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Rearrangement and the weighted logarithmic Sobolev inequality

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Abstract. Here we consider some weighted logarithmic Sobolev inequalities which can be used in the theory of singular Riemannian manifolds. We give the necessary and sufficient conditions such that the 1-dimension weighted logarithmic Sobolev inequality is true and obtain a new estimate on the entropy.

§1 Introduction

Suppose that (M, g) is an n-dimension Riemannian manifold and Σ is a k-dimension submanifold of $M, k < n$. Let φ be a positive function on M that may take ∞ on Σ. The singular manifold $(M, \varphi g)$ is of importance in geometry and PDE. Naturally, the analysis on $(M, \varphi g)$ leads us to studing the following weighted Sobolev inequality

$$(\int_M |f(x)|^q u(x)dV)^{\frac{1}{q}} \le C(\int_M |\nabla f(x)|^p v(x)dV)^{\frac{1}{p}}, 1 \le p \le q \le \infty$$

where u, v are determined by φ and may take ∞ on Σ. In most cases, this inequality can be reduced to

$$(\int_{R^n} |f(x)|^q u(x)dx)^{\frac{1}{q}} \le C(\int_{R^n} |\nabla f(x)|^p v(x)dx)^{\frac{1}{p}}, 1 \le p \le q \le \infty.$$

In [2] and ([16], Chapter 2), the necessary and sufficient conditions were given such that the above inequality is true when $n = 1$ and $n > 1$. But, when $n > 1$, the conditions given in ([16], Chapter 2) are difficult to check in general. So in [8,14,15], they gave some other sufficient conditions. There are many other works [4,1,12,13] on this problem and its applications.

It is well-known that the critical index plays an important role in the Sobolev embedding theorem. For the critical index, we usually need more dedicate estimates, such as the logarithmic Sobolev inequalities. There are some different forms of the weighted logarithmic Sobolev inequalities. See [3,5,10]. Most of them are tightly related with the notion of entropy. The

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notion of entropy was first introduced by Boltzmann in the kinetic theory of gases, and since then played a critical role in many fields, such as statistical physics, information theory and mathematics. For $f \in L^1(X,\mu)$, the entropy of f is defined as

$$\mathrm{Ent}(f)=\int_X |f(x)|\ln\frac{|f(x)|}{\|f\|_{L^1(X,\mu)}}d\mu.$$

By Jessen's inequality, when $\mu(X)\le 1$, the entropy of f is nonnegative. But if $\mu(X)>1$, the entropy of f may be negative. There are many works [3,5,6,7,9] on the estimates of entropies.

Here we consider the following weighted logarithmic Sobolev inequality

$$(\int_{R^n}|f(x)|^q\ln^\lambda(\frac{|f(x)|^q}{\int_{R^n}|f(y)|^qu(y)dy}+e)u(x)dx)^{\frac{1}{q}}\le C(\int_{R^n}|\nabla f(x)|^pv(x)dx)^{\frac{1}{p}}$$

where $1\le p\le q\le\infty,\lambda>0$. When $n=1$, we get the necessary and sufficient conditions such that this weighted logarithmic Sobolev inequality holds. Our main result is stated as

Theorem 1.1. *Assume that u,v are nonnegative and locally integrable on $[0,\infty),\lambda>0,1\le p\le q<\infty$. Then*

$$(\int_0^\infty|f(x)|^q\ln^\lambda(\frac{|f(x)|^q}{\int_0^\infty|f(y)|^qu(y)dy}+e)u(x)dx)^{\frac{1}{q}}\le C(\int_0^\infty|f'(x)|^pv(x)dx)^{\frac{1}{p}}\tag{1}$$

holds for some $C>0$ and all $f\in C_0^1([0,\infty))$ if and only if

$$\sup_{r>0}(\int_0^r u(x)dx)^{\frac{1}{q}}\ln^{\frac{\lambda}{q}}(\frac{1}{\int_0^r u(x)dx}+e)\|v^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([r,\infty))}<\infty.\tag{2}$$

Obviously, the entropy of f on $([0,\infty),\mu)$ is controlled by $\int_0^\infty|f(x)|\ln(\frac{|f(x)|}{\int_0^\infty|f(y)|d\mu(y)}+e)d\mu(x)$. With the help of this theorem, we can generalize some known results for the entropies when $n=1$. For example, we can get the following corollary.

