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Checking weak and strong optimality of the solution to interval convex quadratic program

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Abstract. In this paper, we investigate three canonical forms of interval convex quadratic programming problems. Necessary and sufficient conditions for checking weak and strong optimality of given vector corresponding to various forms of feasible region, are established respectively. By using the concept of feasible direction, these conditions are formulated in the form of linear systems with both equations and inequalities. In addition, we provide two specific examples to illustrate the efficiency of the conditions.

§1 Introduction

Over the past decades, interval systems and interval mathematical programming problems have been studied by many authors, see e.g., [1-17]. Most of the authors dealt with interval systems, see Li et al.[8] Popova[16] and Skalna et al.[21] for latest results, among others. The duality theory for interval-valued optimization problem was proposed by Wu[22, 23]. The problem of computing the range of optimal values of interval convex quadratic programming (IvCQP) was investigated by Liu[14], Li[9], Hladík[5] and Li et al.[12]. Recently, Hladík[7] generalized the above results and proposed an algorithm for computing the optimal value range of IvCQP in a general form.

One of the fundamental problems in interval program is to check whether a given vector is optimal, which is often overlooked. Recently, some authors studied weak and strong optimality of a given point to interval linear programs, see [18-22]. However, little was done on the optimality of a given vector to IvCQP problems. In this paper, we discuss the characterizations for checking weak and strong optimality of the solutions to three canonical forms of interval convex quadratic constrained programming problems. First, we propose the methods to check weak optimality of a given vector to IvCQP problems based on feasible directions. Next, strong

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optimality of a given vector is studied in a similar manner. Finally, we report some illustrated examples.

§2 Preliminaries

Let us introduce some notations. The i-th row of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted by $A_{i,\cdot}$, the j-th column by $A_{\cdot,j}$. Following notations from [3], an interval matrix is defined as

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, $\underline{A} \le \overline{A}$, and "$\le$" is understood componentwise. By

$$A_c = \frac{1}{2}(\underline{A} + \overline{A}), A_\Delta = \frac{1}{2}(\overline{A} - \underline{A}),$$

we denote the center and the radius of A, respectively. Then $\mathbf{A} = [A_c - A_\Delta, A_c + A_\Delta]$. An interval vector $\mathbf{b} = [\underline{b}, \overline{b}] = \{b \in \mathbb{R}^m : \underline{b} \le b \le \overline{b}\}$ is understood as a one-column interval matrix.

Let $\{\pm 1\}^m$ be the set of all $\{-1, 1\}$ m-dimensional vectors, i.e.

$$\{\pm 1\}^m = \{y \in \mathbb{R}^m | \mid y \mid = e\},$$

where $e = (1, \cdots, 1)^T$ is the m-dimensional vector of all 1's and the absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. For a given $y \in \{\pm 1\}^m$, let

$$T_y = diag(y_1, \ldots, y_m)$$

denote the corresponding diagonal matrix. For each $x \in \mathbb{R}^n$, we define its sign vector sgn x by

$$(\text{sgn } x)_i = \begin{cases} 1 & \text{if } x_i \ge 0, \\ -1 & \text{if } x_i < 0, \end{cases}$$

where $i = 1, \cdots, n$. Then we have $|x| = T_z x$, where $z = \text{sgn } x \in \{\pm 1\}^n$.

Given an interval matrix $\mathbf{A} = [A_c - A_\Delta, A_c + A_\Delta]$, for each $y \in \{\pm 1\}^m$ and $z \in \{\pm 1\}^n$, we define matrices

$$A_{yz} = A_c - T_y A_\Delta T_z.$$

Similarly, for an interval vector $\mathbf{b} = [b_c - b_\Delta, b_c + b_\Delta]$ and for each $y \in \{\pm 1\}^m$, we define vectors

$$b_y = b_c + T_y b_\Delta.$$

The set of all m-by-n interval matrices will be denoted by $\mathbb{IR}^{m\times n}$ and the set of all m-dimensional interval vectors by $\mathbb{IR}^m$. Let $\mathbf{Q} \in \mathbb{IR}^{n\times n}$, $\mathbf{A} \in \mathbb{IR}^{m\times n}$, $\mathbf{c} \in \mathbb{IR}^n$ and $\mathbf{b} \in \mathbb{IR}^m$. Assume that Q is positive semidefinite for all $Q \in \mathbf{Q}$.

The interval convex quadratic programming (IvCQP) problem

$$\min \frac{1}{2}x^T \mathbf{Q} x + \mathbf{c}^T x \tag{1a}$$

$$\text{s.t. } x \in M(\mathbf{A},\ \mathbf{b}), \tag{1b}$$

is the family of convex quadratic programming (CQP) problems

$$\min \frac{1}{2}x^T Q x + c^T x \tag{2a}$$

$$\text{s.t. } x \in M(A,\ b), \tag{2b}$$

with data satisfying $Q \in \mathbf{Q}$, $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$. A scenario means a concrete setting of (2).

As we all known, $M(A,\ b)$ is the feasible region associated with a linear system. We can extend the feasible region to interval linear systems, as $M(\mathbf{A},\ \mathbf{b})$. In interval linear programming theory, one of the following canonical forms

(A) $M(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n; \mathbf{A}x = \mathbf{b}, x \geq 0\}$,

(B) $M(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n; \mathbf{A}x \leq \mathbf{b}\}$,

(C) $M(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n; \mathbf{A}x \leq \mathbf{b}, x \geq 0\}$

is usually assumed [1,4]. Similarly, we focus on interval convex quadratic programming problems with the above three canonical forms of feasible regions.

Definition 2.1. A vector x is called a weakly(strongly) feasible solution of the IvCQP problem (1) if it satisfies (2b), for some(each) $A \in \mathbf{A}$, $b \in \mathbf{b}$. Moreover, a vector is called a weakly(strongly) optimal solution to (1) if it is an optimal solution for some(each) concrete settings of (2).

