:

$$y(x) = y_0 + \lambda_1 \int_0^x f(s, y(s))ds + \lambda_2 \int_0^x g(s, y(s))dB(s). \tag{1.2}$$

The main objective of the current paper is to obtain the numerical solution of the Itô type of stochastic Volterra integral equation (1.2). Researchers have presented various methods for numerical solution of stochastic integral equations so far, such as the Taylor series method [5], the block pulse functions [17][18], triangular function method [6], Fibonacci operational matrix [20] and methods based on wavelets [16][21]. In addition, Bernstein polynomials have already been used in the numerical solution of Volterra integral equations [14][15][19], mixed Volterra-Fredholm integral equations [28], differential equations [3], and the numerical solution of stochastic Itô-Volterra integral equations [1]. In this paper, to approximate the solution of the stochastic integral equation (1.2), the operational matrix is generated based on Bernstein multi-scaling polynomials.

The organization of this paper is as follows. In Section 2, stochastic calculations and some of their properties are expressed. In Section 3, both BPs and BMSPs are introduced and some of their properties are described. Also, the operational matrix of integration is defined in this Section. In Section 4, the stochastic operational matrix of integration is generated. In Section 5, the method of solving stochastic Itô-Volterra integral equations through using the generated operational matrices is proposed. Section 6 deals with the convergence analysis of the proposed method. In Section 7, some different examples are given and the efficiency and accuracy of the presented method are evaluated and finally, the last section provides conclusions.

§2 Stochastic calculus and some of its properties

Brownian motion is a random process that occurs irregularly such as repeated momentary vibrations. Suppose X be a random variable with distribution f_X, so for $p \geq 2$ we have:

$$E[X^p] = \int_{-\infty}^{\infty} x^p f_X(x)dx < \infty.$$

Assume that $L^p(\Omega, H)$ be the collection of all strongly measurable, p-th integrable and H-valued random variables. Therefore, $L^p(\Omega, H)$ is a Banach space with norm

$$\|V\|_{L^p(\Omega,H)} = (E[|V|^p])^{1/p}$$

for each $V \in L^p(\Omega, H)$.

Definition 2.1. [8]. *Brownian motion $B(x)$ is a stochastic process, with the following properties:*

(i) *(Brownian motion's initial position)* $P(B(0) = 0) = 1$.

(ii) *(Independence of increments) For each* $0 \leq x_1 < x_2 < ... < x_n$, *the increments* $B(x_1), B(x_2) - B(x_1), ..., B(x_n) - B(x_{n-1})$ *are independent of the past.*

(iii) (Normal increments) $B(x) - B(s)$ *has normal distribution with mean* 0 *and variance* $x - s$. *This implies that* $B(x)$ *has* $N(0, x)$ *distribution.*

(iv) (Continuity of paths) $B(x)$, *for* $0 \leq x$ *are continuous functions of* x.

The initial position of Brownian motion is specified in the definition. When $B(0) = 0$, then the process of Brownian motion is started at 0.

Definition 2.2. [8]. *The sequence* $\{X_n\}$ *converges to* X *in* L^2, *if for each* n, $E[|X_n|^2] < \infty$ *and* $E[\|X_n - X\|^2] \to 0$ *as* $n \to \infty$.
Suppose $0 \leq s \leq T$ *and let* $\nu = \nu(s, T)$ *be the class of functions* $f(x, \omega) : [0, 1] \times \omega \to R^n$, *satisfy:*

(i) the function $(x, \omega) \to f(x, \omega)$ *is* $\beta \times \mathscr{F}$ *measurable, where* β *is the Borel algebra.*

(ii) f *is adapted to* $\mathscr{F}_t$.

(iii) $E[\int_s^T f^2(x, \omega)dx] < \infty$.

Definition 2.3. *(The Itô integral* [3]*). Let* $f \in \nu(s, T)$, *then the Itô integral of* f *is defined by:*

$$\int_s^T f(x, \omega)dB(x)(\omega) = \lim_{n\to\infty} \int_s^T \phi_n(x, \omega)dB(x)(\omega),$$

where $\{\phi_n\}_n$ *is the sequence of elementary functions such that:*

$$E[\int_s^T (f - \phi_n)^2 dx] \to 0, \quad as \quad n \to +\infty.$$

Theorem 2.1. *(The Itô isometry* [3]*). Let* $f \in \nu(s, T)$, *then:*

$$E[(\int_s^T f(x, \omega)dB(x)(\omega))^2] = E[\int_s^T f^2(x, \omega)dx].$$

§3 Bernstein multi-scaling polynomials (BMSPs)

Definition 3.1. *The BPs of degree of integer* n, *on* $[0, 1]$ *are defined as:*

$$\beta_{i,n}(x) = \binom{n}{i} x^i (1 - x)^{n-i}, \quad i = 0, 1, 2, ..., n$$

and for other valuses of i, $\beta_{i,n}(x) = 0$.