Corollary 1.2. *Set $d\mu(x)=(1+x^2)^{-\beta}dx$ and $w(x)=(1+x^2)^2$. Then for any $\beta<\frac{3}{2}$, we have*

$$Ent_\mu(f^2)=\int_0^\infty|f(x)|^2\ln\frac{|f(x)|^2}{\int_0^\infty f^2d\mu}d\mu(x)\le C\int_0^\infty|f'(x)|^2w(x)d\mu(x).$$

Remark. When $\beta\ge 1$, the above inequality was proved in [1, Theorem 3.4]. Thus, this indicates that the estimation in corollary 1.2 holds true for any $\beta\in R$. Note that one can easily show the corresponding inequality in Theorem 1.1 is not true when $\beta\ge 3$. So there are some essential differences between the entropy and the quantity that we consider here. Certainly, in this problem, by virtue of this theorem we can get more generations. We omit the details here.

In Section 2 we shall introduce some definitions and lemmas. The proofs of Theorem 1.1 and Corollary 1.2 will be given in the last section.

Throughout the paper, C,c are used to denote positive constants that are independent of all essential variables and may vary in the different occurrences. We also use the notation $a\approx b$ to mean that there exist two constants C and c such that $ca\le b\le Ca$.

§2 Some definitions and lemmas

In this section we give some definitions and lemmas.

For a real-valued function f on the measure space (X,μ), the distribution and the rearrangement of f are defined as

$$\mu_f(a) = \mu(\{x : |f(x)| > a\}),$$
$$f^*(t) = \inf\{a : \mu_f(a) \le t\}.$$

As μ_f is right-continuous, one can check that f and f^* have the same distributions, i.e., for any $a \ge 0$, there holds

$$|\{t : f^*(t) > a\}| = \mu_f(a).$$

It is easy to see that in order to prove that a non-increasing nonnegative function g is the rearrangement of f, we only need to show that g and f have the same distributions. Below we always use f^*_μ to denote the rearrangement of f on the space $([0,\infty),\mu)$.

The following proposition is probably well-known, but we can not find the related literature. For the sake of completeness, here we give a direct proof, which may be generated to some Orlicz spaces.

Proposition 2.1. *Let μ be a nonnegative Borel measure, $\lambda > 0$ and f^* be the non-increasing rearrangement of f. Then we have*

$$\int_0^\infty f^*(t)\ln^\lambda(\frac{1}{t}+e)dt \approx \int_{R^n} |f(x)|\ln^\lambda(\frac{|f(x)|}{\int_{R^n}|f(y)|d\mu(y)}+e)d\mu(x).$$

Proof. As the integrals are linear, without no generality, we may assume that $\int_{R^n}|f(x)|d\mu(x) = 1$. Set

$$E_k = \{x : e^k < |f(x)| \le e^{1+k}\}, t_k = \mu(E_k), a_k = \sum_{j=k}^\infty t_j.$$

As $e^k a_k \le \sum\limits_{j=k}^{\infty} e^j \mu(E_j) \le \int_{R^n}|f(x)|d\mu(x) = 1$, one get that $a_k \le e^{-k}$. On the other hand, by the definition of a_k and the rearrangement, it is easy to see that

$$\mu_f(e^k) = a_k, e^k < f^*(t) \le e^{1+k} \text{ when } a_{k+1} < t < a_k.$$

At first, we give a fundamental inequality. For $0 \le b < a \le e^{-1}$, there holds

$$\int_b^a \ln^\lambda\frac{1}{t}dt \le C_\lambda(a-b)\ln^\lambda\frac{1}{a}. \tag{3}$$

For any $\epsilon > 0$, as $t^\epsilon \ln\frac{1}{t}$ is increasing on $(0, e^{-\frac{1}{\epsilon}})$ and decreasing on $(e^{-\frac{1}{\epsilon}}, 1)$, for $0 < t < a < e^{-1}$ there holds $t^\epsilon \ln\frac{1}{t} < a^\epsilon \ln\frac{1}{a}$ when $a < e^{-\frac{1}{\epsilon}}$ or

$$t^\epsilon \ln\frac{1}{t} \le \frac{1}{e\epsilon}, a^\epsilon \ln\frac{1}{a} \ge e^{-\epsilon} \text{ when } e^{-\frac{1}{\epsilon}} \le a < e^{-1}.$$

In both cases, we have $t^\epsilon \ln\frac{1}{t} \le \frac{e^{\epsilon-1}}{\epsilon}a^\epsilon \ln\frac{1}{a} = C_\epsilon a^\epsilon \ln\frac{1}{a}$ which means that

$$\ln\frac{1}{t} \le C_\epsilon(\frac{t}{a})^{-\epsilon}\ln\frac{1}{a}. \tag{4}$$

