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Demi-linear analysis II —demi-distributions

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Abstract. In this paper, we develop the theory of demi-distributions which generalizes the usual distribution theory. In particular, we show that many results on differentiations, Fourier transforms, and convolutions can be generalized to demi-distributions theory.

§1 Introduction

Let X be a topological vector space and $\mathcal{N}(X)$ the family of neighborhoods of $0 \in X$, and $C(0)$ the set of complex valued functions γ satisfying

1. $\gamma : \mathbb{C} \to \mathbb{C}$;

2. $\lim_{t\to 0} \gamma(t) = \gamma(0) = 0$;

3. $|\gamma(t)| \geq |t|$ if $|t| \leq 1$.

Let Y be a topological vector space, and $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A mapping $f : X \to Y$ is said to be *demi-linear* if $f(0) = 0$ and there exist $\gamma \in C(0)$ and $U \in \mathcal{N}(X)$ such that every $x \in X$, $u \in U$ and $t \in \{t \in \mathbb{K} : |t| \leq 1\}$ yield $r,\ s \in \mathbb{K}$ for which $|r-1| \leq |\gamma(t)|$, $|s| \leq |\gamma(t)|$ and $f(x+tu) = rf(x) + sf(u)$.

We denote by $\mathscr{L}_{\gamma,U}(X,Y)$ the demi-linear mappings related to $\gamma \in C(0)$ and $U \in \mathcal{N}(X)$, and by $\mathscr{K}_{\gamma,U}(X,Y)$ the subfamily of $\mathscr{L}_{\gamma,U}(X,Y)$ satisfying the following property: if $x \in X,\ u \in U$ and $|t| \leq 1$ then

$$f(x+tu) = f(x) + sf(u)$$

for some s with $|s| \leq |\gamma(t)|$.

As stated in [1, 2], the family of demi-linear mappings is a natural and valuable extension of the family of linear operators.

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For $a > 0$, $\mathscr{D}_a = \{\xi \in \mathbb{C}^{\mathbb{R}^n} : \xi$ is infinitely differentiable and $\xi(x) = 0$ whenever $|x| = \sqrt{x_1^2 + \cdots + x_n^2} > a\}$ has the locally convex Fréchet topology which is given by the norm sequence $\{\|\xi\|_p = \sup_{|x|\le a} \max_{|q|\le p} |D^q \xi(x)|\}_{p=0}^{\infty}$.

Let $\mathscr{D} = \bigcup_{m=1}^{\infty} \mathscr{D}_m$ be the strict inductive limit of $\{\mathscr{D}_m\}$.

Let $\mathscr{S} = \{\xi \in \mathbb{C}^{\mathbb{R}^n} : \xi$ is infinitely differentiable and rapidly decreasing$\}$. Equipped with the norm sequence $\{\|\xi\|_p = \sup_{|k|,|q|\le p, x\in\mathbb{R}^n} |x^k D^q \xi(x)|\}_{p=0}^{\infty}$, $\mathscr{S}$ is a locally convex Fréchet space, where $k = (k_1, k_2, \cdots, k_n)$ is a multi-index and $x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$.

The spaces $\{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$ are called the space of test functions.

The distributions of L. Schwartz, the generalized distributions of Beurling and the ultradistributions of Roumieu are continuous linear functionals defined on some suitable spaces of test functions.

In this paper, $E \in \{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$ is a space of test functions and a function $f : E \to \mathbb{C}$ is called a demi-distribution if f is continuous and demi-linear. Thus, the family of demi-distributions includes all usual distributions and many nonlinear functionals ([1, 2]).

By using the equicontinuity results in [1], we show that the family of demi-linear mappings can be used to develop the theory of distributions. For instance, in the case of the usual distributions the simplest equation $y' = 0$ has solutions $y = constant$ only. However, we will show that the equation $y' = 0$ has extremely many solutions which are nonlinear demi-linear functionals, and the equation $y' = f$ also has extremely many solutions which are demi-distributions. Moreover, we will show that the family of demi-distributions is closed with respect to extremely many of nonlinear transformations such as $|f(\cdot)|$, $\sin|f(\cdot)|$, $e^{|f(\cdot)|} - 1$, etc.

§2 Demi-distributions

Firstly, note that the space $\mathscr{D}$ is an (LF) space and so $\mathscr{D}$ is both barrelled and bornological ([3, p. 222]). Thus, $\mathscr{D}$ is $C-sequential$ ([3, p. 118]). There is an important result which says that every sequentially continuous linear operator from a $C-sequential$ locally convex space to a locally convex space must be continuous ([3, p. 118]). Now, we improve this result as following:

Theorem 2.1. *Let X, Y be locally convex spaces, $U \in \mathcal{N}(X)$ and $\gamma_0(t) = t$, $\forall\, t \in \mathbb{C}$. If X is $C-sequential$ and $f \in \mathscr{K}_{\gamma_0,U}(X,Y)$ is sequentially continuous, then f must be continuous.*

Proof. *Let $V \in \mathcal{N}(Y)$. Pick balanced convex neighborhoods $U_0 \in \mathcal{N}(X)$ and $V_0 \in \mathcal{N}(Y)$ such that $U_0 \subset U$, $V_0 \subset V$.*

Let $W = f^{-1}(V_0)$. For $u, w \in U_0 \bigcap W$ and scalars α, β with $|\alpha| + |\beta| \le 1$, it follows from $f \in \mathscr{K}_{\gamma_0,U}(X,Y)$ that

$$f(\alpha u + \beta w) = f(\alpha u) + s f(w) = s_1 f(u) + s f(w) \in s_1 V_0 + s V_0,$$

where $|s_1| \le |\gamma_0(\alpha)| = |\alpha|$, $|s| \le |\gamma_0(\beta)| = |\beta|$. Then $|s_1| + |s| \le |\alpha| + |\beta| \le 1$ and so $f(\alpha u + \beta w) \in V_0$, $\alpha u + \beta w \in U_0 \bigcap W$. This shows that $U_0 \bigcap W$ is both balanced and convex.

Let $x_k \to 0$ in X. Then $x_k \in U_0$ eventually. Since f is sequentially continuous, $f(x_k) \to f(0) = 0$ and $f(x_k) \in V_0$ eventually, i.e., $x_k \in W$ eventually. Thus, $x_k \in U_0 \bigcap W$ eventually *and so* $U_0 \bigcap W$ *is a sequential neighborhood of* $0 \in X$. *Since* X *is* $C-$ *sequential,* $U_0 \bigcap W \in \mathcal{N}(X)$ *and* $f(U_0 \bigcap W) \subset V_0 \subset V$. *This shows that* f *is continuous at* 0.

Suppose $(x_\alpha)_{\alpha \in I}$ *is a net in* X *such that* $x_\alpha \to x \in X$. *Pick an* $\alpha_0 \in I$ *for which* $x_\alpha - x \in U$, $\forall\, \alpha \geq \alpha_0$. *For* $\alpha \geq \alpha_0$,

$$f(x_\alpha) = f(x + x_\alpha - x) = f(x) + s_\alpha f(x_\alpha - x),\ |s_\alpha| \leq |\gamma_0(1)| = 1.$$

But $f(x_\alpha - x) \to f(0) = 0$ *and so* $f(x_\alpha) \to f(x)$. □

Theorem 2.2. *Suppose that* $E \in \{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$, $\gamma \in C(0)$, $U \in \mathcal{N}(E)$ *and* Y *is a topological vector space. If* $f, f_\nu \in \mathscr{L}_{\gamma,U}(E,Y)$ *are continuous* ($\nu = 1,2,3,\cdots$) *and* $f_\nu(\xi) \to f(\xi)$ *at each* $\xi \in E$, *then for every bounded* $B \subset E$, $\lim_\nu f_\nu(\xi) = f(\xi)$ *uniformly for* $\xi \in B$.

Proof. Since both $\mathscr{D}_a$ *and* $\mathscr{S}$ *are Fréchet spaces having the Montel property, we only need to consider* $\mathscr{D}$. *Suppose that* $B \subset \mathscr{D}$ *is bounded but* $\lim_\nu f_\nu(\xi) = f(\xi)$ *is not uniform for* $\xi \in B$. *Then there exist* $V \in \mathcal{N}(Y)$, $\{\xi_k\} \subset B$ *and integers* $\nu_1 < \nu_2 < \cdots$ *such that*

$$f_{\nu_k}(\xi_k) - f(\xi_k) \notin V,\ k = 1,2,3,\cdots.$$

Pick a balanced $W \in \mathcal{N}(Y)$ *for which* $W + W + W \subset V$. *Since* B *is bounded in* $\mathscr{D}$, $B \subset \mathscr{D}_m$ *for some* $m \in \mathbb{N}$ *([3, p. 219]) and* B *is relatively compact in the Fréchet space* $\mathscr{D}_m$ *([4, Th. 1.6.2]). By passing to a subsequence if necessary, we say that* $\xi_k \to \xi \in \mathscr{D}_m$. *Since* $f_\nu(\eta) \to f(\eta)$ *at each* $\eta \in \mathscr{D}_m$, $\{f_\nu\}_1^\infty$ *is pointwise bounded on* $\mathscr{D}_m$ *and so* $\{f_\nu\}_1^\infty$ *is equicontinuous on* $\mathscr{D}_m$ *by Th. 3.1 of [1]. By Cor. 3.1 of [1],* $\lim_k f_\nu(\xi_k) = f_\nu(\xi)$ *uniformly for* $\nu \in \mathbb{N}$ *and so there is a* $k_0 \in \mathbb{N}$ *such that* $f_\nu(\xi_k) - f_\nu(\xi) \in W$ *for all* $\nu \in \mathbb{N}$ *and* $k > k_0$. *Since* $f : \mathscr{D} \to Y$ *is continuous and* $f_\nu(\xi) \to f(\xi)$, *there exist* $\nu_0,\ k_1 \in \mathbb{N}$ *such that* $f(\xi) - f(\xi_k) \in W$ *for all* $k > k_1$ *and* $f_\nu(\xi) - f(\xi) \in W$ *for all* $\nu > \nu_0$.

Pick an integer $k_2 \geq k_0 + k_1$ *for which* $\nu_k > \nu_0$ *whenever* $k > k_2$. *Then for every* $k > k_2$ *we have that*

$$\begin{aligned} f_{\nu_k}(\xi_k) - f(\xi_k) &= f_{\nu_k}(\xi_k) - f_{\nu_k}(\xi) + f_{\nu_k}(\xi) - f(\xi) + f(\xi) - f(\xi_k) \\ &\in W + W + W \subset V. \end{aligned}$$

This is a contradiction and so $\lim_\nu f_\nu(\xi) = f(\xi)$ *uniformly for* $\xi \in B$. □

In general, $\mathscr{L}_{\gamma,U}(X,Y) \subsetneqq \mathscr{W}_{\gamma,U}(X,Y)$ where Y is locally convex. Using Th. 4.1 of [1] instead of Th. 3.1 of [1], the above proof gives an improved result as follows.

Theorem 2.3. *Let* $E \in \{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$, $\gamma \in C(0)$ *and* $U \in \mathcal{N}(E)$. *Let* Y *be a locally convex space and* $f, f_\nu \in \mathscr{W}_{\gamma,U}(E,Y)$ *are continuous,* $\nu = 1,2,3,\cdots$. *If* $f_\nu(\xi) \to f(\xi)$ *at each* $\xi \in E$, *then for every bounded* $B \subset E$, $\lim_\nu f_\nu(\xi) = f(\xi)$ *uniformly for* $\xi \in B$.

For $\mathscr{K}_{\gamma_0,U}(X,Y)$, we have a much stronger result as follows.

Theorem 2.4. *Let* $E \in \{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$, $U \in \mathcal{N}(E)$ *and* Y *be a locally convex space. Let* $f_\nu \in \mathscr{K}_{\gamma_0,U}(E,Y)$ *be continuous,* $\forall\, \nu \in \mathbb{N}$. *If* $\lim_\nu f_\nu(\xi) = f(\xi)$ *exists at each* $\xi \in E$, *then* f *is also a continuous mapping in* $\mathscr{K}_{\gamma_0,U}(E,Y)$ *and for every bounded* $B \subset E$, $\lim_\nu f_\nu(\xi) = f(\xi)$ *uniformly for* $\xi \in B$.

Proof. *Only need to consider* $E = \mathscr{D}$. *Let* $\xi \in E$, $\eta \in U$ *and* $|t| \le 1$. *Then* $f(\xi + t\eta) = \lim_\nu f_\nu(\xi + t\eta) = \lim_\nu(f_\nu(\xi) + s_\nu f_\nu(\eta))$, *where* $|s_\nu| \le |\gamma_0(t)| = |t| \le 1$. *Say that* $s_{\nu_k} \to s$. *Then* $|s| = \lim_k |s_{\nu_k}| \le |\gamma_0(t)|$ *and*

$$f(\xi + t\eta) = \lim_k f_{\nu_k}(\xi + t\eta) = \lim_k (f_{\nu_k}(\xi) + s_{\nu_k} f_{\nu_k}(\eta)) = f(\xi) + sf(\eta).$$

Thus, $f \in \mathscr{K}_{\gamma_0,U}(E, Y)$.

Let $\xi_k \to \xi$ *in* $\mathscr{D}$. *Then* $\xi_k \to \xi$ *in* $\mathscr{D}_m$ *for some* $m \in \mathbb{N}$ *([3, p. 219]). Since* $f_\nu(\cdot) \to f(\cdot)$, $\{f_\nu\}_1^\infty$ *is pointwise bounded on* $\mathscr{D}_m$ *and, by Cor. 3.1 of [1],* $\lim_k f_\nu(\xi_k) = f_\nu(\xi)$ *uniformly for* $\nu \in \mathbb{N}$. *Then* $\lim_k f(\xi_k) = \lim_k \lim_\nu f_\nu(\xi_k) = \lim_\nu \lim_k f_\nu(\xi_k) = \lim_\nu f_\nu(\xi) = f(\xi)$. *Thus,* $f : \mathscr{D} \to Y$ *is sequentially continuous and so* f *is continuous by Th. 2.1.*

Now the desired follows from Th. 2.2. □

Henceforth, $E \in \{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$.