Some of the reasons that explain the advantages of using BPs for approximation are given in the following theorem.

Theorem 3.1. [11]. *Suppose that* $H = L^2[0, 1]$ *is a Hilbert space with the inner product and* $Y = Span\{\beta_{0,n}(x), \beta_{1,n}(x), ..., \beta_{n,n}(x)\}$ *is a finite dimensional and closed subspace, therefore,* Y *is a complete subspace of* H. *So if* h *is an arbitrary element in* H, *it has a unique best approximation out of* Y *such as* y_0, *that is*

$$\exists y_0 \in Y \quad s.t. \quad \forall y \in Y, \quad \|h - y_0\| \leq \|h - y\|$$

where $\|h\| = \sqrt{(h, h)}$ *and* $(h_1, h_2) = \int_0^1 h_1(x)h_2(x)dx$. *So there exist unique coefficients* $c_0, c_1, ..., c_n$ *such that* $h(x) \approx y_0 = \sum_{i=0}^{n} c_i\beta_{i,n}(x) = C^T\Phi(x)$, *where* $C^T = [c_0, c_1, ..., c_n]$ *and*

$\Phi(x) = [\beta_{0,n}(x), \beta_{1,n}(x), ..., \beta_{n,n}(x)]^T.$

To obtain more information about BPs and their properties, see [13][26]. By compressing BPs on the sub-intervals of $[0,1]$, we reach BMSPs [22].

Definition 3.2. *Suppose that the interval* $[0,1)$ *is divided into* k *sub-intervals* $[j/k,(j+1)/k)$ *for* $j = 0,1,...,k-1$. *BMSPs of degree of integer* n *on this sub-intervals are defined as:*

$$\psi_{i,j,n}(x) = \begin{cases} \beta_{i,n}(kx-j), & j/k \le x < (j+1)/k, \\ 0, & otherwise, \end{cases} \tag{3.1}$$

for $i = 0,1,...,n$.

For convenience, wherever it is need, we put $\psi_{i,j,n}(x) = \psi_{i,j}(x)$.
The BMSPs on $[j/k,(j+1)/k)$ for $j = 0,1,...,k-1$, have the following properties:

(*i*) $\psi_{i,j}(x) \ge 0, \quad i = 0,1,...,n,$

(*ii*) $\sum\limits_{i=0}^{n} \psi_{i,j}(x) = 1,$

(*iii*) $\psi_{i,j,n}(x) = (kx-j)\psi_{i-1,j,n-1}(x) + (j+1-kx)\psi_{i,j,n-1}(x), \quad i = 0,1,...,n,$

(*iv*) $\psi_{i,j,n}(x)$ has maximum at $\frac{i+jn}{kn}, \quad i = 0,1,...,n,$

(*v*) $\psi_{0,j}(x), \psi_{1,j}(x), ..., \psi_{n,j}(x)$ are linear independent polynomials.

In addition, on every sub-intervals $[\frac{j}{k}, \frac{j+1}{k}]$, following relation is hold which shows BMSPs are symmetric,

$$\psi_{i,j,n}(\frac{j}{k}+x) = \psi_{n-i,j,n}(\frac{j+1}{k}-x), \quad i = 0,1,...,n.$$

The advantages stated in Theorem 3.1 for approximation by BPs, also apply to approximation by BMSPs in each sub-interval.
In [31], for the interval $[0,1)$, the operational matrix P has been obtained such that

$$\int_0^x \Phi(s)ds \approx P\Phi(x). \tag{3.2}$$

We define $\Psi(x) = [\Psi_0(x), \Psi_1(x), ..., \Psi_{k-1}(x)]^T$, where $\Psi_j(x)$ is the vector of BMSPs, and according to (3.1), the following equation is obtained:

$$\Psi_j(x) = [\psi_{0,j}(x), \psi_{1,j}(x), ..., \psi_{n,j}(x)]^T.$$

Through considering

$$\int_{j/k}^{(j+1)/k} \psi_{i,j}(x)dx = \frac{1}{k(n+1)},$$

for $i = 0,1,...,n$, and by repeating operational matrix of integration (3.2) for any sub-interval $[\frac{j}{k}, \frac{j+1}{k})$, the following equation is obtained:

$$\int_0^x \Psi(s)ds \approx \mathscr{M}\Psi(x),$$

where $\mathscr{M}$ is a $k(n+1)\times k(n+1)$ operational matrix of integration based on BMSPs as follow:

$$\mathscr{M}=\begin{bmatrix}\frac{1}{k}P & \frac{1}{k(n+1)}\overline{\mathbf{1}} & \cdots & \frac{1}{k(n+1)}\overline{\mathbf{1}}\\ \overline{\mathbf{0}} & \frac{1}{k}P & \cdots & \frac{1}{k(n+1)}\overline{\mathbf{1}}\\ \vdots & \vdots & & \vdots\\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & \cdots & \frac{1}{k}P\end{bmatrix}, \tag{3.3}$$

such that $\overline{\mathbf{0}}$ and $\overline{\mathbf{1}}$ are $(n+1)\times(n+1)$ matrices whose all entries are *zero* and *one*, respectively.