Take $\epsilon = \frac{1}{2\lambda}$ in (4), some direct computations yield that

$$\begin{aligned}\int_b^a \ln^\lambda\frac{1}{t}dt &\le C_\lambda \int_b^a (\frac{t}{a})^{-\frac{1}{2}}\ln^\lambda\frac{1}{a}dt \\ &\le C_\lambda(a-b)\ln^\lambda\frac{1}{a}.\end{aligned}$$

So we get the inequality in (3).

Now we return to the main proof. As $a_k \le e^{-k} \le e^{-1}$ when $k \ge 1$, we have

$$\begin{aligned}
&\int_0^{\infty} f^*(t)\ln^{\lambda}(\frac{1}{t}+e)dt \\
= &\sum_{k=-\infty}^{\infty}\int_{a_{k+1}}^{a_k} f^*(t)\ln^{\lambda}(\frac{1}{t}+e)dt \\
\le &\sum_{k=1}^{\infty}\int_{a_{k+1}}^{a_k} e^{k+1}\ln^{\lambda}\frac{1}{t^2}dt + \int_{a_1}^{1} e\ln^{\lambda}(\frac{1}{t}+e)dt + \int_1^{\infty} f^*(t)\ln^{\lambda}(1+e)dt \\
\le &\sum_{k=1}^{\infty}2^{\lambda}e^{k+1}\int_{a_{k+1}}^{a_k}\ln^{\lambda}\frac{1}{t}dt + e\int_0^1\ln^{\lambda}(\frac{1}{t}+e)dt + \ln^{\lambda}(1+e)\int_0^{\infty} f^*(t)dt \\
\le &C_{\lambda}(\sum_{k=1}^{\infty}e^k\int_{a_{k+1}}^{a_k}\ln^{\lambda}\frac{1}{t}dt + 1) \\
\le &C_{\lambda}(\sum_{k=1}^{\infty}e^k t_k\ln^{\lambda}\frac{1}{a_k} + 1).
\end{aligned} \tag{5}$$

Here we use the fact that $\frac{1}{t}+e < \frac{1}{t^2}$ when $t < e^{-1}$ in the first inequality, $\int_0^{\infty} f^*(t)dt = \int_{R^n}|f(x)|d\mu(x) = 1$ and $\int_0^1 \ln^{\lambda}(\frac{1}{t}+e)dt < \infty$ in the third inequality, and (3) in the last inequality.

For the first term in (5), we need some further estimates.

$$\begin{aligned}
&\sum_{k\ge 1} e^k t_k \ln^{\lambda}\frac{1}{a_k} \\
= &(\sum_{k\ge 1, a_k > e^{-2k}} + \sum_{k\ge 1, a_k\le e^{-2k}})e^k t_k\ln^{\lambda}\frac{1}{a_k} \\
\le &\sum_{k\ge 1, a_k > e^{-2k}}(2k)^{\lambda}e^k t_k + \sum_{k\ge 1, a_k\le e^{-2k}} e^k a_k\ln^{\lambda}\frac{1}{a_k} \\
\le &2^{\lambda}\sum_{k\ge 1}k^{\lambda}e^k t_k + \sum_{k\ge\max\{1,\frac{\lambda}{2}\}} e^k e^{-2k}\ln^{\lambda}\frac{1}{e^{-2k}} + \sum_{\max\{1,\frac{\lambda}{2}\}>k\ge 1} e^k e^{-\lambda}\ln^{\lambda}\frac{1}{e^{-\lambda}} \\
\le &2^{\lambda}\sum_{k\ge 1}k^{\lambda}e^k t_k + \sum_{k\ge\max\{1,\frac{\lambda}{2}\}} e^{-k}(2k)^{\lambda} + \sum_{\frac{\lambda}{2}>k\ge 1} e^{k-\lambda}\lambda^{\lambda} \\
\le &C_{\lambda}(\sum_{k\ge 1}k^{\lambda}e^k t_k + 1).
\end{aligned} \tag{6}$$

In the second inequality we use the fact that $t\ln^{\lambda}\frac{1}{t}$ is increasing on $(0, e^{-\lambda})$ and decreasing on $(e^{-\lambda}, 1)$. From (5) and (6), we can get that

$$\int_0^{\infty} f^*(t)\ln^{\lambda}(\frac{1}{t}+e)dt \le C_{\lambda}(\sum_{k\ge 1}k^{\lambda}e^k t_k + 1). \tag{7}$$