Definition 2.1. $f : E \to \mathbb{C}$ *is called a demi-distribution if* f *is continuous and* $f \in \mathscr{L}_{\gamma,U}(E, \mathbb{C})$ *for some* $\gamma \in C(0)$ *and* $U \in \mathscr{N}(E)$.

Let $E^{(\gamma,U)}$ be the family of demi-distributions related to $\gamma \in C(0)$ and $U \in \mathscr{N}(E)$. Let $[E^{(\gamma,U)}]$ be the span $(E^{(\gamma,U)})$ in $\mathbb{C}^E$, i.e., $[E^{(\gamma,U)}] = \{\text{finite sum } \sum t_k f_k : t_k \in \mathbb{C}, f_k \in E^{(\gamma,U)}\}$.

Let E' be the space of usual distributions, i.e., E' is the space of continuous linear functionals. Obviously, $E' \subset E^{(\gamma,U)}$, $\forall\, U \in \mathscr{N}(E)$, $\gamma \in C(0)$.

Example 2.1. *(1) For every* $f \in L^1_{loc}(\mathbb{R}^n)$ *define* $[f] : \mathscr{D} \to \mathbb{R}$ *by* $[f](\xi) = \int_{\mathbb{R}^n} |f(x)\xi(x)|\, dx$, $\xi \in \mathscr{D}$.

Let $\gamma \in C(0)$ *and* $\xi, \eta \in \mathscr{D}$, $|t| \le 1$. *For every* $x \in \mathbb{R}^n$ *there exists* $\alpha(x) \in [-|t|, |t|]$ *such that* $|\xi(x) + t\eta(x)| = |\xi(x)| + \alpha(x)|\eta(x)|$ *and*

$$\begin{aligned}
[f](\xi + t\eta) &= \int_{\mathbb{R}^n} |f(x)(\xi + t\eta)(x)|\, dx \\
&= \int_{\mathbb{R}^n} |f(x)||\xi(x) + t\eta(x)|\, dx \\
&= \int_{\mathbb{R}^n} |f(x)|[|\xi(x)| + \alpha(x)|\eta(x)|]\, dx \\
&= \int_{\mathbb{R}^n} |f(x)\xi(x)|\, dx + \int_{\mathbb{R}^n} \alpha(x)|f(x)\eta(x)|\, dx \\
&= [f](\xi) + \int_{\mathbb{R}^n} \alpha(x)|f(x)\eta(x)|\, dx.
\end{aligned}$$

If $\int_{\mathbb{R}^n} |f(x)\eta(x)|\, dx = 0$, *then* $0 \le |\int_{\mathbb{R}^n} \alpha(x)|f(x)\eta(x)|\, dx| \le \int_{\mathbb{R}^n} |\alpha(x) f(x)\eta(x)| dx \le |t| \int_{\mathbb{R}^n} |f(x)\eta(x)|\, dx = 0$ *and so* $\int_{\mathbb{R}^n} \alpha(x)|f(x)\eta(x)|\, dx = 0 = 0[f](\eta)$, *where* $0 \le |\gamma(t)|$.

If $\int_{\mathbb{R}^n} |f(x)\eta(x)|\, dx \ne 0$, *then*

$$|\frac{\int_{\mathbb{R}^n} \alpha(x)|f(x)\eta(x)|\, dx}{\int_{\mathbb{R}^n} |f(x)\eta(x)|\, dx}| \le |t| \le |\gamma(t)|,$$

so $\int_{\mathbb{R}^n} \alpha(x)|f(x)\eta(x)|\, dx = s\int_{\mathbb{R}^n} |f(x)\eta(x)|\, dx = s[f](\eta)$ *where* $s = \frac{\int_{\mathbb{R}^n} \alpha(x)|f(x)\eta(x)|\, dx}{\int_{\mathbb{R}^n} |f(x)\eta(x)|\, dx}$, $|s| \le |t| \le |\gamma(t)|$. *Thus,*

$$[f](\xi + t\eta) = [f](\xi) + s[f](\eta),\ \ |s| \le |\gamma(t)|,$$

i.e., $[f] \in \mathscr{K}_{\gamma,\mathscr{D}}(\mathscr{D},\mathbb{R}) \bigcap \mathscr{D}^{(\gamma,\mathscr{D})}$ *but* $[f]$ *is not a usual distribution.*

(2) Let $\mathscr{D}_1(\mathbb{R}) = \{\xi \in \mathbb{R}^{\mathbb{R}} : \xi$ *is infinitely differentiable and* $\xi(x) = 0$ *for* $|x| > 1\}$. *Let* $\gamma(t) = \frac{\pi}{2}t$ *for* $t \in \mathbb{R}$ *and* $U = \{\xi \in \mathscr{D}_1(\mathbb{R}) : \max_{|x|\le 1} |\xi(x)| < 1\}$. *Define* $f : \mathscr{D}_1(\mathbb{R}) \to \mathbb{R}$ *by*

$$f(\xi) = \int_{-\infty}^{\infty} |\sin \xi(x)|\, dx,\ \ \xi \in \mathscr{D}_1(\mathbb{R}).$$

It is easy to show that if $a \in \mathbb{R}$ *and* $u, t \in [-1, 1]$ *then* $\sin(a + tu) = \sin a + s \sin u$ *with* $|s| \le \frac{\pi}{2}|t|$. *Hence, for* $\xi \in \mathscr{D}_1(\mathbb{R}),\ \eta \in U$ *and* $|t| \le 1$ *we have that*

$$\begin{aligned} f(\xi + t\eta) &= \int_{-\infty}^{\infty} |\sin[\xi(x) + t\eta(x)]|\, dx \\ &= \int_{-\infty}^{\infty} |\sin \xi(x) + \alpha(x) \sin \eta(x)|\, dx \qquad (|\alpha(x)| \le \frac{\pi}{2}|t|) \\ &= \int_{-\infty}^{\infty} [|\sin \xi(x)| + \beta(x)|\sin \eta(x)|]\, dx \qquad (|\beta(x)| \le |\alpha(x)| \le \frac{\pi}{2}|t|) \\ &= \int_{-\infty}^{\infty} |\sin \xi(x)|\, dx + \int_{-\infty}^{\infty} \beta(x)|\sin \eta(x)|\, dx \\ &= \int_{-\infty}^{\infty} |\sin \xi(x)|\, dx + s\int_{-\infty}^{\infty} |\sin \eta(x)|\, dx \qquad (|s| \le \frac{\pi}{2}|t| = |\gamma(t)|) \\ &= f(\xi) + sf(\eta), \quad |s| \le \frac{\pi}{2}|t| = |\gamma(t)|. \end{aligned}$$

Thus, $f \in \mathscr{K}_{\gamma,U}(\mathscr{D}_1(\mathbb{R}),\mathbb{R}) \bigcap (\mathscr{D}_1(\mathbb{R}))^{(\gamma,U)}$ *but* f *is not a usual distribution.*

(3) For the case of $\mathbb{R}^n = \mathbb{R}$, *we write that* $\mathscr{S} = \mathscr{S}(\mathbb{R})$. *Let* $U = \{\eta \in \mathscr{S}(\mathbb{R}) : \sup_{x\in\mathbb{R}} |\eta(x)| < 1\}$ *and* $\gamma(t) = \mathrm{e}^t$ *for* $t \in \mathbb{C}$. *Then define* $g : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ *by*

$$g(\xi) = \sqrt{-1}\int_{-1}^{1} (\mathrm{e}^{|\xi(x)|} - 1)\, dx,\ \ \xi \in \mathscr{S}(\mathbb{R}).$$

For $\xi \in \mathscr{S}(\mathbb{R}), \eta \in U$ *and* $|t| \le 1$,

$$\begin{aligned} g(\xi + t\eta) &= \sqrt{-1}\int_{-1}^{1} (\mathrm{e}^{|\xi(x)+t\eta(x)|} - 1)\, dx \\ &= \sqrt{-1}\int_{-1}^{1} (\mathrm{e}^{|\xi(x)|+\alpha(x)|\eta(x)|} - 1)\, dx \qquad (\alpha(x) \in [-|t|, |t|]) \\ &= \sqrt{-1}\int_{-1}^{1} (\mathrm{e}^{|\xi(x)|+\alpha(x)|\eta(x)|} - \mathrm{e}^{\alpha(x)|\eta(x)|} + \mathrm{e}^{\alpha(x)|\eta(x)|} - 1)\, dx \\ &= \sqrt{-1}\int_{-1}^{1} \mathrm{e}^{\alpha(x)|\eta(x)|}(\mathrm{e}^{|\xi(x)|} - 1)\, dx + \sqrt{-1}\int_{-1}^{1} (\mathrm{e}^{\alpha(x)|\eta(x)|} - 1)\, dx. \end{aligned}$$

If $\int_{-1}^{1}(\mathrm{e}^{|\xi(x)|} - 1)\, dx = 0$, *then* $0 \le \int_{-1}^{1} \mathrm{e}^{\alpha(x)|\eta(x)|}(\mathrm{e}^{|\xi(x)|} - 1)\, dx \le \mathrm{e}^{|t|}\int_{-1}^{1}(\mathrm{e}^{|\xi(x)|} - 1)\, dx = 0$ *and so* $\sqrt{-1}\int_{-1}^{1} \mathrm{e}^{\alpha(x)|\eta(x)|}(\mathrm{e}^{|\xi(x)|} - 1)\, dx = 0 = \sqrt{-1}\int_{-1}^{1}(\mathrm{e}^{|\xi(x)|} - 1)\, dx = g(\xi) = rg(\xi)$, *where* $r = 1$, $|r - 1| = 0 \le |\gamma(t)|$. *If* $\int_{-1}^{1}(\mathrm{e}^{|\xi(x)|} - 1)\, dx \ne 0$, *then* $\sqrt{-1}\int_{-1}^{1} \mathrm{e}^{\alpha(x)|\eta(x)|}(\mathrm{e}^{|\xi(x)|} - 1)\, dx =$

$\frac{\int_{-1}^{1} e^{\alpha(x)|\eta(x)|}(e^{|\xi(x)|}-1)\,dx}{\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx}\sqrt{-1}\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx$, *where*

$$|\frac{\int_{-1}^{1} e^{\alpha(x)|\eta(x)|}(e^{|\xi(x)|}-1)\,dx}{\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx}-1| = \frac{|\int_{-1}^{1}(e^{\alpha(x)|\eta(x)|}-1)(e^{|\xi(x)|}-1)\,dx|}{\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx}$$

$$= \frac{|\int_{-1}^{1} e^{\theta(x)\alpha(x)|\eta(x)|}\alpha(x)|\eta(x)|(e^{|\xi(x)|}-1)\,dx|}{\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx} \qquad (0 \le \theta(x) \le 1)$$

$$\le \frac{\int_{-1}^{1} e^{|t|}|t|(e^{|\xi(x)|}-1)\,dx}{\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx} \qquad (\because |\alpha(x)| \le |t|,\ |\eta(x)| \le 1)$$

$$= e^{|t|}|t| \le e|t| = |\gamma(t)|. \qquad (\because |t| \le 1)$$

Thus, $\sqrt{-1}\int_{-1}^{1} e^{\alpha(x)|\eta(x)|}(e^{|\xi(x)|}-1)\,dx = r\sqrt{-1}\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx = rg(\xi)$, *where* $|r-1| \le |\gamma(t)|$.

If $g(\eta) = \sqrt{-1}\int_{-1}^{1}(e^{|\eta(x)|}-1)\,dx = 0$, *then* $\eta(x) = 0$ *a.e. in* $[-1,1]$ *and* $g(\xi+t\eta) = \sqrt{-1}\int_{-1}^{1}(e^{|\xi(x)+t\eta(x)|}-1)\,dx = \sqrt{-1}\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx = g(\xi) = rg(\xi)+sg(\eta)$ *where* $r=1$ *and* $s=0$, $|r-1| = 0 \le |\gamma(t)|$, $|s| = 0 \le |\gamma(t)|$.

Suppose that $g(\eta) = \sqrt{-1}\int_{-1}^{1}(e^{|\eta(x)|}-1)\,dx \ne 0$. *Then*

$$\sqrt{-1}\int_{-1}^{1}(e^{\alpha(x)|\eta(x)|}-1)\,dx = \frac{\int_{-1}^{1}(e^{\alpha(x)|\eta(x)|}-1)\,dx}{\int_{-1}^{1}(e^{|\eta(x)|}-1)\,dx}g(\eta),$$

where

$$|\frac{\int_{-1}^{1}(e^{\alpha(x)|\eta(x)|}-1)\,dx}{\int_{-1}^{1}(e^{|\eta(x)|}-1)\,dx}| = \frac{|\int_{-1}^{1} e^{\delta(x)\alpha(x)|\eta(x)|}\alpha(x)|\eta(x)|\,dx|}{\int_{-1}^{1} e^{\theta(x)|\eta(x)|}|\eta(x)|\,dx} \qquad (0 \le \delta(x),\ \theta(x) \le 1)$$

$$\le \frac{|\int_{-1}^{1} e^{\delta(x)\alpha(x)|\eta(x)|}|\alpha(x)||\eta(x)|\,dx|}{\int_{-1}^{1}|\eta(x)|\,dx}$$

$$\le \frac{\int_{-1}^{1} e^{|t|}|t||\eta(x)|\,dx}{\int_{-1}^{1}|\eta(x)|\,dx}$$

$$= e^{|t|}|t| \le e|t| = |\gamma(t)|.$$

Then $g(\xi+t\eta) = rg(\xi)+sg(\eta)$, *where* $|r-1| \le |\gamma(t)|$, $|s| \le |\gamma(t)|$, *i.e.,* $g \in \mathscr{L}_{\gamma,U}(\mathscr{S}(\mathbb{R}),\mathbb{C})$. *Since* $\xi_k \to \xi$ *in* $\mathscr{S}$ *implies that* $\|\xi_k-\xi\|_0 = \sup_{x\in\mathbb{R}}|\xi_k(x)-\xi(x)| \to 0$ *and so* $g(\xi_k) = \sqrt{-1}\int_{-1}^{1}(e^{|\xi_k(x)|}-1)\,dx \to \sqrt{-1}\int_{-1}^{1}(e^{|\xi(x)|}-1)\,dx = g(\xi)$, *i.e.,* $g: \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ *is continuous. Thus,* $g \in (\mathscr{S}(\mathbb{R}))^{(\gamma,U)}$.