§4 Stochastic operational matrix of integration based on BMSPs

In this section, the stochastic operational matrix of integration, based on BMSPs is generated. First, corresponding to the coefficients of BMSPs, there is a matrix A_j on $[\frac{j}{k},\frac{j+1}{k})$ for $j=0,1,...,k-1$. The $(i+1)-th$ row of matrix A_j is the coefficients of $\psi_{i,j}(x)$. In other words, $(i+1)-th$ row of matrix A_j is as follows, with values calculated at $x=0$,

$$(A_j)_{i+1}=[\psi_{i,j}(x),\frac{1}{1!}\psi'_{i,j}(x),\frac{1}{2!}\psi''_{i,j}(x),...,\frac{1}{n!}\psi^{(n)}_{i,j}(x)].$$

Because the entries of A_j are the coefficients of linear independent polynomials in $\Psi_j(x)$, A_j for $j=0,1,...,k-1$ are invertible. Now, assuming $T_n(x)=[1,x,x^2,...,x^n]^T$, we have:

$$\Psi_j(x)=A_jT_n(x), \tag{4.1}$$

By integrating of $\Psi(x)$ when $x\in[0,\frac{1}{k})$, we have:

$$\int_0^x\Psi_0(s)dB(s)=A_0\int_0^x[1,s,s^2,...,s^n]^TdB(s). \tag{4.2}$$

Using integration by parts formula on the integrals of (4.2), we get

$$\int_0^x s^vdB(s)=x^vB(x)-v\int_0^x s^{v-1}B(s)ds,\quad v=0,1,...,n,$$

and using composite trapezium rule, we will have the following equation:

$$\begin{aligned}\int_0^x s^vdB(s)&\approx x^vB(x)-v\frac{x}{4}[2(\frac{x}{2})^{v-1}B(\frac{x}{2})+x^{v-1}B(x)]\\ &=[(1-\frac{v}{4})B(x)-\frac{v}{2^v}B(\frac{x}{2})]x^v.\end{aligned}$$

Now, by approximating $B(x)$ and $B(\frac{x}{2})$ by $B(\frac{1}{2k})$ and $B(\frac{1}{4k})$, respectively, we have:

$$\int_0^x T_n(s)dB(s)\approx D_{s_0}T_n(x), \tag{4.3}$$

where D_{s_0} is the following matrix:

$$D_{s_0}=\begin{bmatrix}B(\frac{1}{2k}) & 0 & \cdots & 0\\ 0 & \frac{3}{4}B(\frac{1}{2k})-\frac{1}{2}B(\frac{1}{4k}) & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & (1-\frac{n}{4})B(\frac{1}{2k})-\frac{n}{2^n}B(\frac{1}{4k})\end{bmatrix}$$

So, from (4.1-4.3) we have:

$$A_0D_{s_0}T_n(x)=A_0D_{s_0}{A_0}^{-1}\Psi_0(x)=M_{s_0}\Psi_0(x),$$

where $M_{s_0} = A_0 D_{s_0} {A_0}^{-1}$ is the stochastic operational matrix on $[0, \frac{1}{k})$.
Now, assume that $\frac{1}{k} \leq x$, we have:

$$\int_0^x \Psi_0(s)dB(s) = A_0 \int_0^{\frac{1}{k}} T_n(s)dB(s),$$

in which by three interpolation points $\left\{0, \frac{1}{2k}, \frac{1}{k}\right\}$, for $v = 0, 1, ..., n$, we get

$$\int_0^{\frac{1}{k}} s^v dB(s) \approx m_{s_{0,v}} = \frac{1}{k^v}[(1 - \frac{v}{4})B(\frac{1}{k}) - \frac{v}{2^v}B(\frac{1}{2k})],$$

Now, by integrating of $\Psi(x)$ when $x \in [\frac{j}{k}, \frac{j+1}{k})$, we have:

$$\int_0^x \Psi_j(s)dB(s) = A_j \int_{\frac{j}{k}}^x [1, s, s^2, ..., s^n]^T dB(s).$$

Similarly, using integration by parts formula on the recently integrals, we have:

$$\int_{\frac{j}{k}}^x s^v dB(s) = x^v B(x) - (\frac{j}{k})^v B(\frac{j}{k}) - v \int_{\frac{j}{k}}^x s^{v-1} B(s)ds.$$