Definition 2.2. A matrix A is called a feasible matrix if there exists some $b \in \mathbf{b}$ such that $M(A, b) \neq \emptyset$.

Next, we give the conception of feasible directions, which has been discussed in [2].

Definition 2.3. Given an optimization problem with feasible region Ω. A vector $d \in \mathbb{R}^n$, $d \neq 0$, is a feasible direction at $x \in \Omega$ if there exists $\alpha_0 > 0$ such that $x + \alpha d \in \Omega$ for all $\alpha \in [0, \alpha_0]$.

Let $\Omega = \{x | c_i(x) = 0, i \in \Psi;\ c_i(x) \geq 0, i \in \Gamma\}$ be the feasible region of an optimization problem where $c_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \Psi \bigcup \Gamma$ are continuously differentiable functions. If $d \in \mathbb{R}^n$ is the feasible direction at $x \in \Omega$, then

$$\nabla c_i(x)^T d = 0, \quad \forall i \in \Psi; \quad \nabla c_i(x)^T d \geq 0, \quad \forall i \in \Gamma(x) \triangleq \{i \in \Gamma | c_i(x) = 0\}.$$

Moreover, the converse of this result also true if Ω is a linear system.

From Theorem 6.1 in [2], we can get the well-known first-order necessary condition, i.e., if f is a real-valued function on Ω, x^* is a local minimizer of f over Ω, then the rate of increase of f at x^* in any feasible direction d in Ω is nonnegative.

In order to obtain the main results in next section, we give some properties of CQP problems.

Theorem 2.1. *Suppose that x^* be a feasible solution of* (2)*, then x^* is an optimal solution of* (2) *if and only if for any feasible direction d at x^*, we have*

$$(c + Qx^*)^T d \geq 0. \tag{3}$$

Proof. It is obvious by Theorem 6.1, Theorem 22.6 and Theorem 22.7 in [2]. □

In 2013, Hladík[6] considered weak and strong solvability of general interval linear systems consisting of mixed equations and inequalities with mixed free and sign-restricted variables. He generalized the well known weak solvability characterizations by Oettli-Prager (for equations) and Gerlach (for equalities) to a unified framework. From the main results in [6], we can obtain the characterizations of weak and strong solvability for some special systems.

Theorem 2.2. *A vector $x \in \mathbb{R}^n$ is a weak solution of interval linear system $\mathbf{A}x = \mathbf{b}$, $\mathbf{C}x \leq \mathbf{d}$ (the vector satisfies linear system $Ax = b$, $Cx \leq d$, for some $A \in \mathbf{A}$, $C \in \mathbf{C}$, $b \in \mathbf{b}$ and $d \in \mathbf{d}$), if and only if for some $s \in \{\pm 1\}^n$, it satisfies*

$$(A_c - A_\Delta T_s)x \leq \overline{b}, \quad -(A_c + A_\Delta T_s)x \leq -\underline{b}, \ (C_c - C_\Delta T_s)x \leq \overline{d}.$$

Proof. It is clear by Corollary 2 in [6]. □

Theorem 2.3. *A vector $x \in \mathbb{R}^n$ is a strong solution of $\mathbf{A}x = \mathbf{b},\ x \geq 0$ (the vector satisfies linear system $Ax = b,\ x \geq 0$, for each $A \in \mathbf{A},\ b \in \mathbf{b}$) if and only if it satisfies*

$$A_\Delta x = 0,\ A_c x = b_c,\ x \geq 0$$

Proof. It is clear by Theorem 3 in [6]. □

The following theorem from [3,18] characterize the solvability of interval linear systems, which, together with Theorem 2.1, 2.2 and 2.3, will be used to obtain our main results in the next two sections.

Theorem 2.4. *A system $Ax \leq b$ is solvable for each $A \in \mathbf{A},\ b \in \mathbf{b}$ if and only if the system*

$$\overline{A}x_1 - \underline{A}x_2 \leq \underline{b},\ x_1 \geq 0,\ x_2 \geq 0$$

is solvable.

Let $x \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{IR}^{m\times n}$. Obviously, we know $\mathbf{A}x \in \mathbb{IR}^m$. The following lemma from [8] characterizes the lower bound and the upper bound of interval vector $\mathbf{A}x$, which will be used to obtain our main results in the next two sections.

Lemma 2.1. *(Lemma 3.1 in [8]) Let $x \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{IR}^{m\times n}$, and $h = sgn\ x$. Then for each $A \in \mathbf{A}$, we have $A_{eh}x \leq Ax$ and $A_{-eh}x \geq Ax$.*

§3 Checking weakly optimal solution of interval convex quadratic program

In this section, we first propose the method to check weak optimality of a given vector for the interval convex quadratic program, which feasible regions are interval equalities with nonnegativity variables.

Theorem 3.1. *Let $x^* = (x_1^*, \ldots, x_n^*)^T \in \mathbb{R}^n$. Denote $G = \{k_i | i = 1, \cdots, t,\ x_{k_i}^* = 0\}$. Then x^* is a weakly optimal solution to*

$$min\ \frac{1}{2}x^T\mathbf{Q}x + \mathbf{c}^T x \tag{4a}$$

$$s.t.\ \mathbf{A}x = \mathbf{b},\ x \geq 0 \tag{4b}$$

if and only if x^* is a weakly feasible solution to (4), and there exists a feasible matrix A such that the linear system

$$\begin{cases} A(x_1 - x_2) = 0, \\ (x_1 - x_2)_{k_i} \geq 0,\ k_i \in G, \\ (\overline{c} + \overline{Q}x^*)^T x_1 - (\underline{c} + \underline{Q}x^*)^T x_2 < 0, \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases} \tag{5}$$

has no solution.