For every $C \ge 1$ and $\varepsilon > 0$, $\mathscr{K}_{C,\varepsilon}(\mathbb{R},\mathbb{R})$ includes a lot of nonlinear functions. Pick an $h \in \mathscr{K}_{C,\varepsilon}(\mathbb{R},\mathbb{R})$ and let $f(x+iy) = ih(|x+iy|)$, $\forall\, x+iy \in \mathbb{C}$. If $|u+iv| < \varepsilon$ and $|t| \le 1$, then $f[x+iy+t(u+iv)] = ih(|x+iy+t(u+iv)|) = ih(|x+iy|+\alpha|u+iv|) = ih(|x+iy|)+sih(|u+iv|) = f(x+iy)+sf(u+iv)$, where $\alpha \in [-|t|,|t|] \subset [-1,1]$ and $|s| \le C|\alpha| \le C|t|$. This shows that $f \in \mathscr{K}_{C,\varepsilon}(\mathbb{C},\mathbb{C})$ and, therefore, $\mathscr{K}_{C,\varepsilon}(\mathbb{C},\mathbb{C})$ also includes a lot of nonlinear functions.

Let $E^{[\gamma,U]} = \{f \in \mathscr{K}_{\gamma,U}(E,\mathbb{C}) : f \text{ is continuous}\}$. Then $E^{[\gamma,U]} \subset E^{(\gamma,U)}$.

Theorem 2.5. *If* $A \subset E'$ *is an equicontinuous family of distributions and* $\varepsilon > 0$, *then there is* *a* $U \in \mathcal{N}(E)$ *such that*

$$\{h \circ f : h \in \mathscr{L}_{\gamma,\varepsilon}(\mathbb{C},\mathbb{C}), f \in A\} \subset E^{(\gamma,U)},\ \forall \gamma \in C(0),$$
$$\{h \circ f : h \in \mathscr{K}_{\gamma,\varepsilon}(\mathbb{C},\mathbb{C}), f \in A\} \subset E^{[\gamma,U]},\ \forall \gamma \in C(0).$$

Proof. *Since* A *is equicontinuous, there is a* $U \in \mathcal{N}(E)$ *such that* $|f(\eta)| < \varepsilon,\ \forall f \in A, \eta \in U$. *Let* $\xi \in E,\ \eta \in U$ *and* $|t| \le 1$. *For* $h \in \mathscr{L}_{\gamma,\varepsilon}(\mathbb{C},\mathbb{C})$ *and* $f \in A$,

$$\begin{aligned}(h \circ f)(\xi + t\eta) &= h(f(\xi) + tf(\eta)) \\ &= r(h \circ f)(\xi) + s(h \circ f)(\eta), \quad |r-1| \le |\gamma(t)|,\ |s| \le |\gamma(t)|.\end{aligned}$$

Thus, $h \circ f \in \mathscr{L}_{\gamma,U}(E,\mathbb{C})$.

Suppose that $h \in \mathscr{L}_{\gamma,\varepsilon}(\mathbb{C},\mathbb{C})$ *and* $w_k \to w$ *in* $\mathbb{C}$. *Then*

$$\begin{aligned}\lim_k h(w_k) &= \lim_k h(w + w_k - w) = \lim_k h(w + \frac{2(w_k - w)}{\varepsilon}\frac{\varepsilon}{2}) \\ &= \lim_k [r_k h(w) + s_k h(\frac{\varepsilon}{2})],\end{aligned}$$

where $|r_k - 1| \le |\gamma(\frac{2(w_k-w)}{\varepsilon})| \to 0$ *and* $|s_k| \le |\gamma(\frac{2(w_k-w)}{\varepsilon})| \to 0$, *i.e.,* $r_k \to 1,\ s_k \to 0$. *Thus,* $h(w_k) \to h(w)$, h *is continuous. But* $A \subset E'$ *and so* $h \circ f : E \to \mathbb{C}$ *is continuous for* $h \in \mathscr{L}_{\gamma,\varepsilon}(\mathbb{C},\mathbb{C})$ *and* $f \in A$. □

Theorem 2.6. *Let* $\gamma_1 \in C(0)$ *for which* $\sup_{|t|\le 1} |\gamma_1(t)| = 1$, $|\gamma_1(\alpha)| \le |\gamma_1(\beta)|$ *whenever* $|\alpha| \le |\beta| \le 1$, *e.g.,* $\gamma_1(t) = \sqrt{|t|}$. *For every* $U \in \mathcal{N}(E)$, $f \in E^{[\gamma_1,U]}$ *and* $\varepsilon > 0$ *there is a* $V \in \mathcal{N}(E)$ *such that* $\gamma_1 \circ \gamma_1 \in C(0)$ *and*

$$h \circ f \in E^{(\gamma_1 \circ \gamma_1, V)},\ \forall h \in \mathscr{L}_{\gamma_1,\varepsilon}(\mathbb{C},\mathbb{C}),$$
$$h \circ f \in E^{[\gamma_1 \circ \gamma_1, V]},\ \forall h \in \mathscr{K}_{\gamma_1,\varepsilon}(\mathbb{C},\mathbb{C}).$$

Proof. *Pick a* $W \in \mathcal{N}(E)$ *for which* $|f(\eta)| < \varepsilon,\ \forall \eta \in W$. *Let* $h \in \mathscr{L}_{\gamma_1,\varepsilon}(\mathbb{C},\mathbb{C})$, $\xi \in E$, $\eta \in V = U \bigcap W$ *and* $|t| \le 1$. *Then*

$$\begin{aligned}(h \circ f)(\xi + t\eta) &= h(f(\xi + t\eta)) = h(f(\xi) + \alpha f(\eta)) \qquad (|\alpha| \le |\gamma_1(t)| \le 1) \\ &= rh(f(\xi)) + sh(f(\eta)) = r(h \circ f)(\xi) + s(h \circ f)(\eta),\end{aligned}$$

where $|r-1| \le |\gamma_1(\alpha)| \le |\gamma_1(\gamma_1(t))|$, $|s| \le |\gamma_1(\alpha)| \le |\gamma_1(\gamma_1(t))|$.

As in the proof of Th. 2.5, h *is continuous and so* $h \circ f \in E^{(\gamma_1 \circ \gamma_1, V)}$.

Similarly, $h \circ f \in E^{[\gamma_1 \circ \gamma_1, V]}$ *whenever* $h \in \mathscr{K}_{\gamma_1,\varepsilon}(\mathbb{C},\mathbb{C})$. □

Example 2.2. *(1) Let* $h(z) = |z|,\ \forall z \in \mathbb{C}$. *Then* $h \in \mathscr{K}_{\gamma_0,\mathbb{C}}(\mathbb{C},\mathbb{C})$ *where* $\gamma_0(t) = t$. *Let* $U \in \mathcal{N}(E)$ *and* $\gamma_1 \in C(0)$ *as in Th. 2.6. Then for every* $f \in E^{(\gamma_1,U)}$ *and* $a > 0$ *there is a* $V_a \in \mathcal{N}(E)$ *such that* $V_a \subset U$ *and* $|f(\eta)| < a,\ \forall \eta \in V_a$.

Let $a > 0$, $\xi \in E$, $\eta \in V_a$ *and* $|t| \le 1$. *Then*

$$|f(\xi + t\eta)| = |rf(\xi) + \alpha f(\eta)| = |r||f(\xi)| + s|f(\eta)|,$$

where $||r| - 1| \le |r-1| \le |\gamma_1(t)|$, $|s| \le |\alpha| \le |\gamma_1(t)|$. *Thus,* $\gamma_0 \circ \gamma_1 = \gamma_1$ *and* $|f(\cdot)| = h \circ f \in E^{(\gamma_1, V_a)},\ \forall a > 0$.

(2) Let $\gamma_1(t) = \sqrt{|t|}$, $\gamma_2(t) = \frac{\pi}{2}t,\ \forall t \in \mathbb{C}$. *Let* $U \in \mathcal{N}(E)$ *and* $f \in E^{[\gamma_1,U]}$. *There is a* $V \in \mathcal{N}(E)$ *such that* $V \subset U$ *and* $|f(\eta)| < 1,\ \forall \eta \in V$. *Define* $\sin|f(\cdot)| : E \to \mathbb{C}$ *by*

$\sin|f(\cdot)|(\xi) = \sin|f(\xi)|,\ \xi \in E$. *For* $\xi \in E$, $\eta \in V$ *and* $|t| \le 1$,

$$\begin{aligned}\sin|f(\cdot)|(\xi + t\eta) &= \sin|f(\xi + t\eta)| = \sin|f(\xi) + \alpha f(\eta)| \quad (|\alpha| \le |\gamma_1(t)| = \sqrt{|t|} \le 1)\\ &= \sin[|f(\xi)| + \beta|f(\eta)|] \quad (|\beta| \le |\alpha| \le 1)\\ &= \sin|f(\xi)| + s\sin|f(\eta)|\ (|s| \le \frac{\pi}{2}|\beta| \le \frac{\pi}{2}|\alpha| \le \frac{\pi}{2}\sqrt{|t|} = |(\gamma_2 \circ \gamma_1)(t)|)\\ &= \sin|f(\cdot)|(\xi) + s\sin|f(\cdot)|(\eta).\end{aligned}$$

Thus, $\gamma_2 \circ \gamma_1 \in C(0)$ *and* $\sin|f(\cdot)| \in E^{[\gamma_2\circ\gamma_1, V]}$.

(3) If $h(z) = \mathrm{e}^{|z|} - 1,\ \forall z \in \mathbb{C}$, $\gamma_1(t) = \sqrt{|t|}$ *and* $\gamma(t) = \mathrm{e}^{2t}$, *then* $h \in \mathscr{L}_{\gamma,1}(\mathbb{C},\mathbb{C})$ *and for every* $f \in E^{[\gamma_1, U]}$ *there is a* $V \in \mathcal{N}(E)$ *such that* $\mathrm{e}^{|f(\cdot)|} - 1 = h \circ f \in E^{(\gamma\circ\gamma_1, V)}$.

Even f is a nonzero usual distribution, each of $|f(\cdot)|$, $\sin|f(\cdot)|$ and $\mathrm{e}^{|f(\cdot)|} - 1$ can not be a usual distribution. However, Th. 2.6 shows that the family of demi-distributions is closed with respect to infinitely many of nonlinear transformations.

Henceforth, in the notations $E^{(\gamma,U)}$ and $E^{[\gamma,U]}$ we always confess that $\gamma \in C(0)$ and $U \in \mathcal{N}(E)$.

Definition 2.2. $f_k \xrightarrow{w*} f$ *in* $E^{(\gamma,U)}$ *means that* $f_k, f \in E^{(\gamma,U)}$ *for all* $k \in \mathbb{N}$ *and* $f_k(\xi) \to f(\xi)$ *at each* $\xi \in E$, *and* $f_k \to f$ *in* $E^{(\gamma,U)}$ *means that* $f_k, f \in E^{(\gamma,U)}$ *for all* $k \in \mathbb{N}$ *and for every bounded* $B \subset E$, $\lim_k f_k(\xi) = f(\xi)$ *uniformly for* $\xi \in B$.

Now Th. 2.2 gives the following

Theorem 2.7. $f_k \to f$ *in* $E^{(\gamma,U)}$ *if and only if* $f_k \xrightarrow{w*} f$ *in* $E^{(\gamma,U)}$.

Definition 2.3. *A sequence* $\{f_k\} \subset E^{(\gamma,U)}$ *(resp.,* $E^{[\gamma,U]}$*) is* $w*$ *Cauchy if* $\lim_k f_k(\xi)$ *exists at each* $\xi \in E$. $E^{(\gamma,U)}$ *(resp.,* $E^{[\gamma,U]}$*) is said to be sequentially complete if for every* $w*$ *Cauchy sequence* $\{f_k\}$ *in* $E^{(\gamma,U)}$ *(resp.,* $E^{[\gamma,U]}$*) there exists* $f \in E^{(\gamma,U)}$ *(resp.,* $E^{[\gamma,U]}$*) such that* $f_k \to f$, *i.e., for every bounded* $B \subset E$, $\lim_k f_k(\xi) = f(\xi)$ *uniformly for* $\xi \in B$.

Theorem 2.8. *Both* $\mathscr{D}_a^{(\gamma,U)}$ *and* $\mathscr{S}^{(\gamma,V)}$ *are sequentially complete for every* $\gamma \in C(0)$, $U \in \mathcal{N}(\mathscr{D}_a)$ *and* $V \in \mathcal{N}(\mathscr{S})$. *Moreover,* $\mathscr{D}^{[\gamma_0,W]}$ *is also sequentially complete for* $\gamma_0(t) = t$ *and* $W \in \mathcal{N}(\mathscr{D})$.