By using the composite trapezium rule, with $\frac{x-\frac{j}{k}}{2}$ as the step length and approximate $B(x)$ and $B(\frac{x+\frac{j}{k}}{2})$ by $B(\frac{2j+1}{2k})$ and $B(\frac{4j+1}{4k})$, respectively, for $v = 0, 1, ..., n$ and $j = 1, 2, ..., k-1$ we get

$$\begin{aligned}\int_{\frac{j}{k}}^x s^v dB(s) &\approx x^v B(\frac{2j+1}{2k}) - (\frac{j}{k})^v B(\frac{j}{k}) \\ &- v\frac{x-\frac{j}{k}}{4}[(\frac{j}{k})^{v-1}B(\frac{j}{k}) + 2(\frac{4j+1}{4k})^{v-1}B(\frac{4j+1}{4k}) + x^{v-1}B(\frac{2j+1}{2k})].\end{aligned} \tag{4.4}$$

Relation (4.4) can be summarized by:

$$\int_{\frac{j}{k}}^x T_n(s)dB(s) \approx P_{s_j} T_n(x),$$

where P_{s_j} for $j = 1, 2, ..., k-1$, is given by

$$P_{s_j} = \begin{bmatrix} \alpha_{0,j} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{1,j} & \beta_{1,j} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{2,j} & \beta_{2,j} & \gamma_{2,j} & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{3,j} & \beta_{3,j} & \eta_{3,j} & \gamma_{3,j} & 0 & \cdots & 0 & 0 \\ \alpha_{4,j} & \beta_{4,j} & 0 & \eta_{4,j} & \gamma_{4,j} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-1,j} & \beta_{n-1,j} & 0 & 0 & 0 & \cdots & \gamma_{n-1,j} & 0 \\ \alpha_{n,j} & \beta_{n,j} & 0 & 0 & 0 & \cdots & \eta_{n,j} & \gamma_{n,j} \end{bmatrix}.$$

The entries of this matrix are obtained from (4.4) by the following relations:

$$\begin{cases} \alpha_{v,j} = (\frac{j}{k})^v(\frac{v}{4} - 1)B(\frac{j}{k}) + \frac{vj}{2k}(\frac{4j+1}{4k})^{v-1}B(\frac{4j+1}{4k}), & v = 0, 1, ..., n, \\ \beta_{v,j} = -\frac{v}{4}((\frac{j}{k})^{v-1}B(\frac{j}{k}) + 2(\frac{4j+1}{4k})^{v-1}B(\frac{4j+1}{4k})), & v = 1, 2, ..., n, \\ \gamma_{v,j} = (1 - \frac{v}{4})B(\frac{2j+1}{2k}), & v = 2, 3, ..., n, \\ \eta_{v,j} = \frac{vj}{4k}B(\frac{2j+1}{2k}), & v = 3, 4, ..., n. \end{cases}$$

Therefore, we can write:

$$\int_0^x \Psi_j(s)dB(s) \approx A_j P_{s_j} T_n(x) = M_{s_j}\Psi_j(x),$$

where $M_{s_j} = A_j P_{s_j} {A_j}^{-1}$ is the stochastic operational matrix on $[\frac{j}{k}, \frac{j+1}{k})$.
Now, assume that $\frac{j+1}{k} \leq x$, for $j = 1, 2, ..., k-2$, we have:

$$\int_0^x \Psi_j(s)dB(s) = A_j \int_{\frac{j}{k}}^{\frac{j+1}{k}} T_n(s)dB(s).$$

For three interpolation points $\{\frac{j}{k}, \frac{2j+1}{2k}, \frac{j+1}{k}\}$, for $v = 0, 1, ..., n$ and $j = 1, 2, ..., k-2$ we get

$$\begin{aligned}\int_{\frac{j}{k}}^{\frac{j+1}{k}} s^v dB(s) \approx m_{s_{j,v}} &= (\frac{j+1}{k})^v B(\frac{j+1}{k}) - (\frac{j}{k})^v B(\frac{j}{k}) \\ &- \frac{v}{4k}[(\frac{j}{k})^{v-1} B(\frac{j}{k}) + 2(\frac{2j+1}{2k})^{v-1} B(\frac{2j+1}{2k}) \\ &+ (\frac{j+1}{k})^{v-1} B(\frac{j+1}{k})].\end{aligned} \tag{4.5}$$

Then we have:

$$\int_0^x \Psi_0(s)dB(s) = \begin{cases} M_{s_0}\Psi_0(x), & 0 \leq x < \frac{1}{k}, \\ u_{s_0}, & \frac{1}{k} \leq x, \end{cases} \tag{4.6}$$

where in $u_{s_j} = A_j[m_{s_{j,0}}, m_{s_{j,1}}, ..., m_{s_{j,n}}]^T$, and for $j = 1, 2, ..., k-2$ we have:

$$\int_0^x \Psi_j(s)dB(s) = \begin{cases} \mathbf{0}, & 0 \leq x < \frac{j}{k}, \\ M_{s_j}\Psi_j(x), & \frac{j}{k} \leq x < \frac{j+1}{k}, \\ u_{s_j}, & \frac{j+1}{k} \leq x, \end{cases} \tag{4.7}$$

where $\mathbf{0} = [\ \overbrace{0, 0, ..., 0}^{(n+1)\quad times}\ ]^T$. Finally, for the last sub-interval we have:

$$\int_0^x \Psi_{k-1}(s)dB(s) = \begin{cases} \mathbf{0}, & 0 \leq x < \frac{k-1}{k}, \\ M_{s_{k-1}}\Psi_{k-1}(x), & \frac{k-1}{k} \leq x < 1. \end{cases} \tag{4.8}$$