Proof. "Only if": Let x^* be a weakly optimal solution to problem (4), then x^* is a weakly feasible solution to (4). And for some $A \in \mathbf{A},\ b \in \mathbf{b}$, x^* is a weakly optimal solution to problem

$$\min\ \frac{1}{2}x^T\mathbf{Q}x + \mathbf{c}^T x \tag{6a}$$

$$\text{s.t.}\ Ax = b,\ x \geq 0. \tag{6b}$$

Then, from Definition 2.3, the feasible direction to the feasible region (6b) at x^* reads

$$\begin{cases} Ax = 0, \\ x_{k_i} \geq 0, \quad k_i \in G, \end{cases}$$

where $G = \{k_i | i = 1, \cdots, t,\ x^*_{k_i} = 0\}$.

Meanwhile, we know x^* is a weakly optimal solution to problem (6) if and only if it is an optimal solution to problem

$$\min \frac{1}{2} x^T Q x + c^T x \tag{7a}$$

$$\text{s.t. } Ax = b,\ x \geq 0, \tag{7b}$$

for some $Q \in \mathbf{Q}$ and $c \in \mathbf{c}$. Moreover, from Theorem 2.1, x^* is an optimal solution to (7) if and only if there holds (3), where d is any feasible direction to the feasible region (6b) at x^*. Obviously, the solvability of (3) for any feasible direction d is equivalent to the unsolvability of

$$\begin{cases} Ax = 0, \\ x_{k_i} \geq 0 \quad k_i \in G, \\ (c + Qx^*)^T x < 0. \end{cases} \tag{8}$$

That is, x^* is a weakly optimal solution to (6) if and only if the system (8) is unsolvable for some $Q \in \mathbf{Q},\ c \in \mathbf{c}$ or, equivalently, if and only if it is not true that (8) is solvable for all $Q \in \mathbf{Q},\ c \in \mathbf{c}$. Then from Theorem 2.4, we can easily obtain the system

$$\begin{cases} A(x_1 - x_2) = 0, \\ (x_1 - x_2)_{k_i} \geq 0 \quad k_i \in G, \\ (\overline{c + Qx^*})^T x_1 - (\underline{c + Qx^*})^T x_2 < 0,\ x_1 \geq 0,\ x_2 \geq 0 \end{cases} \tag{9}$$

has no solution. Obviously, from the nonnegativity of x^*, we know

$$\begin{cases} \overline{c + Qx^*} = \overline{c} + \overline{Q} x^*, \\ \underline{c + Qx^*} = \underline{c} + \underline{Q} x^*. \end{cases} \tag{10}$$

From the formula (10) and unsolvability of (9), we obtain the system (5) has no solution.

"If": Let x^* be a weakly feasible solution to (4), then for feasible matrix $A \in \mathbf{A}$, there exists $b \in \mathbf{b}$ such that x^* is a feasible solution to problem (6).

Because the linear system (5) has no solution for feasible matrix $A \in \mathbf{A}$, we know that associated system (9) has no solution from the analysis above. According to Theorem 2.4, we have the linear system (8) has no solution for some $Q \in \mathbf{Q},\ c \in \mathbf{c}$. Hence, x^* is an optimal solution to (7) for some $Q \in \mathbf{Q},\ c \in \mathbf{c}$, which means, x^* is a weakly optimal solution to (4). □

In the similar manner, we can prove the next result for the IvCQP, which feasible regions are interval inequalities with free variables.

Theorem 3.2. *Let $x^* \in \mathbb{R}^n$, where $x^* = (x^*_1, \ldots, x^*_n)^T$. Then x^* is a weakly optimal solution to*

$$\min \frac{1}{2} x^T \mathbf{Q} x + \mathbf{c}^T x \tag{11a}$$

$$s.t.\ \mathbf{A} x \leq \mathbf{b} \tag{11b}$$

if and only if x^* is a weakly feasible solution to (11), and there exists a feasible matrix A such *that the linear system*

$$\begin{cases} A_{r_j,\cdot}(x_1 - x_2) \le 0, \ r_j \in F \\ (\overline{c} + Q_{-es}x^*)^T x_1 - (\underline{c} + Q_{es}x^*)^T x_2 < 0, \\ x_1 \ge 0, \ x_2 \ge 0 \end{cases} \tag{12}$$

has no solution, where $s = sgn\ x^*$ *and* $F = \{r_j | j = 1, \cdots, q, \ A_{r_j,\cdot}x^* \ge \underline{b}_{r_j}\}$.

Proof. "Only if": Let x^* be a weakly optimal solution to problem (11), then x^* is a weakly feasible solution to (11). And for some $A \in \mathbf{A}, \ b \in \mathbf{b}$, x^* is a weakly optimal solution to problem

$$\min \frac{1}{2}x^T \mathbf{Q} x + \mathbf{c}^T x \tag{13a}$$

$$\text{s.t. } Ax \le b. \tag{13b}$$

Then feasible direction to the feasible region of (13) at x^* reads

$$A_{r'_j,\cdot}x \le 0, \quad r'_j \in F',$$

where $F' = \{r'_j | j = 1, \cdots, m, \ A_{r'_j,\cdot}x^* = b_{r'_j}\}$. Meanwhile, we know x^* is an optimal solution to problem

$$\min \frac{1}{2}x^T Q x + c^T x \tag{14a}$$

$$\text{s.t. } Ax \le b, \tag{14b}$$

for some $Q \in \mathbf{Q}, \ c \in \mathbf{c}$.

Moreover, from Theorem 2.1, x^* is an optimal solution to (14) if and only if there holds (3), where d is any feasible direction to the feasible region of (13) at x^*. Thus, solvability of (3) for any feasible direction d is equivalent to the unsolvability of

$$\begin{cases} A_{r'_j,\cdot}x \le 0, \ r'_j \in F' \\ (c + Qx^*)^T x < 0. \end{cases} \tag{15}$$

Clearly, for each $r'_j \in F'$ we are easy to obtain $A_{r'_j,\cdot}x^* = b_{r'_j} \ge \underline{b}_{r'_j}$, thus, $r'_j \in F$, which implies $F' \subseteq F$. Hence, the linear system

$$\begin{cases} A_{r_j,\cdot}x \le 0, \ r_j \in F \\ (c + Qx^*)^T x < 0. \end{cases} \tag{16}$$

has no solution. That is, from the weak optimality of x^* for problem (13), we have the system (16) is unsolvable for some $Q \in \mathbf{Q}, \ c \in \mathbf{c}$. Then from Theorem 2.4 and Lemma 2.1, we obtain the system (12) has no solution.