Proof. Let $E \in \{\mathscr{D}_a, \mathscr{S}\}$, $U \in \mathcal{N}(E)$ *and* $\gamma \in C(0)$. *If* $\{f_k\} \subset E^{(\gamma,U)}$ *and* $\lim_k f_k(\xi) = f(\xi)$ *exists at each* $\xi \in E$, *then* $\{f_k\}$ *is equicontinuous by Th. 3.1 of [1]. If* $\xi_\nu \to \xi$ *in* E, *then* $\lim_\nu f_k(\xi_\nu) = f_k(\xi)$ *uniformly for* $k \in \mathbb{N}$ *and* $\lim_\nu f(\xi_\nu) = \lim_\nu \lim_k f_k(\xi_\nu) = \lim_k \lim_\nu f_k(\xi_\nu) = \lim_k f_k(\xi) = f(\xi)$. *Thus,* $f : E \to \mathbb{C}$ *is continuous.*

Let $\xi \in E$, $\eta \in U$ *and* $|t| \le 1$. *Then* $f(\xi + t\eta) = \lim_k f_k(\xi + t\eta) = \lim_k[r_k f_k(\xi) + s_k f_k(\eta)]$ *where* $|r_k - 1| \le |\gamma(t)|$, $|s_k| \le |\gamma(t)|$. *By passing to a subsequence if necessary, we assume that* $r_k \to r$ *and* $s_k \to s$. *Then* $|r-1| \le |\gamma(t)|$, $|s| \le |\gamma(t)|$ *and* $f(\xi+t\eta) = \lim_k[r_k f_k(\xi) + s_k f_k(\eta)] = rf(\xi) + sf(\eta)$, $f \in \mathscr{L}_{\gamma,U}(E,\mathbb{C})$. *Thus,* $f \in E^{(\gamma,U)}$.

Now $f_k \xrightarrow{w*} f$ *in* $E^{(\gamma,U)}$ *and so* $f_k \to f$ *in* $E^{(\gamma,U)}$ *by Th. 2.7.*

The completeness of $\mathscr{D}^{[\gamma_0,W]}$ *follows from Th. 2.4.* □

For $E \in \{\mathscr{D}, \mathscr{S}\}$ and $G \subset \mathbb{R}^n$, let $E_G = \{\xi \in E : supp\,\xi \subset G\}$. Each $f \in E^{(\gamma,U)}$ yields $f|_{E_G} : E_G \to \mathbb{C}$ by $f|_{E_G}(\xi) = f(\xi),\ \forall\, \xi \in E_G$. Then $E_\emptyset = \{0\}$ and $f|_{E_\emptyset} = 0,\ \forall\, f \in E^{(\gamma,U)}$.

Definition 2.4. *$E = \mathscr{D}$. For $f \in E^{(\gamma,U)}$ let*

$$supp\, f = \mathbb{R}^n \backslash \Big[\bigcup \{G \subset \mathbb{R}^n : G \text{ is open, } f|_{E_G} = 0\}\Big].$$

Theorem 2.9. *$E = \mathscr{D}$. For every $f \in E^{(\gamma,U)}$ there is an open $G_0 \subset \mathbb{R}^n$ such that $f|_{E_{G_0}} = 0$ and $supp\, f = \mathbb{R}^n \backslash G_0$.*

Proof. *Let $G_0 = \mathbb{R}^n \backslash supp\, f$ and $\{G_\alpha : \alpha \in I\} = \{G \subset \mathbb{R}^n : G \text{ is open, } f|_{E_G} = 0\}$. Then $G_0 = \bigcup_{\alpha \in I} G_\alpha$ is open and $supp\, f = \mathbb{R}^n \backslash G_0$.*

Suppose that $f|_{E_{G_0}} \neq 0$. There is a $\xi \in E$ such that $supp\,\xi \subset G_0$ but $f(\xi) \neq 0$. Then $\mathbb{R}^n \backslash supp\,\xi \supset \mathbb{R}^n \backslash G_0 = supp\, f$ and $\mathbb{R}^n = (\mathbb{R}^n \backslash supp\,\xi) \bigcup (\bigcup_{\alpha \in I} G_\alpha)$. By the partition of unity, there is a sequence $\{\xi_k\} \subset \mathscr{D}$ such that $\sum_{k=1}^\infty \xi_k(x) = 1$ for all $x \in \mathbb{R}^n$ and each $supp\,\xi_k \subset (\mathbb{R}^n \backslash supp\,\xi)$ or some G_α, and each $x \in supp\,\xi$ has a neighborhood which intersects finitely many of $supp\,\xi_k$ only. But $supp\,\xi$ is compact and so there is an open $G \subset \mathbb{R}^n$ such that $supp\,\xi \subset G$ and G intersects finitely many of $supp\,\xi_k$ only. Hence, $\xi\xi_k = 0$ for all but finitely many of k's. Say that $\{k : \xi\xi_k \neq 0\} = \{1, 2, \cdots, m\}$. Then $\xi(x) = \sum_{k=1}^m \xi(x)\xi_k(x),\ \forall\, x \in \mathbb{R}^n$. For $k \leq m$, $\xi\xi_k \neq 0$ shows that $supp\,\xi_k \not\subset \mathbb{R}^n \backslash supp\,\xi$ and so $supp\xi_k \subset G_\alpha$ for some $\alpha \in I$. Thus, $f(\xi\xi_k) = 0,\ k = 1, 2, \cdots, m$.

Pick a $p \in \mathbb{N}$ such that $\frac{1}{p}\xi\xi_k \in U,\ k = 1, 2, \cdots, m$. Since $supp\,(\frac{1}{p}\xi\xi_k) \subset supp\,\xi_k \subset G_\alpha$ for some $\alpha \in I$, $f(\frac{1}{p}\xi\xi_k) = 0,\ k = 1, 2, \cdots, m$. Then

$$\begin{aligned}
f(\xi) &= f(\sum_{k=1}^{m} \xi\xi_k) = f(\sum_{k=1}^{m-1} \xi\xi_k + (p-1)\frac{1}{p}\xi\xi_m + \frac{1}{p}\xi\xi_m) \\
&= r_1 f(\sum_{k=1}^{m-1} \xi\xi_k + (p-1)\frac{1}{p}\xi\xi_m) + s_1 f(\frac{1}{p}\xi\xi_m) \quad (|r_1 - 1| \leq |\gamma(1)|,\ |s_1| \leq |\gamma(1)|) \\
&= r_1 f(\sum_{k=1}^{m-1} \xi\xi_k + (p-1)\frac{1}{p}\xi\xi_m) \\
&\quad \cdots \quad \cdots \\
&= r_1 r_2 \cdots r_p f(\sum_{k=1}^{m-1} \xi\xi_k) \\
&= r_1 r_2 \cdots r_p f(\sum_{k=1}^{m-2} \xi\xi_k + (p-1)\frac{1}{p}\xi\xi_{m-1} + \frac{1}{p}\xi\xi_{m-1}) \\
&= r_1 \cdots r_p r_{p+1} \cdots r_{2p} f(\sum_{k=1}^{m-2} \xi\xi_k) \\
&\quad \cdots \quad \cdots \\
&= (\prod_{\nu=1}^{mp-1} r_\nu) f(\frac{1}{p}\xi\xi_1) = 0.
\end{aligned}$$

This contradicts that $f(\xi) \neq 0$. Hence, $f|_{E_{G_0}} = 0$. □

§3 Differentiation

$E \in \{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$, $[E^{(\gamma,U)}] = span\,(E^{(\gamma,U)})$ in $\mathbb{C}^E$.

Definition 3.1. *Let $f = \sum_{k=1}^m \alpha_k f_k \in [E^{(\gamma,U)}]$ where each $\alpha_k \in \mathbb{C}$, $f_k \in E^{(\gamma,U)}$. Observing each $\xi \in E$ is a function defined on $\mathbb{R}^n$, for $j \in \{1, 2, \cdots, n\}$ define $\frac{\partial f}{\partial x_j} : E \to \mathbb{C}$ by*

$$\frac{\partial f}{\partial x_j}(\xi) = f(-\frac{\partial \xi}{\partial x_j}),\ \xi \in E.$$

Then $\frac{\partial}{\partial x_j}(\sum_{k=1}^m \alpha_k f_k)(\xi) = (\sum_{k=1}^m \alpha_k f_k)(-\frac{\partial \xi}{\partial x_j}) = \sum_{k=1}^m \alpha_k f_k(-\frac{\partial \xi}{\partial x_j}) = \sum_{k=1}^m \alpha_k \frac{\partial f_k}{\partial x_j}(\xi) = (\sum_{k=1}^m \alpha_k \frac{\partial f_k}{\partial x_j})(\xi)$ and so $\frac{\partial}{\partial x_j}(\sum_{k=1}^m \alpha_k f_k) = \sum_{k=1}^m \alpha_k \frac{\partial f_k}{\partial x_j}$ for $\sum_{k=1}^m \alpha_k f_k \in [E^{(\gamma,U)}]$.

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. As in the case of usual distributions, for $f \in [E^{(\gamma,U)}]$ and every multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $(D^\alpha f)(\xi) = f((-1)^{|\alpha|} D^\alpha \xi)$, $\forall\, \xi \in E$. Evidently, we have that $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, $\frac{\partial^5 f}{\partial x_1 \partial x_2^2 \partial x_3^2} = \frac{\partial^5 f}{\partial x_3^2 \partial x_1 \partial x_2^2}$, etc.

Lemma 3.1. *For every multi-index α, $D^\alpha : E \to E$ is a continuous linear operator.*

Proof. For $E = \mathscr{D}_a$ or $\mathscr{S}$, the conclusion is obvious.

Let $\xi_k \to 0$ in $\mathscr{D}$. Then $\{\xi_k\} \subset \mathscr{D}_m$ for some $m \in \mathbb{N}$ and $\xi_k \to 0$ in $\mathscr{D}_m$ since $\mathscr{D}_m$ is a subspace of $\mathscr{D}$ ([3, p. 219]). Then $\|D^\alpha \xi_k\|_p \le \|\xi_k\|_{|\alpha|+p}$ for all $p \in \mathbb{N}$ and so $D^\alpha \xi_k \to 0$ in $\mathscr{D}_m$, i.e., $D^\alpha \xi_k \to 0$ in $\mathscr{D}$. Thus, $D^\alpha : \mathscr{D} \to \mathscr{D}$ is sequentially continuous. Then $D^\alpha : \mathscr{D} \to \mathscr{D}$ is continuous since $\mathscr{D}$ is bornological and $C-sequential$. See also Th. 2.1. □

Theorem 3.1. *Let α be a multi-index. For every $U \in \mathcal{N}(E)$ there is a $V \in \mathcal{N}(E)$ such that*

$$\{D^\alpha f : f \in E^{(\gamma,U)}\} \subset E^{(\gamma,V)},\ \forall\, \gamma \in C(0),$$
$$\{D^\alpha f : f \in [E^{(\gamma,U)}]\} \subset [E^{(\gamma,V)}],\ \forall\, \gamma \in C(0).$$

Moreover, if $f_k, f \in E^{(\gamma,U)}$ and $f_k \xrightarrow{w*} f$, i.e., $f_k(\xi) \to f(\xi)$ at each $\xi \in E$, then for every bounded $B \subset E$, $\lim_k (D^\alpha f_k)(\xi) = (D^\alpha f)(\xi)$ uniformly for $\xi \in B$.

Proof. Let $U \in \mathcal{N}(E)$. By Lemma 3.1, there is a $V \in \mathcal{N}(E)$ for which $(-1)^{|\alpha|} D^\alpha \eta \in U$, $\forall\, \eta \in V$.

Let $f \in E^{(\gamma,U)}$, $\xi \in E$, $\eta \in V$ and $|t| \le 1$. Then

$$\begin{aligned}(D^\alpha f)(\xi + t\eta) &= f((-1)^{|\alpha|} D^\alpha \xi + t(-1)^{|\alpha|} D^\alpha \eta) \\ &= r f((-1)^{|\alpha|} D^\alpha \xi) + s f((-1)^{|\alpha|} D^\alpha \eta) \\ &= r(D^\alpha f)(\xi) + s(D^\alpha f)(\eta),\end{aligned}$$

where $|r - 1| \le |\gamma(t)|$, $|s| \le |\gamma(t)|$. Thus, $D^\alpha f \in \mathscr{L}_{\gamma,V}(E, \mathbb{C})$.

Since both $(-1)^{|\alpha|} D^\alpha : E \to E$ and $f : E \to \mathbb{C}$ are continuous, $D^\alpha f = f \circ (-1)^{|\alpha|} D^\alpha : E \to \mathbb{C}$ is also continuous and so $D^\alpha f \in E^{(\gamma,V)}$.