Using (4.6-4.8) the stochastic operational matrix of integration based on BMSPs is given by

$$\mathscr{M}_s = \begin{bmatrix} M_{s_0} & U_{s_0} & U_{s_0} & \cdots & U_{s_0} \\ \bar{\mathbf{0}} & M_{s_1} & U_{s_1} & \cdots & U_{s_1} \\ \bar{\mathbf{0}} & \bar{\mathbf{0}} & M_{s_2} & \cdots & U_{s_2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \cdots & M_{s_{k-1}} \end{bmatrix}_{k(n+1)\times k(n+1)} \tag{4.9}$$

so that $U_{s_j} = [u_{s_j}, u_{s_j}, ..., u_{s_j}]$ for $j = 0, 1, ..., k-2$ are $(n+1) \times (n+1)$ matrices. So we have:

$$\int_0^x \Psi(s)dB(s) \approx \mathscr{M}_s \Psi(x).$$

§5 Method of solution

Consider the stochastic integral equation (1.2) and let

$$z_1(x) = f(x, y(x)), \quad z_2(x) = g(x, y(x)). \tag{5.1}$$

By substituting (5.1) into (1.2) the following equations are obtained:

$$\begin{cases} z_1(x) = f(x, y_0 + \lambda_1 \int_0^x z_1(s)ds + \lambda_2 \int_0^x z_2(s)dB(s)), \\ z_2(x) = g(x, y_0 + \lambda_1 \int_0^x z_1(s)ds + \lambda_2 \int_0^x z_2(s)dB(s)). \end{cases} \tag{5.2}$$

Applying BMSPs, the approximation of $z_1(x)$ and $z_2(x)$, can be written as:

$$\begin{cases} z_1(x) \approx \tilde{z_1}(x) = B_{n,k}(z_1(x)) = {C_1}^T \Psi(x), \\ z_2(x) \approx \tilde{z_2}(x) = B_{n,k}(z_2(x)) = {C_2}^T \Psi(x). \end{cases} \tag{5.3}$$

Note that $B_{n,k}(z_i(x))$, $i = 1, 2$, are linear combination of BMSPs and approximate form of $z_i(x)$. Also C_1 and C_2 are vectors of unknown coefficients. By integrating the sides of (5.3), and using the operational matrices (3.3,4.9), we get the following relations:

$$\int_0^x z_1(s)ds \approx {C_1}^T \int_0^x \Psi(s)ds = {C_1}^T \mathscr{M} \Psi(x), \tag{5.4}$$

and

$$\int_0^x z_2(s)dB(s) \approx {C_2}^T \int_0^x \Psi(s)dB(s) = {C_2}^T \mathscr{M}_s \Psi(x). \tag{5.5}$$

By substituting (5.3-5.5) into (5.2), the following equations are obtained:

$$\begin{cases} {C_1}^T \Psi(x) = f(x, y_0 + \lambda_1 {C_1}^T \mathscr{M} \Psi(x) + \lambda_2 {C_2}^T \mathscr{M}_s \Psi(x)), \\ {C_2}^T \Psi(x) = g(x, y_0 + \lambda_1 {C_1}^T \mathscr{M} \Psi(x) + \lambda_2 {C_2}^T \mathscr{M}_s \Psi(x)). \end{cases} \tag{5.6}$$

Now, using Newton-Cotes points $x_i = \frac{2i-1}{2k(n+1)}$, for $i = 1, 2, ..., k(n+1)$, (5.6) can be rewritten as

$$\begin{cases} {C_1}^T \Psi(x_i) = f(x_i, y_0 + \lambda_1 {C_1}^T \mathscr{M} \Psi(x_i) + \lambda_2 {C_2}^T \mathscr{M}_s \Psi(x_i)), \\ {C_2}^T \Psi(x_i) = g(x_i, y_0 + \lambda_1 {C_1}^T \mathscr{M} \Psi(x_i) + \lambda_2 {C_2}^T \mathscr{M}_s \Psi(x_i)). \end{cases} \tag{5.7}$$

Solving the non-linear system (5.7) with Newton's method, C_1 and C_2 are obtained. Finally, the approximate solution of (1.2) is obtained as follows:

$$y_{n,k}(x) = y_0 + \lambda_1 {C_1}^T \mathscr{M} \Psi(x) + \lambda_2 {C_2}^T \mathscr{M}_s \Psi(x).$$

§6 Convergence Analysis

In this section, the convergence analysis of the proposed method is discussed.