On the other hand, one can yield that

$$\int_{R^n}|f(x)|\ln^{\lambda}(|f(x)|+e)d\mu(x) \ge \sum_{k=-\infty}^{\infty}\int_{E_k} e^k\ln^{\lambda}(e^k+e)d\mu(x) \ge \sum_{k=-\infty}^{\infty}k^{\lambda}e^k t_k. \tag{8}$$

Obviously, there holds $\int_{R^n} |f(x)| \ln^\lambda(|f(x)|+e)d\mu(x) \geq 1$. So, by (7) and (8) we have

$$\begin{aligned}\int_0^\infty f^*(t)\ln^\lambda(\frac{1}{t}+e)dt &\leq C_\lambda(\sum_{k\geq 1} k^\lambda e^k t_k + 1)\\ &\leq C_\lambda(\int_{R^n} |f(x)|\ln^\lambda(|f(x)|+e)d\mu(x)+1)\\ &\leq C_\lambda \int_{R^n} |f(x)|\ln^\lambda(|f(x)|+e)d\mu(x). \end{aligned} \tag{9}$$

To complete the proof of this proposition, it is left for us to show that

$$\int_{R^n} |f(x)|\ln^\lambda(|f(x)|+e)d\mu(x) \leq C_\lambda \int_0^\infty f^*(t)\ln^\lambda(\frac{1}{t}+e)dt. \tag{10}$$

When $k \geq 1$ and $x \in E_k$, as $a_k \leq e^{-k}$, we obtain that $e^{k+1}+e < (\frac{1}{a_k}+e)^2$ and

$$|f(x)|\ln^\lambda(|f(x)|+e) \leq e^{1+k}\ln^\lambda(e^{1+k}+e) \leq e^{1+k}2^\lambda \ln^\lambda(\frac{1}{a_k}+e) \leq C_\lambda e^k \ln^\lambda(\frac{1}{a_k}+e).$$

By virtue of the fact that $e^k < f^*(t) \leq e^{1+k}$ for $a_{k+1} < t < a_k$ and $a_k - a_{k+1} = t_k = \mu(E_k)$, we can get that

$$\begin{aligned}&\int_{R^n} |f(x)|\ln^\lambda(|f(x)|+e)d\mu(x)\\ =\ & (\sum_{k\geq 1}+\sum_{k\leq 0})\int_{E_k} |f(x)|\ln^\lambda(|f(x)|+e)d\mu(x)\\ \leq\ & \sum_{k\geq 1} C_\lambda e^k \ln^\lambda(\frac{1}{a_k}+e)t_k + \sum_{k\leq 0}\int_{E_k} |f(x)|\ln^\lambda(e+e)d\mu(x)\\ \leq\ & C_\lambda(\sum_{k\geq 1}\int_{a_{k+1}}^{a_k} f^*(t)\ln^\lambda(\frac{1}{t}+e)dt + \int_{R^n}|f(x)|d\mu(x))\\ \leq\ & C_\lambda(\int_0^\infty f^*(t)\ln^\lambda(\frac{1}{t}+e)dt+1).\end{aligned}$$

It is easy to see that $\int_0^\infty f^*(t)\ln^\lambda(\frac{1}{t}+e)dt \geq \int_0^\infty f^*(t)dt = 1$. So we prove (10).

From (9) and (10), we obtain that

$$c_\lambda \int_0^\infty f^*(t)\ln^\lambda(\frac{1}{t}+e)dt \leq \int_{R^n} |f(x)|\ln^\lambda(|f(x)|+e)d\mu(x) \leq C_\lambda \int_0^\infty f^*(t)\ln^\lambda(\frac{1}{t}+e)dt.$$

The proof of this proposition is completed.□

The following lemma is fundamental.

Lemma 2.2. *Suppose that u is nonnegative and locally integrable on $[0,\infty)$ and $d\mu(x) = u(x)dx$. Set $\mu^{-1}(t) = \sup\{a : \int_0^a u(x)dx \leq t\}, t > 0$. If f is nonnegative and non-increasing on $(0,\infty)$, then the rearrangement of f on $([0,\infty),\mu)$ is*

$$f^*_\mu(t) = f\circ\mu^{-1}(t) = f(\mu^{-1}(t)),\ \ a.e.\ t \in (0,\infty).$$

Remark. From the definition we can see that $\mu^{-1}(t) = \infty$ when $t \geq \int_0^\infty u(x)dx$. If $0 < t < \int_0^\infty u(x)dx$, by the continuity of integral, $\mu^{-1}(t)$ is well-defined and

$$\int_0^{\mu^{-1}(t)} u(x)dx = t. \tag{11}$$

It is easy to check that μ^{-1} is strictly increasing when $0 < t < \int_0^\infty u(x)dx$.