Suppose that $f_k, f \in E^{(\gamma,U)}$, $f_k(\xi) \to f(\xi)$, $\forall\, \xi \in E$, and $B \subset E$ is bounded. Then $(-1)^{|\alpha|} D^\alpha(B) = \{(-1)^{|\alpha|} D^\alpha \xi : \xi \in B\}$ is bounded and, by Th. 2.7, $\lim_k (D^\alpha f_k)(\xi) = \lim_k f_k((-1)^{|\alpha|} D^\alpha \xi) = f((-1)^{|\alpha|} D^\alpha \xi) = (D^\alpha f)(\xi)$ uniformly for $\xi \in B$. □

Example 3.1. *(1) Let* $f \in L^1_{loc}(\mathbb{R}^n)$, $\gamma \in C(0)$ *and*

$$[f](\xi) = \int_{\mathbb{R}^n} |f(x)\xi(x)|\,dx,\ \xi \in \mathscr{D}.$$

Then $[f] \in \mathscr{D}^{[\gamma,\mathscr{D}]}$ *(see Exam. 2.1(1)), and*

$$(D^\alpha[f])(\xi) = \int_{\mathbb{R}^n} |f(x)(D^\alpha \xi)(x)|\,dx,\ \forall\, \xi \in \mathscr{D}.$$

(2) γ *and* U *as in Exam. 2.1(2), and*

$$f(\xi) = \int_{-\infty}^{\infty} |\sin \xi(x)|\,dx,\ \forall\, \xi \in \mathscr{D}_1(\mathbb{R}).$$

Then $f \in (\mathscr{D}_1(\mathbb{R}))^{[\gamma,U]}$ *and*

$$\begin{aligned}(D^\alpha f)(\xi) &= \int_{-\infty}^{\infty} |\sin[(-1)^{|\alpha|}(D^\alpha \xi)(x)]|\,dx \\ &= \int_{-\infty}^{\infty} |\sin(D^\alpha \xi)(x)|\,dx,\ \forall\, \xi \in \mathscr{D}_1(\mathbb{R}).\end{aligned}$$

(3) Let $U \in \mathcal{N}(E)$ *and* $\gamma(t) = \sqrt{|t|}$, $\forall\, t \in \mathbb{C}$. *For every* $f \in E^{[\gamma,U]}$, *both* $\sin|f(\cdot)|$ *and* $\mathrm{e}^{|f(\cdot)|} - 1$ *are demi-distributions, see Exam. 2.2. Then for every multi-index* α,

$$D^\alpha \sin|f(\cdot)| = \sin|D^\alpha f(\cdot)|,\quad D^\alpha(\mathrm{e}^{|f(\cdot)|} - 1) = \mathrm{e}^{|D^\alpha f(\cdot)|} - 1.$$

In general, Th. 2.6 shows that $h \circ f$ *is a demi-distribution for every* $h \in \mathscr{L}_{\gamma,\varepsilon}(\mathbb{C},\mathbb{C})$. *Then*

$$D^\alpha(h \circ f) = h \circ D^\alpha f.$$

In fact, $(D^\alpha(h \circ f))(\xi) = (h \circ f)((-1)^{|\alpha|} D^\alpha \xi) = h[f((-1)^{|\alpha|} D^\alpha \xi)] = h[(D^\alpha f)(\xi)] = (h \circ D^\alpha f)(\xi)$, $\forall\, \xi \in E$.

(4) $E \in \{\mathscr{D}_a, \mathscr{S}\}$ *and* $\{\|\cdot\|_p\}_{p=0}^\infty$ *is the usual norm sequence on* E. *For* $p \in \mathbb{N}$ *and* $\varepsilon > 0$, *let* $U_{p,\varepsilon} = \{\eta \in E : \|\eta\|_p < \varepsilon\}$. *Then for every multi-index* α *and* $\varepsilon > 0$,

$$D^\alpha f \in E^{(\gamma, U_{p+|\alpha|,\varepsilon})},\ \forall\, f \in E^{(\gamma,U_{p,\varepsilon})},\ \gamma \in C(0),\ p \in \mathbb{N},$$
$$D^\alpha f \in [E^{(\gamma, U_{p+|\alpha|,\varepsilon})}],\ \forall\, f \in [E^{(\gamma,U_{p,\varepsilon})}],\ \gamma \in C(0),\ p \in \mathbb{N}.$$

In fact, $\|\eta\|_{p+|\alpha|} < \varepsilon$ *implies* $\|(-1)^{|\alpha|} D^\alpha \eta\|_p \le \|\eta\|_{p+|\alpha|} < \varepsilon$.

Definition 3.2. $\zeta : \mathbb{R}^n \to \mathbb{C}$ *is called a multiplier in* E *if for every* $\xi \in E$ *the pointwise product* $\zeta\xi \in E$ *and* $\zeta\xi_k \to 0$ *in* E *whenever* $\xi_k \to 0$. *For a multiplier* ζ *in* E *and* $f \in [E^{(\gamma,U)}]$, *define* $\zeta f : E \to \mathbb{C}$ *by* $(\zeta f)(\xi) = f(\zeta\xi)$, $\forall\, \xi \in E$.

Theorem 3.2. *If* ζ *is a multiplier in* E, *then for every* $U \in \mathcal{N}(E)$ *there is a* $V \in \mathcal{N}(E)$ *such that*

$$\{\zeta f : f \in E^{(\gamma,U)}\} \subset E^{(\gamma,V)},\ \forall\, \gamma \in C(0),$$
$$\{\zeta f : f \in [E^{(\gamma,U)}]\} \subset [E^{(\gamma,V)}],\ \forall\, \gamma \in C(0).$$

Proof. *The correspondence* $\xi \mapsto \zeta\xi$ *is a continuous linear operator from* E *into* E *and so there is a* $V \in \mathcal{N}(E)$ *such that* $\zeta\eta \in U$ *for all* $\eta \in V$.

Let $f \in E^{(\gamma,U)}$ *and* $\xi \in E$, $\eta \in V$, $|t| \le 1$. *Then*

$$\begin{aligned}(\zeta f)(\xi + t\eta) &= f(\zeta\xi + t\zeta\eta) = r f(\zeta\xi) + s f(\zeta\eta) \\ &= r(\zeta f)(\xi) + s(\zeta f)(\eta),\quad |r - 1| \le |\gamma(t)|,\ |s| \le |\gamma(t)|.\end{aligned}$$

Thus, $\zeta f \in \mathscr{L}_{\gamma,V}(E,\mathbb{C})$. The continuity of $\zeta f : E \to \mathbb{C}$ follows from the continuity of f and the continuity of the correspondence $\xi \mapsto \zeta\xi$. Hence, $\zeta f \in E^{(\gamma,V)}$. □

Lemma 3.2. *Let $E \in \{\mathscr{D}_a(\mathbb{R}), \mathscr{D}(\mathbb{R}), \mathscr{S}(\mathbb{R})\}$ be a space of test functions defined on $\mathbb{R}$, i.e., $n = 1$. Pick a $\zeta \in E$ for which $\int_{-\infty}^{\infty} \zeta(x)\,dx = 1$ and define $A : E \to E$ by $A(\xi) = \xi - (\int_{-\infty}^{\infty} \xi(x)\,dx)\zeta$, $\xi \in E$. Then A is a continuous linear operator, $\int_{-\infty}^{\infty} A(\xi)(x)\,dx = 0$ for all $\xi \in E$ and $A(\xi^{(k)}) = \xi^{(k)}$, $\forall \xi \in E$, $k \in \mathbb{N}$.*

Proof. For $\xi, \eta \in E$ and $t \in \mathbb{C}$, $A(\xi + t\eta) = \xi + t\eta - (\int_{-\infty}^{\infty} (\xi + t\eta)(x)\,dx)\zeta = \xi - (\int_{-\infty}^{\infty} \xi(x)\,dx)\zeta + t\eta - t(\int_{-\infty}^{\infty} \eta(x)\,dx)\zeta = A(\xi) + tA(\eta)$.

Since $1 \in E'$, if $\xi_k \to \xi$ in E, then $\int_{-\infty}^{\infty} \xi_k(x)\,dx \to \int_{-\infty}^{\infty} \xi(x)\,dx$, $A(\xi_k) = \xi_k - (\int_{-\infty}^{\infty} \xi_k(x)\,dx)\zeta \to \xi - (\int_{-\infty}^{\infty} \xi(x)\,dx)\zeta = A(\xi)$ and so A is sequentially continuous. Since E is bornological, A is continuous.

For $\xi \in E$ and $k \geq 1$, $A(\xi^{(k)}) = \xi^{(k)} - (\int_{-\infty}^{\infty} \xi^{(k)}(x)\,dx)\zeta = \xi^{(k)}$. □

For usual distributions, the equation $y' = 0$ has solutions $y = const$ only. However, for demi-distributions in $E^{(\gamma,U)}$, the equation $y' = 0$ has extremely many solutions which are not constants, and the equation $y' = f$ also has extremely many solutions which are demi-distributions.

Lemma 3.3. *Let E be a space of test functions defined on $\mathbb{R}$. Let $\gamma \in C(0)$ and $U \in \mathcal{N}(E)$. For $y \in E^{(\gamma,U)}$, $y' = 0$ if and only if $y(\xi) = 0$ whenever $\int_{-\infty}^{\infty} \xi(x)\,dx = 0$.*

Proof. Suppose that $y' = 0$ and $\xi \in E$ for which $\int_{-\infty}^{\infty} \xi(x)\,dx = 0$. Letting $\eta(x) = \int_{-\infty}^{x} \xi(t)\,dt$, $\eta \in E$ and $\xi = \eta'$. Then $y(\xi) = y(-(-\eta)') = y'(-\eta) = 0$.

The converse is obvious. □

In general, we have

Theorem 3.3. *Let $E \in \{\mathscr{D}_a(\mathbb{R}), \mathscr{D}(\mathbb{R}), \mathscr{S}(\mathbb{R})\}$, a space of test functions defined on $\mathbb{R}$. Let $U \in \mathcal{N}(E)$, $\gamma \in C(0)$. Then for every $\xi_0 \in E$ and $f_0 \in E^{(\gamma,U)}$ there is a $V \in \mathcal{N}(E)$ such that the equation $y' = 0$ has a solution $f \in E^{(\gamma,V)}$ which is given by $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(x)\,dx)\xi_0]$ for $\xi \in E$. If $f_0 = 1$, then $f_0 \in E'$ and $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(x)\,dx)\xi_0] = (\int_{-\infty}^{\infty} \xi(x)\,dx) f_0(\xi_0) = (\int_{-\infty}^{\infty} \xi(x)\,dx)(\int_{-\infty}^{\infty} \xi_0(\tau)\,d\tau) = \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} \xi_0(\tau)\,d\tau)\xi(x)\,dx$, $\forall \xi \in E$, i.e., $f = \int_{-\infty}^{\infty} \xi_0(\tau)\,d\tau \in E'$, a constant which is a usual solution of the equation $y' = 0$.*

The solutions of the equation $y' = 0$ have an interesting property as follows.

Theorem 3.4. *Let $E \in \{\mathscr{D}_a(\mathbb{R}), \mathscr{D}(\mathbb{R})\}$, a space of test functions defined on $\mathbb{R}$. Let $U \in \mathcal{N}(E)$, $\gamma \in C(0)$ and $y \in E^{[\gamma,U]}$. If $y' = 0$, then for every $\zeta \in E$ with $\int_{-\infty}^{\infty} \zeta(x)\,dx = 1$,*

$$y(\xi) = y[(\int_{-\infty}^{\infty} \xi(x)\,dx)\zeta], \ \forall \xi \in E,$$

i.e., if $\zeta_1, \zeta_2 \in E$ such that $\int_{-\infty}^{\infty} \zeta_1(x)\,dx = \int_{-\infty}^{\infty} \zeta_2(x)\,dx = 1$, then $y[(\int_{-\infty}^{\infty} \xi(x)\,dx)\zeta_1] = y[(\int_{-\infty}^{\infty} \xi(x)\,dx)\zeta_2] = y(\xi)$ for all $\xi \in E$ and, in particular,

$$y(\xi) = y(\eta) \text{ whenever } \int_{-\infty}^{\infty} \xi(x)\,dx = \int_{-\infty}^{\infty} \eta(x)\,dx = 1,$$

$$y(\frac{\xi}{\int_{-\infty}^{\infty}\xi(x)\,dx}) = y(\frac{\eta}{\int_{-\infty}^{\infty}\eta(x)\,dx}) \text{ whenever } \int_{-\infty}^{\infty}\xi(x)\,dx \neq 0 \text{ and } \int_{-\infty}^{\infty}\eta(x)\,dx \neq 0.$$

Proof. If $\zeta \in E$ such that $\zeta \neq \xi'$, $\forall\,\xi \in E$, i.e., $\int_{-\infty}^{\infty}\zeta(x) \neq 0$, then $\frac{1}{\int_{-\infty}^{\infty}\zeta(x)\,dx}\zeta \in E$ and $\int_{-\infty}^{\infty}\frac{\zeta(x)}{\int_{-\infty}^{\infty}\zeta(t)\,dt}\,dx = 1$. Pick a $\zeta \in E$ for which $\int_{-\infty}^{\infty}\zeta(x)\,dx = 1$ and let $A(\xi) = \xi - (\int_{-\infty}^{\infty}\xi(x)\,dx)\zeta$ for $\xi \in E$. By Lemma 3.2, $A : E \to E$ is a continuous linear operator and $\int_{-\infty}^{\infty}A(\xi)(x)\,dx = 0$, $\forall\,\xi \in E$. Moreover,

$$A(\xi)(x) = \Big(\int_{-\infty}^{x}A(\xi)(t)\,dt\Big)', \ \forall\,\xi \in E,\ x \in \mathbb{R}.$$

Let $\xi \in E$ and pick a $p \in \mathbb{N}$ for which $\frac{1}{p}A(\xi) \in U$. Then

$$\begin{aligned} y(\xi) &= y[(\int_{-\infty}^{\infty}\xi(x)\,dx)\zeta + A(\xi)] \\ &= y[(\int_{-\infty}^{\infty}\xi(x)\,dx)\zeta + (p-1)\frac{1}{p}A(\xi) + \frac{1}{p}A(\xi)] \\ &= y[(\int_{-\infty}^{\infty}\xi(x)\,dx)\zeta + (p-1)\frac{1}{p}A(\xi)] + s_1 y(\frac{1}{p}A(\xi)) \\ &\cdots\quad\cdots \\ &= y[(\int_{-\infty}^{\infty}\xi(x)\,dx)\zeta] + sy(\frac{1}{p}A(\xi)). \end{aligned}$$

But $\frac{1}{p}A(\xi)(x) = (\frac{1}{p}\int_{-\infty}^{x}A(\xi)(t)\,dt)'$ for all $x \in \mathbb{R}$ and so $y(\frac{1}{p}A(\xi)) = y[-(-\frac{1}{p}\int_{-\infty}^{x}A(\xi)(t)\,dt)'] = y'(-\frac{1}{p}\int_{-\infty}^{x}A(\xi)(t)\,dt) = 0$ since $y' = 0$. Therefore,

$$y(\xi) = y[(\int_{-\infty}^{\infty}\xi(x)\,dx)\zeta],\ \forall\,\xi \in E.$$

If $\int_{-\infty}^{\infty}\xi(x)\,dx = \int_{-\infty}^{\infty}\eta(x)\,dx = 1$, then $y(\xi) = y[(\int_{-\infty}^{\infty}\xi(x)\,dx)\zeta] = y(\zeta) = y[(\int_{-\infty}^{\infty}\eta(x)\,dx)\zeta] = y(\eta)$. □

For $E \in \{\mathscr{D}_a(\mathbb{R}), \mathscr{D}(\mathbb{R})\}$ let

$$E_1 = \{\xi \in E : \int_{-\infty}^{\infty}\xi(x)\,dx = 1\}.$$

If $y \in E'$ is a usual distribution such that $y' = 0$, then y must be a constant $C \in \mathbb{R}$, i.e., $y(\xi) = \int_{-\infty}^{\infty}C\xi(x)\,dx$ for all $\xi \in E$. Hence, $y(\xi) = C$, $\forall\,\xi \in E_1$. Th. 3.3 shows that the same fact holds for the case of $E^{[\gamma,U]}$.