"If": Let x^* be a weakly feasible solution to (11), then for feasible matrix $A \in \mathbf{A}$, there exists some $b \in \mathbf{b}$ such that x^* is a feasible solution to (13).

Because the linear system (12) has no solution, we know that the associated linear system (17) has no solution from the above analysis. According to Theorem 2.4, we know that (16) has no solution for some $Q \in \mathbf{Q}, \ c \in \mathbf{c}$. Denote

$$\tilde{b_k} = \begin{cases} A_{k,\cdot}x^*, \ k \in F \\ b_k \in \mathbf{b}_k, \ k \notin F, \end{cases}$$

obviously, $\tilde{b} \in \mathbf{b}$. Moreover, we have

$$A_{k,\cdot}x^* = \tilde{b_k}, \ k \in F$$

and

$$A_{k,\cdot}x^* < \underline{b}_k \le \tilde{b_k},\ k \notin F,$$

thus, $Ax^* \le \tilde{b}$, meanwhile, $F' = F$. Hence, we know the linear system (15) has no solution, which means x^* is an optimal solution to (14), for some $Q \in \mathbf{Q},\ c \in \mathbf{c}$. Thus, x^* is a weakly optimal solution to problem (11). □

Theorem 3.3. *Let $x^* = (x_1^*, \ldots, x_n^*)^T \in \mathbb{R}^n$, Then x^* is a weakly optimal solution to*

$$min\ \frac{1}{2}x^T\mathbf{Q}x + \mathbf{c}^Tx \tag{17a}$$

$$s.t.\ \mathbf{A}x \le \mathbf{b},\ x \ge 0 \tag{17b}$$

if and only if x^* is a weakly feasible solution to (17), and there exists a feasible matrix A such that the linear system

$$\begin{cases} A_{r_j,\cdot}(x_1 - x_2) \le 0,\ r_j \in F \\ (x_1 - x_2)_{k_i} \ge 0,\ k_i \in G, \\ (\overline{c} + \overline{Q}x^*)^Tx_1 - (\underline{c} + \underline{Q}x^*)^Tx_2 < 0, \\ x_1 \ge 0,\ x_2 \ge 0 \end{cases} \tag{18}$$

has no solution, where $F = \{r_j | j = 1, \cdots, q,\ A_{r_j,\cdot}x^* \ge \underline{b}_{r_j}\}$, and $G = \{k_i | i = 1, \cdots, t,\ x_{k_i}^* = 0\}$.

Proof: The proof is similar to that of the Theorem 3.1 and Theorem 3.2 and is thus omitted here. □

§4 Checking strongly optimal solution of interval convex quadratic program

In this section, we first propose the method to check strong optimality of a given vector for the interval convex quadratic program. In this case, the feasible regions are interval equalities with nonnegative variables.

Theorem 4.1. *Let $x^* \in \mathbb{R}^n$, where $x^* = (x_1^*, \ldots, x_n^*)^T$. Denote*

$$G = \{k_i | i = 1, \cdots, t,\ x_{k_i}^* = 0\}.$$

Then x^* is a strongly optimal solution to (4) if and only if it is a strongly feasible solution to (4), and for each $h_i \in \{\pm 1\},\ i = 1, 2, \cdots, n - t$, the linear system

$$\begin{cases} \underline{A}x \le 0, \\ \overline{A}x \ge 0, \\ x_{k_i} \ge 0,\ k_i \in G, \\ \sum_{i=1}^{t}(\underline{c} + \underline{Q}x^*)_{k_i}x_{k_i} + \\ \sum_{i=t+1}^{n}[(c_c + Q_cx^*)_{k_i} + h_{i-t}(c_\Delta + Q_\Delta x^*)_{k_i}]x_{k_i} < 0 \end{cases} \tag{19}$$

has no solution.

Proof. "Only if": Let x^* be a strongly optimal solution to (4), then x^* is a strongly feasible solution to (4), and for each $A \in \mathbf{A},\ b \in \mathbf{b},\ Q \in \mathbf{Q}$ and $c \in \mathbf{c}$, x^* is an optimal solution to CQP problem

$$\min\ \frac{1}{2}x^TQx + c^Tx \tag{20a}$$

$$\text{s.t.}\ Ax = b,\ x \ge 0. \tag{20b}$$

Clearly, the feasible direction to the feasible region of (20) at x^* reads

$$\begin{cases} Ax = 0, \\ x_{k_i} \geq 0, \quad k_i \in G, \end{cases}$$

and from the discussion in "Only if" part of Theorem 3.1, we have the following linear system

$$\begin{cases} Ax = 0, \\ x_{k_i} \geq 0, \quad k_i \in G, \\ (c + Qx^*)^T x < 0 \end{cases} \tag{21}$$

has no solution. Obviously, for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $Q \in \mathbf{Q}$ and $c \in \mathbf{c}$, the system (21) which has no solution is equivalent to the conditions that

$$\begin{cases} \mathbf{A}x = 0, \\ x_{k_i} \geq 0 \quad k_i \in G, \\ (\mathbf{c} + \mathbf{Q}x^*)^T x < 0 \end{cases} \tag{22}$$

has no weak solution. Moreover, from Theorem 2.2, the interval linear system (22) has no weak solution if and only if for each $s \in \{\pm 1\}^n$, the following system

$$\begin{cases} (A_c - A_\Delta T_s)x \leq 0, \\ -(A_c + A_\Delta T_s)x \leq 0, \\ x_{k_i} \geq 0 \quad k_i \in G, \\ ((c_c + Q_c x^*)^T - (c_\Delta + Q_\Delta x^*)^T T_s)x < 0, \end{cases} \tag{23}$$

has no solution.