Proof. Because f is nonnegative and non-increasing on $(0,\infty)$, $f(\infty) = \lim_{x\to\infty} f(x)$ can be defined suitable.

As $\mu^{-1} : (0,\infty) \to (0,\infty]$ is increasing and f is non-increasing on $(0,\infty]$, we know that $f \circ \mu^{-1}$ is nonnegative and non-increasing on $(0,\infty)$. So, it left for us to show that $f \circ \mu^{-1}$ on $(0,\infty)$ and f on $((0,\infty),\mu)$ have the same distributions, i.e., for any $a > 0$, there holds

$$|\{t : f \circ \mu^{-1}(t) > a\}| = \mu_f(a).$$

Set $r = \sup\{x : f(x) > a\}$ and $t = \int_0^r u(x)dx$. As f is non-increasing on $(0,\infty)$, we have $f(x) > a$ for $0 < x < r$ and $f(x) \le a$ when $x > r$. The fact that u is locally integrable implies that $\mu(\{r\}) = 0$, so there holds

$$\mu_f(a) = \mu((0,r)) = \int_0^r u(x)dx = t. \tag{12}$$

On the other hand, if $0 < s < t$, as $0 < s < t = \int_0^r u(x)dx \le \int_0^\infty u(x)dx$, we have

$$\int_0^{\mu^{-1}(s)} u(x)dx = s < t = \int_0^r u(x)dx$$

which yields that $\mu^{-1}(s) < r$. So we have

$$f \circ \mu^{-1}(s) = f(\mu^{-1}(s)) > a.$$

If $s > t$ and $\int_0^\infty u(x)dx \le s$, then $\mu^{-1}(s) = \infty > r$. If $s > t$ and $\int_0^\infty u(x)dx < s$, then

$$\int_0^{\mu^{-1}(s)} u(x)dx = s > t = \int_0^r u(x)dx$$

which means that $\mu^{-1}(s) > r$. So in both cases we get that $\mu^{-1}(s) > r$ when $s > t$ and

$$f \circ \mu^{-1}(s) = f(\mu^{-1}(s)) \le a.$$

Therefore we prove that $(0,t) \subset \{s : f \circ \mu^{-1}(s) > a\} \subset (0,t]$ which yields that

$$|\{s : f \circ \mu^{-1}(s) > a\}| = t = \mu_f(a).$$

So we complete the proof of this lemma.□

Lemma 2.3. *([2], P407, Theorem 2) Assume that w, v are nonnegative and locally integrable on $[0,\infty)$ and $1 \le p \le q < \infty$. Then*

$$\Big(\int_0^\infty \Big(\int_x^\infty |g(t)|dt\Big)^q w(x)dx\Big)^{\frac{1}{q}} \le C\Big(\int_0^\infty |g(x)|^p v(x)dx\Big)^{\frac{1}{p}}$$

holds for some $C > 0$ and all $g \in L_{loc}([0,\infty))$ if and only if

$$\sup_{r>0}\Big(\int_0^r w(x)dx\Big)^{\frac{1}{q}} \|v^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([r,\infty))} < \infty. \tag{13}$$

Remark. In [2], they proved the dual form of this lemma and this lemma was considered as a corollary. In fact, if we set $w(x) = x^{-2}\tilde{w}(x^{-1}), v(x) = x^{2p-2}\tilde{v}(x^{-1})$ and $g(x) = x^{-2}\tilde{g}(x^{-1})$, then this lemma can be derived from Theorem 1 in ([2], P405) directly.