Corollary 3.1. *$E \in \{\mathscr{D}_a(\mathbb{R}), \mathscr{D}(\mathbb{R})\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. If $y \in E^{[\gamma,U]}$ such that $y' = 0$, then $y(\cdot)$ is an invariant on E_1, i.e., there is a $C \in \mathbb{R}$ such that*

$$y(\xi) = C,\ \forall\,\xi \in E_1.$$

Although Th. 3.3 gives a lot of various solutions of the equation $y' = 0$ for the case of $E^{(\gamma,U)}$, Th. 3.3 does not give all solutions. However, for the case of $E^{[\gamma,U]}$ we can give all solutions of $y' = 0$.

Theorem 3.5. *Let $E \in \{\mathscr{D}_a(\mathbb{R}), \mathscr{D}(\mathbb{R})\}$ be a space of test functions defined on $\mathbb{R}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. Then for every $\xi_0 \in E$ and $f_0 \in E^{[\gamma,U]}$ there is a $V \in \mathcal{N}(E)$ such that the* *equation $y'=0$ has a solution $f\in E^{[\gamma,V]}$ which is given by $f(\xi)=f_0[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0]$ for $\xi\in E$. Conversely, if $f\in E^{[\gamma,U]}$ is a solution of the equation $y'=0$, then there exist $\xi_0\in E$ and $f_0\in E^{[\gamma,U]}$ such that $f(\xi)=f_0[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0]$, $\forall\,\xi\in E$.*

Proof. Let $\xi_0\in E$, $f_0\in E^{[\gamma,U]}$. There is a $V\in\mathcal{N}(E)$ such that $(\int_{-\infty}^{\infty}\eta(x)\,dx)\xi_0\in U$ for all $\eta\in V$. Let $f(\xi)=f_0[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0]$ for $\xi\in E$. If $\xi\in E$, $\eta\in V$ and $|t|\le 1$, then $f(\xi+t\eta)=f_0[(\int_{-\infty}^{\infty}(\xi+t\eta)(x)\,dx)\xi_0]=f_0[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0+t(\int_{-\infty}^{\infty}\eta(x)\,dx)\xi_0]=f_0[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0]+sf_0[(\int_{-\infty}^{\infty}\eta(x)\,dx)\xi_0]=f(\xi)+sf(\eta)$, where $|s|\le|\gamma(t)|$. Thus, $f\in E^{[\gamma,V]}$ and $f'=0$:

$$f'(\xi)=f(-\xi')=f_0[(\int_{-\infty}^{\infty}-\xi'(x)\,dx)\xi_0]=f_0(0)=0,\ \forall\,\xi\in E.$$

Conversely, suppose that $f\in E^{[\gamma,U]}$ and $f'=0$. Pick a $\xi_0\in E$ with $\int_{-\infty}^{\infty}\xi_0(x)\,dx=1$, and let $f_0=f$. By Th. 3.4,

$$f(\xi)=f[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0]=f_0[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0],\ \forall\,\xi\in E.\ \Box$$

We now consider the equation $y'=f$ where $f\in E^{(\gamma,U)}$.

Theorem 3.6. *Let $E\in\{\mathscr{D}_a(\mathbb{R}),\mathscr{D}(\mathbb{R})\}$ be a space of test functions defined on $\mathbb{R}$, $E_1=\{\xi\in E:\int_{-\infty}^{\infty}\xi(x)\,dx=1\}$, $U\in\mathcal{N}(E)$ and $\gamma\in C(0)$. Let $f\in E^{(\gamma,U)}$ be arbitrary. Then every $\zeta\in E_1$ gives $U_\zeta\in\mathcal{N}(E)$ and $y_\zeta\in E^{(\gamma,U_\zeta)}$ such that $y'_\zeta=f$ and*

$$y_\zeta(\xi)=f(-\int_{-\infty}^{x}[\xi(\tau)-(\int_{-\infty}^{\infty}\xi(s)\,ds)\zeta(\tau)]\,d\tau),\ \forall\,\xi\in E.$$

Proof. Only need to consider $E=\mathscr{D}(\mathbb{R})$. Pick a $\zeta\in E_1$ and define $A_\zeta:\mathscr{D}(\mathbb{R})\to\mathscr{D}(\mathbb{R})$ by $A_\zeta(\xi)=\xi-(\int_{-\infty}^{\infty}\xi(\tau)\,d\tau)\zeta$, $\forall\,\xi\in\mathscr{D}(\mathbb{R})$. By Lemma 3.2, A_ζ is a continuous linear operator and $\int_{-\infty}^{\infty}A_\zeta(\xi)(\tau)\,d\tau=0$, $\forall\,\xi\in\mathscr{D}(\mathbb{R})$. For every $\xi\in\mathscr{D}(\mathbb{R})$, $A_\zeta(\xi)(x)=\frac{d}{dx}[\int_{-\infty}^{x}A_\zeta(\xi)(\tau)\,d\tau]$, $\forall\,x\in\mathbb{R}$. Since $A_\zeta(\xi)\in\mathscr{D}(\mathbb{R})$, $\forall\,\xi\in\mathscr{D}(\mathbb{R})$, there ia an $a_\xi>0$ such that $A_\zeta(\xi)(x)=0$, $\forall\,|x|>a_\xi$ and so $\int_{-\infty}^{x}A_\zeta(\xi)(\tau)\,d\tau=0$ for $x<-a_\xi$ and $\int_{-\infty}^{x}A_\zeta(\xi)(\tau)\,d\tau=\int_{-\infty}^{\infty}A_\zeta(\xi)(\tau)\,d\tau=0$ for $x>a_\xi$. Thus, $\int_{-\infty}^{x}A_\zeta(\xi)(\tau)\,d\tau$ gives a test function in $\mathscr{D}(\mathbb{R})$, $\forall\,\xi\in\mathscr{D}(\mathbb{R})$.

Let $T(\xi)(x)=\int_{-\infty}^{x}A_\zeta(\xi)(\tau)\,d\tau$, $\forall\,\xi\in\mathscr{D}(\mathbb{R})$, $x\in\mathbb{R}$. Since A_ζ is linear, $T:\mathscr{D}(\mathbb{R})\to\mathscr{D}(\mathbb{R})$ is a linear operator. Let $\xi_k\to 0$ in $\mathscr{D}(\mathbb{R})$. By Lemma 3.2, $A_\zeta(\xi_k)\to 0$ in $\mathscr{D}(\mathbb{R})$ and so $A_\zeta(\xi_k)\to 0$ in $\mathscr{D}_{m_0}(\mathbb{R})$ for some $m_0\in\mathbb{N}$ ([3, p. 219]). Then $\{T(\xi_k)\}\subset\mathscr{D}_{m_0}(\mathbb{R})$. In fact, $\{A_\zeta(\xi_k)\}\subset\mathscr{D}_{m_0}(\mathbb{R})$, i.e., $A_\zeta(\xi_k)(x)=0$, $\forall\,|x|>m_0$, $k\in\mathbb{N}$, hence $T(\xi_k)(x)=\int_{-\infty}^{x}A_\zeta(\xi_k)(\tau)\,d\tau=0$ for $x<-m_0$, $k\in\mathbb{N}$ and $T(\xi_k)(x)=\int_{-\infty}^{x}A_\zeta(\xi_k)(\tau)\,d\tau=\int_{-\infty}^{\infty}A_\zeta(\xi_k)(\tau)\,d\tau=0$ whenever $x>m_0$ and $k\in\mathbb{N}$. Since $A_\zeta(\xi_k)\to 0$ in $\mathscr{D}_{m_0}(\mathbb{R})$,

$$\|T(\xi_k)\|_0=\sup_{|x|\le m_0}|T(\xi_k)(x)|=\sup_{|x|\le m_0}|\int_{-m_0}^{x}A_\zeta(\xi_k)(\tau)\,d\tau|$$
$$\le\int_{-m_0}^{m_0}|A_\zeta(\xi_k)(\tau)|\,d\tau\le 2m_0\|A_\zeta(\xi_k)\|_0\to 0,\quad i.e.,\ \|T(\xi_k)\|_0\to 0.$$

Moreover, $\frac{dT(\xi)}{dx}(x)=A_\zeta(\xi)(x)$, $\forall\,\xi\in\mathscr{D}(\mathbb{R})$, $x\in\mathbb{R}$, i.e., $\frac{dT(\xi)}{dx}=A_\zeta(\xi)$, $\forall\,\xi\in\mathscr{D}(\mathbb{R})$. Since $\{T(\xi_k)\}\subset\mathscr{D}_{m_0}(\mathbb{R})$ and $A_\zeta(\xi_k)\to 0$ in $\mathscr{D}_{m_0}(\mathbb{R})$, $\|T(\xi_k)\|_p\le\max\{\|T(\xi_k)\|_0,\ \|A_\zeta(\xi_k)\|_{p-1}\}\to 0$ for each $p\in\mathbb{N}$. Thus, $T(\xi_k)\to 0$ in $\mathscr{D}_{m_0}(\mathbb{R})$, i.e., $T(\xi_k)\to 0$ in $\mathscr{D}(\mathbb{R})$. Therefore, $T:\mathscr{D}(\mathbb{R})\to\mathscr{D}(\mathbb{R})$ is sequentially continuous. Since T is linear and $\mathscr{D}(\mathbb{R})$ is bornological, i.e.,

$\mathscr{D}(\mathbb{R})$ is $C-$ sequential, T is continuous and so there is a balanced $U_\zeta \in \mathcal{N}(\mathscr{D}(\mathbb{R}))$ such that $T(U_\zeta) \subset U$.

Define $y_\zeta : \mathscr{D}(\mathbb{R}) \to \mathbb{C}$ by

$$y_\zeta(\xi) = f(-T(\xi)) = f(-\int_{-\infty}^{x}[\xi(\tau) - (\int_{-\infty}^{\infty}\xi(s)\,ds)\zeta(\tau)]d\tau),\ \forall\,\xi \in \mathscr{D}(\mathbb{R}).$$

Since both f and T are continuous, y_ζ is continuous.

Let $\xi \in \mathscr{D}(\mathbb{R})$, $\eta \in U_\zeta$ and $|t| \le 1$. Since U_ζ is balanced and $f \in \mathscr{D}(\mathbb{R})^{(\gamma,U)}$, $T(-\eta) \in T(U_\zeta) \subset U$ and

$$\begin{aligned} y_\zeta(\xi + t\eta) &= f(-T(\xi + t\eta)) = f(-T(\xi) + tT(-\eta)) = rf(-T(\xi)) + sf(T(-\eta)) \\ &= rf(-T(\xi)) + sf(-T(\eta)) = ry_\zeta(\xi) + sy_\zeta(\eta),\ |r-1| \le |\gamma(t)|,\ |s| \le |\gamma(t)|. \end{aligned}$$

Thus, $y_\zeta \in \mathscr{D}(\mathbb{R})^{(\gamma,U_\zeta)}$.

For every $\xi \in \mathscr{D}(\mathbb{R})$, $T(\xi')(x) = \int_{-\infty}^{x} A_\zeta(\xi')(\tau)\,d\tau = \int_{-\infty}^{x}[\xi'(\tau) - (\int_{-\infty}^{\infty}\xi'(s)\,ds)\zeta(\tau)]\,d\tau = \int_{-\infty}^{x}\xi'(\tau)\,d\tau = \xi(x)$, $\forall\,x \in \mathbb{R}$, i.e., $T(\xi') = \xi$, $\forall\,\xi \in \mathscr{D}(\mathbb{R})$. Then

$$y_\zeta'(\xi) = y_\zeta(-\xi') = f(-T(-\xi')) = f(T(\xi')) = f(\xi),\ \forall\,\xi \in \mathscr{D}(\mathbb{R}),$$

i.e., $y_\zeta' = f$. □

Theorem 3.7. *Let $E \in \{\mathscr{D}_a(\mathbb{R}), \mathscr{D}(\mathbb{R})\}$ and $f \in E^{(\gamma,U)}$. For every $\xi_0 \in E$, $f_0 \in E^{(\gamma,U)}$ and $\zeta \in E$ with $\int_{-\infty}^{\infty}\zeta(x)\,dx \ne 0$ let*

$$g(\xi) = f_0[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0] + f(-\int_{-\infty}^{x}[\xi(\tau) - \frac{\int_{-\infty}^{\infty}\xi(s)\,ds}{\int_{-\infty}^{\infty}\zeta(s)\,ds}\zeta(\tau)]\,d\tau),\ \forall\,\xi \in E,$$

then $g \in [E^{(\gamma,W)}]$ for some $W \in \mathcal{N}(E)$ and $g' = f$.