By a contradiction, if there exists some $h_i \in \{\pm 1\}$, $i = 1, 2, \cdots, n-t$, such that the linear system (19) has a solution $\hat{x}$, which implies

$$\begin{cases} \underline{A}\hat{x} \leq 0, \\ \overline{A}\hat{x} \geq 0, \\ \hat{x}_{k_i} \geq 0, \ k_i \in G, \\ \sum_{i=1}^t (\underline{c} + \underline{Q}x^*)_{k_i} \hat{x}_{k_i} + \\ \sum_{i=t+1}^n [(c_c + Q_c x^*)_{k_i} + h_{i-t}(c_\Delta + Q_\Delta x^*)_{k_i}]\hat{x}_{k_i} < 0, \end{cases} \tag{24}$$

Let $s_{k_i} = 1$, $i = 1, 2, \cdots, t$; $s_{k_i} = \operatorname{sgn} \hat{x}_{k_i}$, $i = t+1, \cdots, n$, obviously, $s \in \{\pm 1\}^n$. From Lemma 2.1, we have

$$\begin{aligned} (A_c - A_\Delta T_s)\hat{x} &= \sum_{i=1}^n (A_c - A_\Delta T_s)_{\cdot,k_i} \hat{x}_{k_i} \\ &= \sum_{i=1}^t [(A_c)_{\cdot,k_i} - (A_\Delta)_{\cdot,k_i} s_{k_i}]\hat{x}_{k_i} + \sum_{i=t+1}^n [(A_c)_{\cdot,k_i} - (A_\Delta)_{\cdot,k_i} s_{k_i}]\hat{x}_{k_i} \\ &= \sum_{i=1}^t \underline{A}_{\cdot,k_i} \hat{x}_{k_i} + \sum_{i=t+1}^n (A_{es})_{\cdot,k_i} \hat{x}_{k_i} \leq \sum_{i=1}^t \underline{A}_{\cdot,k_i} \hat{x}_{k_i} + \sum_{i=t+1}^n \underline{A}_{\cdot,k_i} \hat{x}_{k_i} \\ &= \underline{A}\hat{x}. \end{aligned}$$

Similarly, there holds

$$(A_c + A_\Delta T_s)\hat{x} \geq \overline{A}\hat{x}.$$

Thus, from formula (24), there holds

$$\begin{cases} (A_c - A_\Delta T_s)\hat{x} \leq 0, \\ (A_c + A_\Delta T_s)\hat{x} \geq 0. \end{cases}$$

Meanwhile,

$$\begin{aligned}
& [(c_c+Q_cx^*)^T-(c_\Delta+Q_\Delta x^*)^T T_s]\hat{x} \\
= & [(c_c+Q_cx^*)-T_s(c_\Delta+Q_\Delta x^*)]^T\hat{x} \\
= & \sum_{i=1}^{n}[(c_c+Q_cx^*)_{k_i}-s_{k_i}(c_\Delta+Q_\Delta x^*)_{k_i}]\hat{x}_{k_i} \qquad (25)\\
= & \sum_{i=1}^{t}[(c_c+Q_cx^*)_{k_i}-s_{k_i}(c_\Delta+Q_\Delta x^*)_{k_i}]\hat{x}_{k_i}+\sum_{i=t+1}^{n}[(c_c+Q_cx^*)_{k_i}-s_{k_i}(c_\Delta+Q_\Delta x^*)_{k_i}]\hat{x}_{k_i} \\
= & \sum_{i=1}^{t}(\underline{c}+\underline{Q}x^*)_{k_i}\hat{x}_{k_i}+\sum_{i=t+1}^{n}[(c_c+Q_cx^*)_{k_i}-s_{k_i}(c_\Delta+Q_\Delta x^*)_{k_i}]\hat{x}_{k_i}.
\end{aligned}$$

Moreover, for each vector $\hat{x}\in\mathbb{R}^n$, we know $|\hat{x}|=(\text{sgn }\hat{x})\cdot\hat{x}$, then

$$\begin{aligned}
& \sum_{i=t+1}^{n}[(c_c+Q_cx^*)_{k_i}-s_{k_i}(c_\Delta+Q_\Delta x^*)_{k_i}]\hat{x}_{k_i} \\
= & \sum_{i=t+1}^{n}[(c_c+Q_cx^*)_{k_i}\hat{x}_{k_i}-|\hat{x}_{k_i}|(c_\Delta+Q_\Delta x^*)_{k_i}] \\
\le & \sum_{i=t+1}^{n}[(c_c+Q_cx^*)_{k_i}\hat{x}_{k_i}+h_{i-t}\hat{x}_{k_i}(c_\Delta+Q_\Delta x^*)_{k_i}] \qquad (26)\\
= & \sum_{i=t+1}^{n}[(c_c+Q_cx^*)_{k_i}+h_{i-t}(c_\Delta+Q_\Delta x^*)_{k_i}]\hat{x}_{k_i}
\end{aligned}$$

where $h_{i-t}\in\{\pm1\}$. Thus, together with (24) (25) and (26), there holds

$$\begin{aligned}
& [(c_c+Q_cx^*)^T-(c_\Delta+Q_\Delta x^*)^T T_s]\hat{x} \\
\le & \sum_{i=1}^{t}[(\underline{c}+\underline{Q}x^*)_{k_i}]\hat{x}_{k_i}+\sum_{i=t+1}^{n}[(c_c+Q_cx^*)_{k_i}+h_{i-t}(c_\Delta+Q_\Delta x^*)_{k_i}]\hat{x}_{k_i}<0.
\end{aligned}$$

Which implies $\hat{x}$ satisfies (23), for some $s\in\{\pm1\}^n$. This is a contradiction, thus, the linear system (19) has no solution, for each $h_i\in\{\pm1\},\ i=1,2,\cdots,n-t$.

"If": From the "Only if" part in Theorem 4.1, we know that x^* is a strongly optimal solution to (4) if and only if x^* is a strongly feasible solution to (4) and for each $s\in\{\pm1\}^n$, the linear system (23) has no solution.