§3 The main proof

Now we give the proof of our main theorem. At first, we show that

$$(2) \Rightarrow (1).$$

For $\lambda > 0, a > e^{-1}$, there holds

$$\int_0^a \ln^\lambda(\frac{1}{t}+e)dt \le \int_{e^{-1}}^a \ln^\lambda(e+e)dt + \int_0^{e^{-1}} \ln^\lambda(\frac{1}{t}+e) \le C_\lambda a \ln^\lambda(\frac{1}{a}+e).$$

Therefore, by (3), for any $a > 0$ we get that

$$a\ln^\lambda(\frac{1}{a}+e) \le \int_0^a \ln^\lambda(\frac{1}{t}+e)dt \le C_\lambda a \ln^\lambda(\frac{1}{a}+e). \quad (14)$$

Set $a = \mu(r) = \int_0^r u(x)dx$ in (14), we have

$$(\int_0^r u(x)\ln^\lambda(\frac{1}{\int_0^x u(y)dy}+e)dx)^{\frac{1}{q}} \approx (\int_0^r u(x)dx)^{\frac{1}{q}} \ln^{\frac{\lambda}{q}}(\frac{1}{\int_0^r u(x)dx}+e).$$

So, the inequality in (2) is equivalent to

$$\sup_{r>0}(\int_0^r u(x)\ln^\lambda(\frac{1}{\int_0^x u(s)ds}+e)dx)^{\frac{1}{q}} \|v^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([r,\infty))} < \infty. \quad (15)$$

Take $d\mu(x) = u(x)dx$. By Proposition 2.1, we know that

$$\int_0^\infty |f(x)|^q \ln^\lambda(\frac{|f(x)|^q}{\int_0^\infty |f(y)|^q u(y)dy}+e)u(x)dx \approx \int_0^\infty f_\mu^*(t)^q \ln^\lambda(\frac{1}{t}+e)dt.$$

So (1) is equivalent to

$$(\int_0^\infty f_\mu^*(t)^q \ln^\lambda(\frac{1}{t}+e)dt)^{\frac{1}{q}} \le C(\int_0^\infty |f'(x)|^p v(x)dx)^{\frac{1}{p}}. \quad (16)$$

As $f \in C_0^1([0,\infty))$, there holds $|f(t)| \le G(t) = \int_t^\infty |f'(s)|ds, t \in (0,\infty)$. By Lemma 2.2, we have

$$f_\mu^*(t) \le G_\mu^*(t) = G(\mu^{-1}(t)) = \int_{\mu^{-1}(t)}^\infty |f'(s)|ds. \quad (17)$$

Set $x = \mu^{-1}(t)$. From the remark of Lemma 2.2, when $0 < t < \int_0^\infty u(s)ds$, there holds $t = \mu(x) = \int_0^x u(s)ds$. On the other hand, when $t \ge \mu(\infty) = \int_0^\infty u(s)ds$, we have $\mu^{-1}(t) = \infty$ and $f_\mu^*(t) \le G(\mu^{-1}(t)) = G(\infty) = 0$. So we obtain that

$$\begin{aligned}
&\int_0^\infty f_\mu^*(t)^q \ln^\lambda(\frac{1}{t}+e)dt \\
\le\ & \int_0^{\mu(\infty)} (\int_{\mu^{-1}(t)}^\infty |f'(s)|ds)^q \ln^\lambda(\frac{1}{t}+e)dt \\
=\ & \int_0^\infty (\int_x^\infty |f'(s)|ds)^q \ln^\lambda(\frac{1}{\mu(x)}+e)d\mu(x) \\
=\ & \int_0^\infty (\int_x^\infty |f'(s)|ds)^q u(x)\ln^\lambda(\frac{1}{\int_0^x u(s)ds}+e)dx. \quad (18)
\end{aligned}$$

Take $g(x) = |f'(x)|, w(x) = u(x)\ln^\lambda(\frac{1}{\int_0^x u(s)ds}+e)$ in Lemma 2.3. We get that if (15) holds, then

$$(\int_0^\infty (\int_x^\infty |f'(s)|ds)^q u(x)\ln^\lambda(\frac{1}{\int_0^x u(s)ds}+e)dx)^{\frac{1}{q}} \le C(\int_0^\infty |f'(x)|^p v(x)dx)^{\frac{1}{p}}.$$

From (18), we get the inequality in (16). So one can obtain that

$$(2) \Rightarrow (15) \Rightarrow (16) \Rightarrow (1).$$

At last, we prove that

$$(1) \Rightarrow (2).$$

When $p > 1$, for any $r > 0$, set

$$f_r(x) = \begin{cases} \int_r^\infty v(s)^{-\frac{1}{p-1}} ds, & 0 \le x < r; \\ \int_x^\infty v(s)^{-\frac{1}{p-1}} ds, & x \ge r. \end{cases}$$