Proof. *Let $\xi_0 \in E$, $f_0 \in E^{(\gamma,U)}$ and $\zeta \in E$ with $\int_{-\infty}^{\infty}\zeta(x)\,dx \ne 0$. Let*

$$g_0(\xi) = f_0[(\int_{-\infty}^{\infty}\xi(x)\,dx)\xi_0],\ \xi \in E,$$

$$y_\zeta(\xi) = f(-\int_{-\infty}^{x}[\xi(\tau) - \frac{\int_{-\infty}^{\infty}\xi(s)\,ds}{\int_{-\infty}^{\infty}\zeta(s)\,ds}\zeta(\tau)]\,d\tau),\ \xi \in E.$$

Then $g_0 \in E^{(\gamma,V)}$ for some $V \in \mathcal{N}(E)$ and $g_0' = 0$ by Th. 3.3, and $y_\zeta \in E^{(\gamma,U_\zeta)}$ for some balanced $U_\zeta \in \mathcal{N}(E)$ and $y_\zeta' = f$ by Cor. 3.3.

Let $W = V \bigcap U_\zeta$. Then $W \in \mathcal{N}(E)$ and $E^{(\gamma,V)} \bigcup E^{(\gamma,U_\zeta)} \subset E^{(\gamma,W)}$. Thus, $g_0 \in E^{(\gamma,W)}$, $y_\zeta \in E^{(\gamma,W)}$, $g = g_0 + y_\zeta \in [E^{(\gamma,W)}]$ and $g' = (g_0 + y_\zeta)' = g_0' + y_\zeta' = f$. □

Further discussions of ordinary and partial differential equations of demi-distributions will be interesting but we reserve these discussions for another paper.

§4 Fourier Transform

Let $x + iy = (x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{C}^n$, $|y| = |y_1| + \cdots + |y_n|$. For $a > 0$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, a multi-index, let $(x+iy)^\alpha = \prod_{k=1}^{n}(x_k + iy_k)^{\alpha_k}$ and

$$Z(a) = \left\{\zeta \in \mathbb{C}^{\mathbb{C}^n} : \zeta \text{ is entire; for every multi-index } \alpha,\ |(x+iy)^\alpha\zeta(x+iy)| \le C_\alpha(\zeta)\mathrm{e}^{a|y|}\right\},$$

$$\|\zeta\|_p = \sup_{x+iy\in\mathbb{C}^n, |\alpha|\le p} |(x+iy)^\alpha\zeta(x+iy)|\mathrm{e}^{-a|y|},\ p = 0, 1, 2, 3, \cdots,$$

$$Z = \left\{\zeta \in \mathbb{C}^{\mathbb{C}^n} : \zeta \text{ is entire, } \exists\, a(\zeta) > 0 \text{ such that } |(x+iy)^\alpha \zeta(x+iy)| \le C_\alpha(\zeta) \mathrm{e}^{a(\zeta)|y|}\right\}.$$

The Fourier transform $F(\xi)$ of $\xi \in \mathscr{D}$ is given by

$$F(\xi)(\sigma + i\tau) = \zeta(\sigma + i\tau) = \int \xi(x) \mathrm{e}^{i(x,\sigma)-(x,\tau)}\, dx,\ (x,\sigma) = \sum_{j=1}^n x_j\sigma_j,\ (x,\tau) = \sum_{j=1}^n x_j\tau_j.$$

Then $F[\mathscr{D}_a] = Z_a$, $F[\mathscr{D}] = Z$, and operators

$$F : \mathscr{D}_a \to Z_a,\quad F : \mathscr{D} \to Z,\quad F^{-1} : Z_a \to \mathscr{D}_a,\quad F^{-1} : Z \to \mathscr{D}$$

are both continuous and linear ([4, 3.1.1—3.1.2]).

For $\xi \in \mathscr{S}$ let

$$F(\xi)(\sigma) = \zeta(\sigma) = \int \xi(x) \mathrm{e}^{i(x,\sigma)}\, dx,\ \forall\, \sigma \in \mathbb{R}^n.$$

Then $F(\mathscr{S}) = \mathscr{S}$ and both F and F^{-1} are continuous and linear.

Definition 4.1. *Let $E \in \{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. For $f \in E^{(\gamma,U)}$ define $\hat{f} : F(E) \to \mathbb{C}$ by $\hat{f}(\zeta) = (2\pi)^n f(F^{-1}(\zeta))$, $\forall\, \zeta \in F(E)$. We write $\hat{f} = F(f)$ and so*

$$F(f)(F(\xi)) = (2\pi)^n f(\xi),\ \forall\, f \in E^{(\gamma,U)},\ \xi \in E.$$

Henceforth, $E \in \{\mathscr{D}_a, \mathscr{D}, \mathscr{S}\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$.

Theorem 4.1. $F(E^{(\gamma,U)}) = (F(E))^{(\gamma,F(U))}$.

Proof. *Since both F and F^{-1} are continuous linear operators, $F(U) \in \mathcal{N}(F(E))$. Let $f \in E^{(\gamma,U)}$, $\zeta \in F(E)$, $\eta \in F(U)$ and $|t| \le 1$. Then*

$$\begin{aligned}F(f)(\zeta + t\eta) &= (2\pi)^n f(F^{-1}(\zeta + t\eta)) = (2\pi)^n f(F^{-1}(\zeta) + tF^{-1}(\eta))\\ &= (2\pi)^n [rf(F^{-1}(\zeta)) + sf(F^{-1}(\eta))]\\ &= r(2\pi)^n f(F^{-1}(\zeta)) + s(2\pi)^n f(F^{-1}(\eta))\\ &= rF(f)(\zeta) + sF(f)(\eta), \qquad |r-1| \le |\gamma(t)|,\ |s| \le |\gamma(t)|.\end{aligned}$$

Thus, $F(f) \in \mathscr{L}_{\gamma,F(U)}(F(E), \mathbb{C})$.

Let $\zeta_\alpha \to \zeta$ in $F(E)$. Then $F^{-1}(\zeta_\alpha) \to F^{-1}(\zeta)$ in E and so $F(f)(\zeta_\alpha) = (2\pi)^n f(F^{-1}(\zeta_\alpha)) \to (2\pi)^n f(F^{-1}(\zeta)) = F(f)(\zeta)$. This shows that $F(f)$ is continuous and $F(f) \in (F(E))^{(\gamma,F(U))}$.

Conversely, for $g \in (F(E))^{(\gamma,F(U))}$ define

$$f(\xi) = (2\pi)^{-n} g(F(\xi)),\ \forall\, \xi \in E.$$

If $\xi \in E$, $\eta \in U$ and $|t| \le 1$, then

$$\begin{aligned}f(\xi + t\eta) &= (2\pi)^{-n} g(F(\xi + t\eta)) = (2\pi)^{-n} g(F(\xi) + tF(\eta))\\ &= (2\pi)^{-n}[rg(F(\xi)) + sg(F(\eta))]\\ &= rf(\xi) + sf(\eta), \qquad |r-1| \le |\gamma(t)|,\ |s| \le |\gamma(t)|,\end{aligned}$$

i.e., $f \in \mathscr{L}_{\gamma,U}(E, U)$.

Since both g and F are continuous, f is continuous so $f \in E^{(\gamma,U)}$ and

$$F(f)(\zeta) = (2\pi)^n f(F^{-1}(\zeta)) = g(\zeta),\ \forall\, \zeta \in F(E),$$

i.e., $g = F(f)$. □

Definition 4.2. *Let* $[(F(E))^{(\gamma,F(U))}] = span\,(F(E))^{(\gamma,F(U))}$ *in* $\mathbb{C}^{F(E)}$. *For* $f \in [E^{(\gamma,U)}]$ *define* $F(f) : F(E) \to \mathbb{C}$ *by*

$$F(f)(F(\xi)) = (2\pi)^n f(\xi),\ \forall\, \xi \in E.$$

Theorem 4.2. *If* $f = \sum_{k=1}^m \alpha_k f_k$ *where* $\alpha_k \in \mathbb{C}$ *and* $f_k \in E^{(\gamma,E)}$, *then* $F(f) = \sum_{k=1}^m \alpha_k F(f_k) \in [(F(E))^{(\gamma,F(U))}]$, *and*

$$F([E^{(\gamma,U)}]) = [(F(E))^{(\gamma,F(U))}].$$

Moreover, $F : [E^{(\gamma,U)}] \to [(F(E))^{(\gamma,F(U))}]$ *is* $w*-w*$ *continuous and linear.*

Now we consider the case of $n = 1$. Let $\mathscr{S}(\mathbb{R})$ be the space of infinitely differentiable but rapidly decreasing functions defined on $\mathbb{R}$. Then $F(\mathscr{S}(\mathbb{R})) = \mathscr{S}(\mathbb{R})$, $F((\mathscr{S}(\mathbb{R}))') = (\mathscr{S}(\mathbb{R}))'$.

A constant $C \in (\mathscr{S}(\mathbb{R}))'$ means that $C(\zeta) = \int_{-\infty}^{\infty} C\zeta(\sigma)\,d\sigma$ for all $\zeta \in \mathscr{S}(\mathbb{R})$ ([4, 3.2.1]), and $C = F(C\delta) = CF(\delta)$. In fact, $CF(\delta)(F(\xi)) = 2\pi C\delta(\xi) = 2\pi C\xi(0) = 2\pi C\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i0\sigma} F(\xi)(\sigma)d\sigma = \int_{-\infty}^{\infty} CF(\xi)(\sigma)\,d\sigma = C(F(\xi))$ for all $\xi \in \mathscr{S}(\mathbb{R})$.

Lemma 4.1. *Let* $y \in (\mathscr{S}(\mathbb{R}))'$, *a usual distribution. Then*

$$y(i\sigma\zeta(\sigma)) = 0 \text{ for all } \zeta \in \mathscr{S}(\mathbb{R})$$

if and only if $y = C\delta$, *where* C *is a constant.*

Proof. Suppose that $y \in (\mathscr{S}(\mathbb{R}))'$ *and* $y(i\sigma\zeta(\sigma)) = 0,\ \forall\,\zeta \in \mathscr{S}(\mathbb{R})$. *Since* $(\mathscr{S}(\mathbb{R}))' = F((\mathscr{S}(\mathbb{R}))')$, *there is a usual distribution* $f \in (\mathscr{S}(\mathbb{R}))'$ *such that* $y = F(f)$ *and*

$$f'(\xi) = f(-\xi') = \frac{1}{2\pi}F(f)(F((-\xi)')) = \frac{1}{2\pi}F(f)(-i\sigma F(-\xi)(\sigma))$$
$$= \frac{1}{2\pi}y(i\sigma F(\xi)(\sigma)) = 0,\ \forall\,\xi \in \mathscr{S}(\mathbb{R}),$$

i.e., $f' = 0$. *But* f *is a usual distribution so* $f = C$, *a constant. Thus,* $y = F(f) = F(C) = CF(1) = C\delta$.

Conversely, if $y = C\delta$ *where* C *is a constant, then*

$$y(i\sigma\zeta(\sigma)) = C\delta(i\sigma\zeta(\sigma)) = 0,\ \forall\,\zeta \in \mathscr{S}(\mathbb{R}). \ \square$$

However, there exists a lot of various demi-distributions on $\mathscr{S}(\mathbb{R})$ which satisfy the condition $y(i\sigma\zeta(\sigma)) = 0,\ \forall\,\zeta \in \mathscr{S}(\mathbb{R})$.

Theorem 4.3. *Let* $U \in \mathcal{N}(\mathscr{S}(\mathbb{R}))$ *and* $\gamma \in C(0)$. *Pick an arbitrary* $f_0 \in (\mathscr{S}(\mathbb{R}))^{(\gamma,U)}$ *and* $\xi_0 \in \mathscr{S}(\mathbb{R})$ *and let* $f(\xi) = f_0[(\int_{-\infty}^{\infty}\xi(t)\,dt)\xi_0],\ \forall\,\xi \in \mathscr{S}(\mathbb{R})$. *Then* $f \in (\mathscr{S}(\mathbb{R}))^{(\gamma,V)}$ *for some* $V \in \mathcal{N}(\mathscr{S}(\mathbb{R}))$ *and* $F(f) \in (\mathscr{S}(\mathbb{R}))^{(\gamma,F(V))}$ *such that*

$$F(f)(i\sigma\zeta(\sigma)) = 0,\ \forall\,\zeta \in \mathscr{S}(\mathbb{R}).$$

Proof. By Th. 3.3, there is a $V \in \mathcal{N}(\mathscr{S}(\mathbb{R}))$ *such that* $f \in (\mathscr{S}(\mathbb{R}))^{(\gamma,V)}$ *and* $f' = 0$. *If* $\zeta \in \mathscr{S}(\mathbb{R})$, *then* $\zeta = F(\xi)$ *for some* $\xi \in \mathscr{S}(\mathbb{R})$ *and*

$$F(f)(i\sigma\zeta(\sigma)) = F(f)(i\sigma F(\xi)(\sigma)) = F(f)(-i\sigma F(-\xi)(\sigma))$$
$$= F(f)(F((-\xi)')) = 2\pi f(-\xi') = 2\pi f'(\xi) = 0. \ \square$$

§5 Convolutions

In this section, $E \in \{\mathscr{D}, \mathscr{S}\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$.