Assume that x^* is a strongly feasible solution to (4), and for each $h_i\in\{\pm1\},\ i=1,2,\cdots,n-t$ the system (19) has no solution, we only need to prove (23) has no solution for each $s\in\{\pm1\}^n$, in this part.

By a contradiction, if there exists $s\in\{\pm1\}^n$ such that the linear system (23) has a solution $\hat{x}$, which implies

$$\begin{cases}
(A_c-A_\Delta T_s)\hat{x}\le 0,\\
-(A_c+A_\Delta T_s)\hat{x}\le 0,\\
\hat{x}_{k_i}\ge 0 \quad k_i\in G,\\
((c_c+Q_cx^*)^T-(c_\Delta+Q_\Delta x^*)^T T_s)\hat{x}<0.
\end{cases} \qquad (27)$$

Meanwhile, from Theorem 2.3, we know

$$A_\Delta x^*=0,$$

since x^* is a strongly feasible solution to (4). Then we have $(A_\Delta)_{\cdot,j} = 0$ if $x_j^* \neq 0$, due to $A_\Delta \geq 0$ and $x^* \geq 0$. Because $x_{k_i}^* > 0,\ k_i \notin G$, hence, $(A_\Delta)_{\cdot,k_i} = 0,\ k_i \notin G$. Thus, we have

$$\underline{A}_{\cdot,k_i} = \overline{A}_{\cdot,k_i} = (A_c)_{\cdot,k_i},\ i = t+1, \cdots, n, \tag{28}$$

meanwhile,

$$\hat{x}_{k_i} \geq 0,\ i = 1, \cdots, t. \tag{29}$$

Then from (28) (29) and combing with (27) , there holds

$$\begin{aligned} \underline{A}\hat{x} &= \sum_{i=1}^{t} \underline{A}_{\cdot,k_i}\hat{x}_{k_i} + \sum_{i=t+1}^{n} (A_c)_{\cdot,k_i}\hat{x}_{k_i} \\ &\leq \sum_{i=1}^{t}(A_c - A_\Delta T_s)_{\cdot,k_i}\hat{x}_{k_i} + \sum_{i=t+1}^{n} (A_c)_{\cdot,k_i}\hat{x}_{k_i} = (A_c - A_\Delta T_s)\hat{x} \leq 0. \end{aligned}$$

Similarly, we have

$$\overline{A}\hat{x} \geq (A_c + A_\Delta T_s)\hat{x} \geq 0$$

Moreover, due to $x^* \geq 0$ and $\hat{x}_{k_i} \geq 0,\ i = 1, 2, \cdots, t$, then there holds

$$\begin{aligned} &[(c_c + Q_c x^*)^T - (c_\Delta + Q_\Delta x^*)^T T_s]\hat{x} \\ = &\sum_{i=1}^{t}[(c_c + Q_c x^*)_{k_i} - s_{k_i}(c_\Delta + Q_\Delta x^*)_{k_i}]\hat{x}_{k_i} + \sum_{i=t+1}^{n}[(c_c + Q_c x^*)_{k_i} - s_{k_i}(c_\Delta + Q_\Delta x^*)_{k_i}]\hat{x}_{k_i} \\ \geq &\sum_{i=1}^{t}[(\underline{c} + \underline{Q}x^*)_{k_i}]\hat{x}_{k_i} + \sum_{i=t+1}^{n}[(c_c + Q_c x^*)_{k_i} - s_{k_i}(c_\Delta + Q_\Delta x^*)_{k_i}]\hat{x}_{k_i}. \end{aligned}$$

Let $s_{k_i} = -h_{i-t},\ i = t+1, \cdots, n$, then from formula (27) we have

$$\sum_{i=1}^{t}[(\underline{c} + \underline{Q}x^*)_{k_i}]\hat{x}_{k_i} + \sum_{i=t+1}^{n}[(c_c + Q_c x^*)_{k_i} + h_{i-t}(c_\Delta + Q_\Delta x^*)_{k_i}]\hat{x}_{k_i} < 0,$$

which implies $\hat{x}$ satisfies (19) for some $h_i \in \{\pm 1\},\ i = 1, \cdots, n-t$. This is a contradiction, thus, for each $s \in \{\pm 1\}^n$ the interval linear system (23) has no solution. Hence, x^* is a strongly optimal solution to (4). □

Obviously, from Theorem 4.1, we can checking the optimality of a strongly feasible solution for IvCQP (4) by 2^{n-t} linear systems. In the similar manner, we can prove the optimality of a given vector for the IvCQP (11).

Theorem 4.2. *Let $x^* \in \mathbb{R}^n$. Denote*

$$F = \{r_j | j = 1, \cdots, q,\ (A_{eh})_{r_j,\cdot} x^* = \overline{b}_{r_j}\},$$

where $h = sgn\ x^*$. Then x^* is a strongly optimal solution to (11) *if and only if it is a strongly feasible solution to* (11)*, and for each $s \in \{\pm 1\}^n$, the system*

$$\begin{cases} (A_{es})_{r_j,\cdot} x \leq 0,\ r_j \in F, \\ (c + Qx^*)_{es}^T x < 0 \end{cases} \tag{30}$$

has no solution.