By approximation, here we may assume that v^{-1} is continuous on $[0, \infty)$, so $f_r \in C_0^1([0, \infty))$ and it is non-increasing on $(0, \infty)$. By Lemma 2.2, if $t < \mu(r) = \int_0^r u(s)ds$, then $\mu^{-1}(t) < r$ and

$$(f_r)_\mu^*(t) = f_r(\mu^{-1}(t)) = \int_r^\infty v(s)^{-\frac{1}{p-1}} ds.$$

Some direct computations yield that

$$\begin{aligned} & (\int_0^\infty (f_r)_\mu^*(t)^q \ln^\lambda(\frac{1}{t} + e)dt)^{\frac{1}{q}} \\ \ge\ & (\int_0^{\mu(r)} (f_r)_\mu^*(t)^q \ln^\lambda(\frac{1}{t} + e)dt)^{\frac{1}{q}} \\ =\ & (\int_0^{\mu(r)} (\int_r^\infty v(s)^{-\frac{1}{p-1}} ds)^q \ln^\lambda(\frac{1}{t} + e)dt)^{\frac{1}{q}} \\ =\ & (\int_0^r \ln^\lambda(\frac{1}{\mu(x)} + e)d\mu(x))^{\frac{1}{q}} (\int_r^\infty v(s)^{-\frac{1}{p-1}} ds) \\ =\ & (\int_0^r u(x) \ln^\lambda(\frac{1}{\int_0^x u(s)ds} + e)dx)^{\frac{1}{q}} \|v^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([r,\infty))}^{\frac{p}{p-1}}. \end{aligned}$$

On the other hand, $f_r'(x) = v(x)^{-\frac{1}{p-1}} \chi_{(r,\infty)}(x)$, so (16) implies that

$$\begin{aligned} & (\int_0^r u(x) \ln^\lambda(\frac{1}{\int_0^x u(s)ds} + e)dx)^{\frac{1}{q}} \|v^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([r,\infty))}^{\frac{p}{p-1}} \\ \le\ & C(\int_r^\infty v(s)^{-\frac{p}{p-1}+1} ds)^{\frac{1}{p}} \\ =\ & C\|v^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([r,\infty))}^{\frac{1}{p-1}} \end{aligned}$$

which means that

$$(\int_0^r u(x) \ln^\lambda(\frac{1}{\int_0^x u(s)ds} + e)dx)^{\frac{1}{q}} \|v^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([r,\infty))} \le C.$$

On the other hand, when $p = 1$, for any $r > 0$, there holds

$$|\{x \in (r, \infty) : v^{-1}(x) \ge \frac{2\|v^{-1}\|_{L^\infty([r,\infty))}}{3}\}| > 0$$

which implies that $|\{x \in (r, \infty) : v(x) \le \frac{3\|v^{-1}\|_{L^\infty([r,\infty))}^{-1}}{2}\}| > 0$. So, by the differential theorem, we always can find an interval $(a, b) \subset (r, \infty)$ such that

$$\int_a^b v(x)dx < 2(b - a)\|v^{-1}\|_{L^\infty([r,\infty))}^{-1}.$$

By approximation, we may set $f_r' = \chi_{(a,b)}$. Then for any $x < r$ we have $f_r(x) = b - a$. In this case, if $t < \mu(r)$, by Lemma 2.2, then $\mu^{-1}(t) < r$ and

$$(f_r)^*_\mu(t) = f_r(\mu^{-1}(t)) = b - a.$$

Some simple computations yield that

$$\begin{aligned} & (\int_0^\infty (f_r)^*_\mu(t)^q \ln^\lambda(\frac{1}{t}+e)dt)^{\frac{1}{q}} \\ \geq\ & (b-a)(\int_0^{\mu(r)} \ln^\lambda(\frac{1}{t}+e)dt)^{\frac{1}{q}} \\ =\ & (b-a)(\int_0^r u(x)\ln^\lambda(\frac{1}{\int_0^x u(s)ds}+e)dx)^{\frac{1}{q}}. \end{aligned}$$

Obviously, there holds

$$\int_0^\infty |f'_r(x)|v(x)dx = \int_a^b v(x)dx < 2(b-a)\|v^{-1}\|^{-1}_{L^\infty([r,\infty))}.$$