Definition 5.1. *([4, 3.3.2]) A distribution $f_0 \in E'$ is called a convolution multiplier on E if the following (i) and (ii) hold for f_0:*

(i) if for each $\xi \in E$ define $f_0 * \xi : \mathbb{R}^n \to \mathbb{C}$ by

$$(f_0 * \xi)(x) = f_0(\xi(x + \cdot)),\ \forall\, x \in \mathbb{R}^n,$$

then $f_0 * \xi \in E$;

(ii) if $\xi_k \to 0$ in E, then $f_0 * \xi_k \to 0$ in E.

Lemma 5.1. *If f_0 is a convolution multiplier on E, then $f_0 * \cdot : E \to E$ is a continuous linear operator.*

Following [4], $P(D) = \sum a_\alpha D^\alpha = \sum a_{\alpha_1,\cdots,\alpha_n} \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ is a finite sum, where $a_\alpha \in \mathbb{C}$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index. For $\xi \in E$ and $x \in \mathbb{R}^n$, $\frac{\partial \xi(x+\tau)}{\partial \tau_j} = \frac{\partial \xi(x+\tau)}{\partial (x_j+\tau_j)} \frac{\partial (x_j+\tau_j)}{\partial \tau_j} = \frac{\partial \xi(x+\tau)}{\partial (x_j+\tau_j)}(\frac{\partial x_j}{\partial \tau_j} + \frac{\partial \tau_j}{\partial \tau_j}) = \frac{\partial \xi(x+\tau)}{\partial (x_j+\tau_j)}$ and, by induction, it is easy to see that

$$\frac{\partial^{|\alpha|}\xi(x+\tau)}{\partial \tau_1^{\alpha_1}\cdots\partial\tau_n^{\alpha_n}} = \frac{\partial^{|\alpha|}\xi(x+\tau)}{\partial(x_1+\tau_1)^{\alpha_1}\cdots\partial(x_n+\tau_n)^{\alpha_n}}$$

for every multi-index α and so there is no any ambiguity for the notation $D^\alpha \xi(x + \cdot)$, i.e.,

$$D^\alpha \xi(x+\cdot) = \frac{\partial^{|\alpha|}\xi(x+\tau)}{\partial \tau_1^{\alpha_1}\cdots\partial\tau_n^{\alpha_n}} = \frac{\partial^{|\alpha|}\xi(x+\tau)}{\partial(x_1+\tau_1)^{\alpha_1}\cdots\partial(x_n+\tau_n)^{\alpha_n}} = (D^\alpha\xi)(x+\cdot).$$

Theorem 5.1. *If $f_0 \in E'$ is a convolution multiplier on E, then $P(D)f_0$ is also a convolution multiplier on E, and*

$$(\sum a_\alpha D^\alpha f_0) * \xi = \sum a_\alpha (-1)^{|\alpha|} f_0 * D^\alpha \xi,\ \forall\, \xi \in E.$$

Definition 5.2. *For every convolution multiplier $f_0 \in E'$ and $f \in [E^{(\gamma,U)}]$ define the convolution $f_0 * f : E \to \mathbb{C}$ by*

$$(f_0 * f)(\xi) = f(f_0 * \xi),\ \forall\, \xi \in E.$$

Example 5.1. *(1) For every $\xi \in E$, $\delta * \xi = \xi$, $(D^\alpha\delta) * \xi = (-1)^{|\alpha|} D^\alpha \xi$, and for every $f \in [E^{(\gamma,U)}]$, $\delta * f = f$, $(D^\alpha \delta) * f = D^\alpha f$.*

In fact, for $\xi \in E$ and $x \in \mathbb{R}^n$, $(\delta * \xi)(x) = \delta(\xi(x+\cdot)) = \xi(x+0) = \xi(x)$, $((D^\alpha\delta) * \xi)(x) = (D^\alpha\delta)(\xi(x+\cdot)) = \delta((-1)^{|\alpha|}\frac{\partial^{|\alpha|}\xi(x+\tau)}{\partial\tau_1^{\alpha_1}\cdots\partial\tau_n^{\alpha_n}}) = (-1)^{|\alpha|}(D^\alpha\xi)(x)$, i.e., $\delta * \xi = \xi$, $(D^\alpha\delta) * \xi = (-1)^{|\alpha|}D^\alpha\xi$. Then for every $\xi \in E$,

$$(\delta * f)(\xi) = f(\delta * \xi) = f(\xi),\ ((D^\alpha\delta) * f)(\xi) = f((D^\alpha\delta) * \xi) = f((-1)^{|\alpha|}D^\alpha\xi) = (D^\alpha f)(\xi).$$

(2) Let $f \in E'$ and $U = \{\eta \in E : |f(\eta)| < 1\}$. By Cor. 2.1, $h \circ f \in E^{(\gamma,U)}$ for each $h \in \mathscr{L}_{\gamma,1}(\mathbb{C},\mathbb{C})$ and $f_0 * (h \circ f) = h \circ (f_0 * f)$ for every convolution multiplier $f_0 \in E'$. In fact,

$$(f_0 * (h \circ f))(\xi) = (h \circ f)(f_0 * \xi) = h[f(f_0 * \xi)] = h[(f_0 * f)(\xi)] = [h \circ (f_0 * f)](\xi),\ \forall\, \xi \in E.$$

Theorem 5.2. *Let $f_0 \in E'$ be a convolution multiplier. For every $U \in \mathcal{N}(E)$ there is a $V \in \mathcal{N}(E)$ such that*

$$f_0 * f \in E^{(\gamma,V)},\ \forall\, f \in E^{(\gamma,U)},$$
$$f_0 * f \in [E^{(\gamma,V)}],\ \forall\, f \in [E^{(\gamma,U)}].$$

Henceforth, we write $f_0 * \xi = f_0(\xi(x+\cdot)) = (f_0 * \xi)(x)$, see [4, 3.3.2]. Observe that $tf \in E^{(\gamma,U)}$ whenever $t \in \mathbb{C}$ and $f \in E^{(\gamma,U)}$.

Lemma 5.2. *If $f_0 \in E'$ is a convolution multiplier and $t \in \mathbb{C}$, then*

$$t(f_0 * \xi) = (tf_0) * \xi,\ \forall\, \xi \in E;$$
$$t(f_0 * f) = f_0 * (tf),\ \forall\, f \in [E^{(\gamma,U)}];$$
$$t(f_0 * f) = (tf_0) * f,\ \forall\, f \in E'.$$

Proof. $t(f_0 * \xi) = tf_0(\xi(x+\cdot)) = (tf_0)(\xi(x+\cdot)) = (tf_0) * \xi,\ \forall\, \xi \in E$. *For* $f \in [E^{(\gamma,U)}]$ *and* $\xi \in E$, $t(f_0 * f)(\xi) = tf(f_0 * \xi) = (tf)(f_0 * \xi) = (f_0 * (tf))(\xi)$. *If* $f \in E'$, *then* $t(f_0 * f)(\xi) = tf(f_0 * \xi) = f(t(f_0 * \xi)) = f((tf_0) * \xi) = ((tf_0) * f)(\xi)$ *for all* $\xi \in E$. □

As usual, $e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $e_n = (0, \cdots, 0, 1)$.

Theorem 5.3. *If $f_0 \in E'$ is a convolution multiplier and α is a multi-index, then $D^\alpha(f_0 * \xi) = f_0 * D^\alpha \xi$ for $\xi \in E$, and*

$$D^\alpha(f_0 * f) = (D^\alpha f_0) * f = f_0 * D^\alpha f,\ \forall\, f \in [E^{(\gamma,U)}].$$

Recall that if $f \in E'$ for which *supp* f is bounded in $\mathbb{R}^n$, then f must be a convolution multiplier ([4, Th. 3.3.4]). Then we can develop the result of continuity of convolution ([4, Th. 3.3.5]).

First, we give an improvement of Th. 3.3.5 of [4] as follows.

Theorem 5.4. *If $\{f_k\} \subset E'$ such that $f_k \xrightarrow{w*} f$, i.e., $f_k(\xi) \to f(\xi)$ at each $\xi \in E$ ($f \in E'$ by Th. 2.8) and there is a bounded $F \subset \mathbb{R}^n$ such that supp $f_k \subseteq F$, $\forall\, k \in \mathbb{N}$, then for every $g \in [E^{(\gamma,U)}]$ and bounded $B \subset E$,*

$$\lim_k (f_k * g)(\xi) = (f * g)(\xi) \text{ uniformly for } \xi \in B.$$

We also give some simple facts before our main result Th. 5.6.

Theorem 5.5. *Let $f_0 \in E'$ be a convolution multiplier, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. There is a $V \in \mathcal{N}(E)$ for which $f_0 * \cdot : [E^{(\gamma,U)}] \to [E^{(\gamma,V)}]$ is a linear operator such that $f_0 * f \in E^{(\gamma,V)}$ for each $f \in E^{(\gamma,U)}$. Moreover, if $f_k \xrightarrow{w*} f$ in $E^{(\gamma,U)}$, i.e., $f, f_k \in E^{(\gamma,U)}$ and $f_k(\xi) \to f(\xi)$ at each $\xi \in E$, then for every bounded $B \subset E$, $\lim_k (f_0 * f_k)(\xi) = (f_0 * f)(\xi)$ uniformly for $\xi \in B$.*

We now have a strong continuity result for convolution as follows.

Theorem 5.6. *Let $\{f_k\} \subset E'$ be a sequence of usual distributions such that $f_k \xrightarrow{w*} f$, i.e., $f_k(\xi) \to f(\xi)$ at each $\xi \in E$ ($f \in E'$ by Th. 2.8) and there is a bounded $F \subset \mathbb{R}^n$ such that supp $f_k \subseteq F$, $\forall\, k \in \mathbb{N}$. If $g_k \xrightarrow{w*} g$ in $E^{(\gamma,U)}$, i.e., $g, g_k \in E^{(\gamma,U)}$ for all k and $g_k(\xi) \to g(\xi)$ at each $\xi \in E$, then for every bounded $B \subset E$, $\lim_{k,m\to+\infty}(f_k * g_m)(\xi) = (f * g)(\xi)$ uniformly for $\xi \in B$ and, in particular, $\lim_k (f_k * g_k)(\xi) = (f * g)(\xi)$ uniformly for $\xi \in B$, and $(f_k * g_k)(\xi_k) \to (f * g)(\xi)$ whenever $\xi_k \to \xi$ in E.*

Proof. As in the proof of Th. 5.4, it follows from $f_k \xrightarrow{w*} f$ in E' and $g, g_k \in E^{(\gamma,U)}$ that there is a $V \in \mathcal{N}(E)$ such that $f * g,\ f_m * g_k \in E^{(\gamma,V)}$ for all $k, m \in \mathbb{N}$.

Let $\xi \in E$. As was noticed in the proof of Th. 5.4, $f_m * \xi \to f * \xi$ in E and so $\lim_m (f_m * g_k)(\xi) = \lim_m g_k(f_m * \xi) = g_k(f * \xi)$, $\forall\, k \in \mathbb{N}$. *But $\{f_m * \xi\}$ is bounded in E and, by Th. 2.2,* $\lim_k (f_m * g_k)(\xi) = \lim_k g_k(f_m * \xi) = g(f_m * \xi) = (f_m * g)(\xi)$ *uniformly for $m \in \mathbb{N}$. Then* $\lim_{k,m\to+\infty}(f_m * g_k)(\xi) = \lim_m \lim_k (f_m * g_k)(\xi) = \lim_m (f_m * g)(\xi) = \lim_m g(f_m * \xi) = g(f * \xi) = (f * g)(\xi)$, $\forall\, \xi \in E$.

Let B be a bounded subset of E. If $\lim_{k,m\to+\infty}(f_m * g_k)(\xi) = (f * g)(\xi)$ *is not uniformly for $\xi \in B$, then there exist $\varepsilon > 0$, $\{\xi_\nu\} \subset B$ and integer sequences $k_1 < k_2 < \cdots$ and $m_1 < m_2 < \cdots$ such that*

$$(*) \qquad |(f_{m_\nu} * g_{k_\nu})(\xi_\nu) - (f * g)(\xi_\nu)| \ge \varepsilon, \ \nu = 1, 2, 3, \cdots .$$

Since $f * g$, $f_{m_\nu} * g_{k_\nu} \in E^{(\gamma, V)}$ for all $\nu \in \mathbb{N}$ and

$$\lim_\nu (f_{m_\nu} * g_{k_\nu})(\xi) = \lim_{k,m\to+\infty}(f_m * g_k)(\xi) = (f * g)(\xi), \ \forall\, \xi \in E,$$

it follows from Th. 2.2 or Th. 2.7 that $\lim_\nu (f_{m_\nu} * g_{k_\nu})(\xi) = (f * g)(\xi)$ *uniformly for $\xi \in B$ and so there is a $\nu_0 \in \mathbb{N}$ such that*

$$|(f_{m_\nu} * g_{k_\nu})(\xi_\nu) - (f * g)(\xi_\nu)| < \varepsilon, \ \forall\, \nu > \nu_0.$$

This contradicts $(*)$ and so $\lim_{k,m\to+\infty}(f_m * g_k)(\xi) = (f * g)(\xi)$ *uniformly for $\xi \in B$.* □

References

[1] Ronglu Li, Shuhui Zhong, Linsong Li. *Demi-linear analysis I—basic principles*, J Korean Math Soc, 2009, 46(3): 643-656.

[2] Ronglu Li, Shuhui Zhong, Chengri Cui. *New basic principles of functional analysis (Abstract)*, J Yanbian Univ, 2004, 30(3): 157-160.

[3] A Wilansky. *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York, 1978.

[5] I M Gelfand, et al. *Generalized Functions II*, Academic Press, New York, 1968.

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