Proof. "Only if": Let x^* be a strongly optimal solution to (11), then x^* is a strongly feasible solution to (11), and for each $A \in \mathbf{A},\ Q \in \mathbf{Q}$ and $c \in \mathbf{c}$, x^* is an optimal solution to

CQP problem

$$\min \frac{1}{2}x^TQx + c^Tx \tag{31a}$$

$$\text{s.t. } Ax \leq \bar{b}. \tag{31b}$$

Specially, consider $A \in \mathbf{A}$, where A is defined as

$$A_{r_j,\cdot} = \begin{cases} A_{r_j,\cdot} \in \mathbf{A}_{r_j,\cdot}, & r_j \in F \\ (A_{eh})_{r_j,\cdot}, & r_j \in \{1,\ 2, \cdots, n\} \backslash F. \end{cases} \tag{32}$$

Let the set

$$F' = \{r_j' | j = 1, \cdots, t,\ A_{r_j',\cdot} x^* = \bar{b}_{r_j'}\},$$

the feasible direction to the feasible region of (31) at x^* reads

$$A_{r_j',\cdot} x \leq 0,\ r_j' \in F'.$$

Clearly, from Lemma 2.1, we are easy to obtain that for each $r_j \in F$

$$\bar{b}_{r_j} = (A_{eh})_{r_j,\cdot} x^* \leq A_{r_j,\cdot} x^* \leq \bar{b}_{r_j},$$

obviously,

$$A_{r_j,\cdot} x^* = \bar{b}_{r_j}.$$

Thus, $r_j \in F'$, which implies $F \subseteq F'$.

Now, we prove $F' \backslash F$ is empty by a contradiction. For some $r_j' \in F' \backslash F$, from the formula (32), we have

$$A_{r_j',\cdot} = (A_{eh})_{r_j',\cdot},$$

then

$$(A_{eh})_{r_j',\cdot} x^* = A_{r_j',\cdot} x^* = \bar{b}_{r_j'}.$$

Equivalently, $r_j' \in F$, a contradiction. Thus, we have $F = F'$.

Note that x^* is an optimal solution to (31), then from the discussion in "Only if" part of Theorem 3.2, we know that the system

$$\begin{cases} A_{r_j',\cdot} x \leq 0,\ r_j' \in F' \\ (c + Qx^*)^T x < 0. \end{cases}$$

has no solution. Thus, for each $A_{r_j,\cdot} \in \mathbf{A}_{r_j,\cdot},\ r_j \in F$ and $Q \in \mathbf{Q},\ c \in \mathbf{c}$, the system

$$\begin{cases} A_{r_j,\cdot} x \leq 0,\ r_j \in F \\ (c + Qx^*)^T x < 0. \end{cases} \tag{33}$$

has no solution that is nothing to do with $A_{r_j,\cdot} \in \mathbf{A}_{r_j,\cdot},\ r_j \notin F$, due to $F = F'$. Hence, for each $A \in \mathbf{A},\ c \in \mathbf{c},\ Q \in \mathbf{Q}$, the linear system (33) has no solution, obviously, it is equivalent to the conditions that

$$\begin{cases} \mathbf{A}_{r_j,\cdot} x \leq 0,\ r_j \in F \\ (\mathbf{c} + \mathbf{Q}x^*)^T x < 0. \end{cases}$$

has no weak solution. Thus, from Theorem 2.2, the interval linear system has no weak solution if and only if for each $s \in \{\pm 1\}^n$, the following system

$$\begin{cases} (A_c - A_\Delta T_s)_{r_j,\cdot} x \leq 0,\ r_j \in F \\ ((c_c + Q_c x^*)^T - (c_\Delta + Q_\Delta x^*)^T T_s) x < 0. \end{cases}$$

has no solution, which implies (30) holds.

"If": Let x^* be a strongly feasible solution to (11), then for each $A \in \mathbf{A},\ b \in \mathbf{b},\ Q \in \mathbf{Q},\ c \in$

$\mathbf{c}$, x^* is a feasible solution to CQP problem

$$\min\ \frac{1}{2}x^TQx + c^Tx \tag{34a}$$

$$\text{s.t. } Ax \le b. \tag{34b}$$

Then the feasible direction to the feasible region of (34) at x^* reads

$$A_{r'_j,\cdot}x \le 0,\ r'_j \in F',$$

where

$$F' = \{r'_j | j = 1, \cdots, m,\ A_{r'_j,\cdot}x^* = b_{r'_j}\}.$$

Similarly, from Lemma 3.1, we are easy to obtain that for each $r_j \in F$, then

$$b_{r_j} \le \overline{b}_{r_j} = (A_{eh})_{r_j,\cdot}x^* \le A_{r_j,\cdot}x^* \le b_{r_j},$$

which implies

$$A_{r_j,\cdot}x^* = b_{r_j}.$$

Clearly, we know $r_j \in F'$, thus, $F \subseteq F'$.

Because the linear system (30) has no solution for each $s \in \{\pm 1\}^n$, the linear system (33) has no solution for each $A \in \mathbf{A},\ b \in \mathbf{b},\ Q \in \mathbf{Q},\ c \in \mathbf{c}$ by Theorem 2.2. Moreover, we are easy to obtain that for each $A,\ b,\ Q,\ c$, the linear system

$$\begin{cases} A_{r'_j,\cdot}x \le 0,\ r'_j \in F' \\ (c + Qx^*)^Tx < 0 \end{cases}$$

has no solution since $F \subseteq F'$. Hence, from Theorem 2.1, the x^* is an optimal solution to CQP (34), which means x^* is a strongly optimal solution to problem (11). □

Theorem 4.3. *Let $x^* \in \mathbb{R}^n$, where $x^* = (x_1^*, \ldots, x_n^*)^T$. Denote $F = \{r_j | j = 1, \cdots, q,\ \underline{A}_{r_j,\cdot}x^* = \overline{b}_{r_j}\}$ and $G = \{k_i | i = 1, \cdots, t,\ x^*_{k_i} = 0\}$. Then x^* is a strongly optimal solution to* (17) *if and only if x^* is a strongly feasible solution to (17), and for each $h_i \in \{\pm 1\},\ i = 1, 2, \cdots, n - t$, the system*

$$\begin{cases} \underline{A}_{r_j,\cdot}x \le 0,\ r_j \in F, \\ x_{k_i} \ge 0,\ k_i \in G, \\ \sum_{i=1}^{t}(\underline{c} + \underline{Q}x^*)_{k_i}x_{k_i} + \\ \sum_{i=t+1}^{n}[(c_c + Q_cx^*)_{k_i} + h_{i-t}(c_\Delta + Q_\Delta x^*)_{k_i}]x_{k_i} < 0, \end{cases} \tag{35}$$

has no solution.