So (16) implies that

$$\begin{aligned} & (b-a)(\int_0^r u(x)\ln^\lambda(\frac{1}{\int_0^x u(s)ds}+e)dx)^{\frac{1}{q}} \\ \leq\ & C(b-a)\|v^{-1}\|^{-1}_{L^\infty([r,\infty))} \end{aligned}$$

which means that

$$(\int_0^r u(x)\ln^\lambda(\frac{1}{\int_0^x u(s)ds}+e)dx)^{\frac{1}{q}}\|v^{-1}\|_{L^\infty([r,\infty))} \leq C.$$

Now, whether $p > 1$ or $p = 1$, we both obtain that

$$(\int_0^r u(x)\ln^\lambda(\frac{1}{\int_0^x u(s)ds}+e)dx)^{\frac{1}{q}}\|v^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([r,\infty))} \leq C.$$

Since r is arbitrary, we obtain that

$$(1) \Rightarrow (16) \Rightarrow (15) \Rightarrow (2).$$

The proof of Theorem 1.1 is completed. □

Finally, we show how to get Corollary 1.2. Take $p = q = 2, u(x) = (1+x^2)^{-\beta}$ and $v(x) = (1+x^2)^{2-\beta}$. Then by Theorem 1.1,

$$\int_0^\infty |f(x)|^2 \ln(\frac{|f(x)|^2}{\int_0^\infty f^2(y)(1+y^2)^{-\beta}dy}+e)(1+x^2)^{-\beta}dx \leq C\int_0^\infty |f'(x)|^2(1+x^2)^{2-\beta}dx$$

holds if and only if

$$\sup_{r>0}[\int_0^r (1+x^2)^{-\beta}dx][\ln(\frac{1}{\int_0^r (1+x^2)^{-\beta}dx}+e)][\int_r^\infty (1+x^2)^{\beta-2}dx] < \infty.$$

Some simple computations yield that

$$\int_0^r (1+x^2)^{-\beta}dx \approx \begin{cases} r, & r \leq 2; \\ 1, & r > 2, \beta > \frac{1}{2}; \\ \ln r, & r > 2, \beta = \frac{1}{2}; \\ r^{1-2\beta}, & r > 2, \beta < \frac{1}{2}. \end{cases}$$

In particular, for $r > 2$ and any $\beta \in R$ we obtain that

$$c \leq \int_0^r (1+x^2)^{-\beta}dx \leq C\max\{\ln r, r^{1-2\beta}\}. \tag{19}$$

On the other hand, when $\beta < \frac{3}{2}$, we have

$$\int_r^\infty (1+x^2)^{\beta-2}dx \approx \begin{cases} 1, & r \le 2; \\ r^{2\beta-3}, & r > 2. \end{cases}$$

Now, when $\beta < \frac{3}{2}, 0 < r \le 2$, there holds

$$[\int_0^r (1+x^2)^{-\beta}dx][\ln(\frac{1}{\int_0^r (1+x^2)^{-\beta}dx} + e)][\int_r^\infty (1+x^2)^{\beta-2}dx] \le Cr\ln(\frac{1}{r}+e) \le C.$$

When $\beta < \frac{3}{2}$ and $r > 2$, from (19) one can get that

$$\begin{aligned} & [\int_0^r (1+x^2)^{-\beta}dx][\ln(\frac{1}{\int_0^r (1+x^2)^{-\beta}dx} + e)][\int_r^\infty (1+x^2)^{\beta-2}dx] \\ \le\ & C\max\{\ln r, r^{1-2\beta}\}\ln(\frac{1}{c}+e)r^{2\beta-3} \\ \le\ & C\max\{r^{2\beta-3}\ln r, r^{-2}\} \le C. \end{aligned}$$

So we derive that

$$\sup_{r>0}[\int_0^r (1+x^2)^{-\beta}dx][\ln(\frac{1}{\int_0^r (1+x^2)^{-\beta}dx} + e)][\int_r^\infty (1+x^2)^{\beta-2}dx] < \infty$$

which implies that

$$\int_0^\infty |f(x)|^2 \ln(\frac{|f(x)|^2}{\int_0^\infty f^2(y)(1+y^2)^{-\beta}dy} + e)(1+x^2)^{-\beta}dx \le C\int_0^\infty |f'(x)|^2(1+x^2)^{2-\beta}dx.$$

Obviously, $\text{Ent}_\mu(f^2)$ is controlled by $\int_0^\infty |f(x)|^2 \ln(\frac{|f(x)|^2}{\int_0^\infty f^2 d\mu} + e)d\mu(x)$ where $d\mu(x) = (1+x^2)^{-\beta}dx$. So the proof is completed. □

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