Theorem 6.1. [25]. *The sequence $\{B_n(h); n = 1, 2, \cdots\}$ for all function h in $C[0, 1]$, converges uniformly to h.*
Note that $B_n(h)$, is a linear combination of BPs that is an approximate form of the function h.

Theorem 6.2. *Let $h = [h_1, h_2, \cdots, h_k]$, such that function h_j is defined on $C[0, \frac{j}{k}]$, for $j = 1, 2, \cdots, k$, the sequence $\{B_{n,k}(h); n = 1, 2, \cdots\}$ converges uniformly to h.*

Proof. Here, $B_{n,k}(h)$ as defined in (5.3). According to Theorem 6.1, for any function h_j on $[0, \frac{j}{k}]$, and for any ϵ_j, there exists n_j such that inequality $\|B_{n,k}(h_j) - h_j\| < \epsilon_j$ holds. Therefore, we have:

$$\begin{aligned}\|B_{n,k}(h) - h\| &\leq \|B_{n,k}(h_1) - h_1\| + \|B_{n,k}(h_2) - h_2\| + \cdots + \|B_{n,k}(h_k) - h_k\| \\ &< \epsilon_1 + \epsilon_2 + \cdots + \epsilon_k,\end{aligned}$$

so, from Theorem 6.1, there exists $n > 0$ such that for any $\epsilon = \sum_{j=1}^{k} \epsilon_j$, the following inequality on $[0, 1]$ is established,

$$\|B_{n,k}(h) - h\| < \epsilon.$$

$\square$

Theorem 6.3. [1]. *Let $y(x)$ be analytical solution and $y_n(x)$ be the approximation solution of (1.2), based on BPs. Also assume that*

(i) For every T and N, there is a constant D depending only on T and N such that for all $|y|, |z| \leq N$ and all $0 \leq x \leq T$,

$$|f(x, y) - f(x, z)| + |g(x, y) - g(x, z)| \leq D|y - z|.$$

(ii) Coefficients satisfy the linear growth condition

$$|f(x, y)| + |g(x, y)| \leq D(1 + |y|).$$

(iii) $E(|y|^2) < \infty$.

Then $y_n(x)$ converges to $y(x)$ in L^2.

By using Theorem 6.3 on $[0, \frac{j}{k})$ for $j = 1, 2, \cdots, k$, the convergence $y_{n,k}(x)$ to $y(x)$ is obtained.

§7 Numerical examples

What follows are examples of linear and non-linear stochastic integral equations to illustrate the accuracy of the proposed method. In these examples, n is the degree of BPs and k is the number of sub-intervals in [0,1). To obtain a %95 confidence interval for errors, we solve the problem m times and calculate the absolute error, which is the difference between the analytical and approximate solutions. To do this, a new set of random numbers with a normal distribution is generated in each iteration. In the end we obtain the mean of the errors ($\bar{y}_E$), mean of the standard deviations of the errors (s_E), and construct the confidence intervals for errors by relation $(L, U) = \bar{y}_E \pm 1.96 \frac{s_E}{\sqrt{m}}$. For comparison, these results are presented in the table, along with the error results of other numerical methods.

Example 1: Consider the linear stochastic Itô-Volterra integral equation as follows:

$$y(x) = \frac{1}{12} + \int_0^x cos(s)y(s)ds + \int_0^x sin(s)y(s)dB(s).$$

The analytical solution of this equation is:

$$y(x) = \frac{1}{12} e^{-\frac{x}{4} + \sin(x) + \frac{1}{8}\sin(2x) + \int_0^x sin(s)dB(s)}.$$

The numerical results of Example 1 are presented in Table 1 and Figure 1.

Example 2: In this example we consider the following non-linear stochastic Itô-Volterra inte-

Table 1. Mean, standard deviation and Confidence Interval (C.I)

for error mean of Example 1 with n=5, k=3 and m=100.

x	$\bar{y}_E$ with method [1]	$\bar{y}_E$ in our method	s_E	%95 C.I	
				Lower	Upper
0.0	0.0000057016	0.0000000002	8.96e-10	2.43e-11	3.75e-10
0.1	0.0019436568	0.0012789771	9.52e-04	1.09e-03	1.47e-03
0.2	0.0059398372	0.0038522283	2.94e-03	3.28e-03	4.43e-03
0.3	0.0115472991	0.0083686210	6.45e-03	7.10e-03	9.63e-03
0.4	0.0198483288	0.0149808749	1.19e-02	1.26e-02	1.73e-02
0.5	0.0287292718	0.0208816636	1.67e-02	1.76e-02	2.42e-02
0.6	0.0398943512	0.0269929750	2.34e-02	2.24e-02	3.16e-02
0.7	0.0626147700	0.0360247360	3.11e-02	2.99e-02	4.21e-02
0.8	0.0846395033	0.0471369323	4.14e-02	3.90e-02	5.53e-02
0.9	0.1071397756	0.0559632362	5.34e-02	4.55e-02	6.64e-02