Proof: The proof is similar to that of the Theorem 4.1 and Theorem 4.2 and is thus omitted here. □

§5 Illustrative examples

In this section, we present two examples to illustrate the methods proposed in Section 3 and 4.

Example 1 Consider the IvCQP problem

$$\begin{aligned} min\ & \frac{1}{2}x_1^2 + 2x_2^2 + [-1, 0]x_1x_2 + [-3, 0]x_1 + [0, 1]x_2 \\ s.t.\ & [0, 1]x_1 + [1, 3]x_2 \le [1, 2], \\ & x_1, x_2 \ge 0. \end{aligned} \tag{36}$$

Let $x^* = (2,0)^T$. Clearly, we are easy to obtain that x^* is a weakly feasible solution, and $A = (1,\ 2)$ is a feasible matrix of the x^*.

Now construct the corresponding system (18). Note that the set $F = \{r_j | j = 1, \cdots, q,\ A_{r_j,}.x^* \geq \underline{b}_{r_j}\} = \{1\}$ and $G = \{k_i | i = 1, \cdots, t,\ x^*_{k_i} = 0\} = \{2\}$ in this example. Thus, the linear system (18) is

$$\begin{cases} (1,\ 2)(x_1 - x_2) \leq 0, \\ (x_1 - x_2)_2 \geq 0, \\ (\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix})^T x_1 - (\begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix})^T x_2 < 0, \\ x_1,\ x_2 \geq 0. \end{cases}$$

where $x_1 = \begin{pmatrix} x_1^1 \\ x_1^2 \end{pmatrix}$, $x_2 = \begin{pmatrix} x_2^1 \\ x_2^2 \end{pmatrix}$. The linear system above can be written as

$$\begin{cases} x_1^1 + 2x_1^2 - x_2^1 - 2x_2^2 \leq 0, \\ x_1^2 - x_2^2 \geq 0, \\ 2x_1^1 + x_1^2 + x_2^1 + 2x_2^2 < 0, \\ x_1^1, x_1^2, x_2^1, x_2^2 \geq 0. \end{cases} \tag{37}$$

Obviously the third inequality in the system (41) contradicts with the nonnegativity of variables $x_1^1,\ x_1^2,\ x_2^1,\ x_2^2$. Hence, linear system (37) has no solution. From Theorem 3.3 we know that x^* is a weak optimal solution to (36).

Moreover, we choose $A = (1,\ 2)$, $b = 2$, $c = (-2,\ 1)^T$ and $Q = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 4 \end{pmatrix}$. Then $x^* = (2,0)^T$ is optimal for the scenario

$$\begin{aligned} &min\ \frac{1}{2}x_1^2 + 2x_2^2 - 0.5x_1x_2 - 2x_1 + x_2 \\ &s.t.\ \ x_1 + 2x_2 \leq 2, \\ &\qquad x_1, x_2 \geq 0. \end{aligned}$$

Example 2 Consider the IvLP problem

$$\begin{aligned} min\ &\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_3^2 + [-1,0]x_1x_3 + x_2x_3 - 3x_1 + [-1,0]x_2 + x_3 \\ s.t.\ \ &x_1 + x_2 + [0,2]x_3 = 2, \\ &2x_1 - x_2 + [1,3]x_3 = 1, \\ &x_1, x_2, x_3 \geq 0. \end{aligned} \tag{38}$$

Let $x^* = (1,\ 1,\ 0)^T$. Clearly, from Theorem 2.3 we know that x^* is a strongly feasible solution to (38).

Now construct the corresponding system (19). Note that the set $G = \{k_i | i = 1, \cdots, t,\ x^*_{k_i} = 0\} = \{3\}$ in this example. Thus, the linear system (19) is

$$\begin{cases} \underline{A}x \leq 0, \\ \overline{A}x \geq 0, \\ x_3 \geq 0, \\ (\underline{c} + \underline{Q}x^*)_3 x_3 + [(c_c + Q_c x^*)_1 + h_1(c_\Delta + Q_\Delta x^*)_1]x_1 \\ \quad + [(c_c + Q_c x^*)_2 + h_2(c_\Delta + Q_\Delta x^*)_2]x_2 < 0, \end{cases} \tag{39}$$

where $h_1,\ h_2 \in \{\pm 1\}$, $x = (x_1,\ x_2,\ x_3)^T$. Note that $(c_\Delta + Q_\Delta x^*)_1 = 0$ and hence $h_1(c_\Delta + Q_\Delta x^*)_1 = 0$, so we only discuss the cases $h_2 = \pm 1$. When $h_2 = 1$, the linear system (39) can be written as

$$\begin{cases} x_1 + x_2 \leq 0, \\ 2x_1 - x_2 + x_3 \leq 0, \\ x_1 + x_2 + 2x_3 \geq 0, \\ 2x_1 - x_2 + 3x_3 \geq 0, \\ x_3 \geq 0, \\ x_3 - 2x_1 + x_2 < 0. \end{cases} \tag{40}$$

Adding the second inequality and the sixth inequality in the system (40), we have $2x_3 < 0$, which contradicts with the nonnegativity of variable x_3. Hence, the linear system (40) has no solution. When $h_2 = -1$, we can similarly find that the corresponding system has no solution. From Theorem 4.1 we know that x^* is a strongly optimal solution to (38).

§6 Conclusion

This paper derives various methods to check weak and strong optimality of a given vector to IvCQP with three canonical forms. All methods of three decision problems of interval convex quadratic programming are established separately, since the equivalent transformation between the interval linear systems is generally impossible due to dependency. Apart from these three canonical forms of IvCQP, there are some other types of IvCQP, for example the general problem can be formulated by using equations, inequalities or both. Moreover, by using sign-restricted variables, the most general model with variables unrestricted in sign has been considered. The methodology of this paper can be applicable to make a generalization of the solution concepts for these interval convex quadratic programming problems.

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