Table 2. Mean, standard deviation and Confidence Interval (C.I)

for error mean of Example 2 with n=4, k=2 and m=100.

x	$\bar{y}_E$ with method [20]	$\bar{y}_E$ in our method	s_E	%95 C.I	
				Lower	Upper
0.0	0.0093556656	0.0085324816	9.70e-03	6.63e-03	1.04e-02
0.1	0.0312170021	0.0272566994	2.13e-02	2.31e-02	3.15e-02
0.2	0.0408973955	0.0361847405	2.86e-02	3.06e-02	4.18e-02
0.3	0.0487956063	0.0455502644	3.57e-02	3.86e-02	5.25e-02
0.4	0.0551883932	0.0535570030	3.95e-02	4.59e-02	6.13e-02
0.5	0.0610913014	0.0587266172	4.52e-02	4.99e-02	6.76e-02
0.6	0.0665501662	0.0646431428	4.66e-02	5.55e-02	7.38e-02
0.7	0.0747982971	0.0744666971	5.80e-02	6.31e-02	8.58e-02
0.8	0.0840172390	0.0789687798	6.26e-02	6.67e-02	9.12e-02
0.9	0.0972814702	0.0897067219	7.10e-02	7.58e-02	1.04e-01

gral equation:

$$y(x) = \frac{1}{8} - \frac{1}{64}\int_0^x y(s)(1-y^2(s))ds + \frac{1}{8}\int_0^x (1-y^2(s))dB(s),$$

with the analytical solution, $y(x) = \frac{9e^{0.25B(x)}-7}{9e^{0.125B(x)}+7}$. The numerical results of Example 2 are presented in Table 2 and Figure 1.

Figure 1 for any examples presents a comparison between the mean of analytical solutions and the mean of approximate solutions obtained by m times repeating the method. The values of these solutions were calculated and plotted at points with a step length of 0.05.

§8 Conclusions

Since often the analytical solution of many stochastic integral equations cannot be achieved, it is important to obtain an approximate solution of them. In this paper, a numerical method is

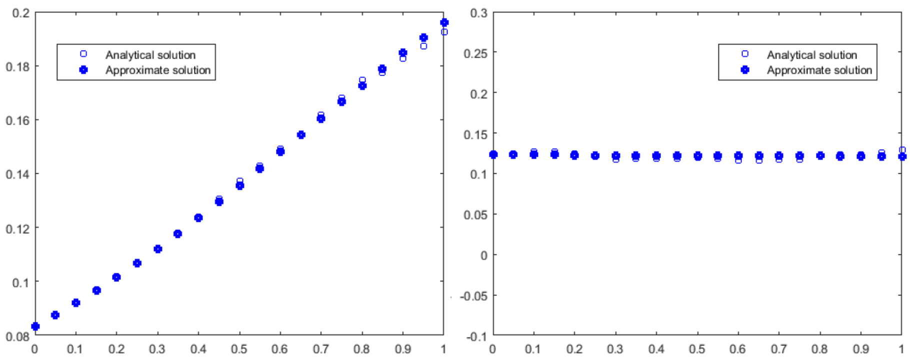

Figure 1. Mean of the analytical and approximate solutions, (left: Example 1, right: Example 2).

proposed to solve the stochastic Itô-Volterra integral equations. For this purpose, using BMSPs, a stochastic operational matrix of integration has been introduced. Then through applying it, the stochastic Itô-Volterra integral equation became an algebraic equation system and solved using the Newton's iteration method. Moreover, it has been shown that in the proposed method, using BMSPs allows us to obtain a more appropriate approximation for stochastic integral equations by smaller values of n as the degree of BPs and this is an advantage of using BMSPs. Comparing the results of this method with some other methods, it has been concluded that the method presented in the current paper has higher accuracy and efficiency.

References

[1] M Asgari, E Hashemizadeh, M Khodabin, K Maleknejad. *Numerical solution of nonlinear stochastic integral equation by stochastic operational matrix based on Bernstein polynomials*, Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, 2014: 3-12.

[2] J C Cortés, L Jódar, L Villafuerte. *Mean square numerical solution of random differential equations: facts and possibilities*, Computers & Mathematics with Applications, 2007, 53(7): 1098-1106.

[3] E H Doha, A Bhrawy, M Saker. *Integrals of Bernstein polynomials: an application for the solution of high even-order differential equations*, Applied Mathematics Letters, 2011, 24(4): 559-565.

[4] P Glasserman. *Monte Carlo methods in financial engineering*, Springer Science & Business Media, 2013.

[5] M Khodabin, K Maleknejad, T Damercheli. *Approximate solution of the stochastic Volterra integral equations via expansion method*, International Journal of Industrial Mathematics, 2014, 6(1): 41-48.

[6] M Khodabin, K Maleknejad, F Hosseini Shekarabi. *Application of triangular functions to numerical solution of stochastic Volterra integral equations,*, IAENG International Journal of Applied Mathematics, 2013, 43(1): 1-9.

[7] M Khodabin, K Maleknejad, M Rosami, M Nouri. *Numerical solution of stochastic differential equations by second order Runge-Kutta methods*, Mathematical and Computer Modelling, 2011, 53(9-10): 1910-1920.

[8] F C Klebaner. *Introduction to stochastic calculus with applications*, World Scientific Publishing Company, 2012.

[9] P E Kloeden, R Pearson. *The numerical solution of stochastic differential equations*, The ANZIAM Journal, 1977, 20(1): 8-12.

[10] P E Kloeden, E Platen. *Numerical solution of stochastic differential equations*, Springer-Verlage, Berlin, 1995.

[11] F Kreyszig. *Introductory functional analysis with applications*, Wiley New York, 1978.

[12] H H Kuo. *Introduction to Stochastic Integration*, Springer-Verlag New York, 2006.

[13] G G Lorentz. *Bernstein polynomials*, American Mathematical Soc, 2012.

[14] K Maleknejad, E Hashemizadeh, B Basirat. *Computational method based on Bernstein operational matrices for nonlinear Volterra-Fredholm-Hammerstein integral equations*, Communications in Nonlinear Science and Numerical Simulation, 2012, 17(1): 52-61.

[15] K Maleknejad, E Hashemmizadeh, R Ezzati. *A new approach to the numerical solution of Volterra integral equations by using Bernsteins approximation*, Communications in Nonlinear Science and Numerical Simulation, 2011, 16(2): 647-655.

[16] K Maleknejad, M Khodabin, M Fallahpour. *Approximation solution of two-dimensional linear stochastic Fredholm integral equation by applying the Haar wavelet*, International Journal of Mathematical Modelling & Computations, 2015, 5(4): 361-372.

[17] K Maleknejad, M Khodabin, F Hosseini Shekarabi. *Modiffed block pulse functions for numerical solution of stochastic Volterra integral equations*, Journal of Applied Mathematics, 2014.

[18] K Maleknejad, M Khodabin, M Rostami. *Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions*, Mathematical and Computer Modelling, 2012, 55(3-4): 791-800.

[19] B Mandal, S Bhattacharya. *Numerical solution of some classes of integral equations using Bernstein polynomials*, Applied Mathematics and computation, 2007, 190(2): 1707-1716.

[20] M Mirzaee, S Hoseini. *Numerical approach for solving nonlinear stochastic Itô-Volterra integral equations using Fibonacci operational matrices*, Scientia Iranica Transaction D: Computer Science & Engineering and Electrical Engineering, 2015, 22(6): 2472-2481.

[21] F Mohammadi. *A wavelet-based computational method for solving stochastic Itô-Volterra integral equations*, Journal of Computational Physics, 2015, 298: 254-265.

[22] M Mohamadi, E Babolian, S Yousefi. *Bernstein multiscaling polynomials and application by solving Volterra integral equations*, Mathematical Sciences, 2017, 11(1): 27-37.

[23] B Øksendal. *Stochastic differential equations: An introduction with applications*, Springer, Berlin, 2003.

[24] E Platen. *An introduction to numerical methods for stochastic differential equations*, Acta numerica, 1999, 8: 197-246.

[25] M J D Powell. *Approximation theory and methods*, Cambridge university press, 1981.

[26] T J Rivilin. *An introduction to the approximation of functions*, Courier Corporation, 2003.

[27] Y Saito, T Mitsui. *Simulation of stochastic differential equations*, Annals of the Institute of Statistical Mathematics, 1993, 45(3): 419-432.

[28] F H Shekarabi, K Maleknejad, R Ezzati. *Application of two-dimensional Bernstein polynomials for solving mixed Volterra-Fredholm integral equations*, Afrika Matematika, 2015, 26(7-8): 1237-1251.

[29] D Talay. *Numerical solution of stochastic differential equations*, Taylor & Francis, 1994.

[30] C P Tsokos, W J Padgett. *Random integral equations with applications to life sciences and engineering*, Academic Press, 1974, 108.

[31] S Yousefi, M Behroozifar. *Operational matrices of Bernstein polynomials and their applications*, International Journal of Systems Science, 2010, 41(6): 709-716.

[32] X Zhang. *Euler schemes and large deviations for stochastic Volterra equations with singular kernels*, Journal of Differential Equations, 2008, 244(9): 2226-2250.

Department of mathematics, karaj Branch, Islamic Azad University, karaj, Iran.
Email:m-khodabin@kiau.ac.ir