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Log-logistic parameters estimation using moving extremes ranked set sampling design

HE Xiao-fang CHEN Wang-xue* YANG Rui

Abstract. In statistical parameter estimation problems, how well the parameters are estimated largely depends on the sampling design used. In the current paper, a modification of ranked set sampling (RSS) called moving extremes RSS (MERSS) is considered for the estimation of the scale and shape parameters for the log-logistic distribution. Several traditional estimators and ad hoc estimators will be studied under MERSS. The estimators under MERSS are compared to the corresponding ones under SRS. The simulation results show that the estimators under MERSS are significantly more efficient than the ones under SRS.

§1 Introduction

Cost effective sampling is a problem of major concern in some experiments especially when the measurement of the characteristic of interest is costly or painful or time consuming. The method of ranked set sampling (RSS) provides an effective way to achieve observational economy in terms of precision achieved per unit of sampling. Initially the concept of RSS was introduced by McIntyre (1952) as a process of increasing the precision of the sample mean as an estimator of population mean. Ranking can be performed based on expert judgment, visual inspection or any means that does not involve actually quantifying the observations. In RSS one first draws n^2 units at random from the population and partitions them into n sets of n units. The n units in each set are ranked without making actual measurements. From the first set of n units the unit ranked lowest is chosen for actual quantification. From the second set of n units the unit ranked second lowest is measured. The process is continued until the unit ranked largest is

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*Corresponding author.

measured from the $n-$th set of n units. If a larger sample size is required then the procedure can be repeated k times.

Takahasi and Wakimoto (1968) established a very important statistical foundation for the theory of RSS. Later estimation of parameters of various commonly used distributions has been carried out using RSS (for details see Stokes (1995), Al-Saleh et al. (2003), Chen et al. (2004), Al-Saleh et al. (2009), Abu-Dayyeh et al. (2013), Chen et al. (2017), Chen et al. (2018) and Qian et al. (2019)). However, ranking accuracy affects the efficiency of the estimator. When the set size is large, ranking error tends to occur. In order to reduce the error of ranking and keep optimality inherited in the original RSS procedure, Al-Odat et al. (2001) introduced the concept of varied set size RSS, which is coined here as moving extremes RSS (MERSS).

The procedure of MERSS is described as follows:

1. Select n simple random samples of sizes $1, 2, 3, \cdots, n$, respectively.

2. Order the elements of each set by visual inspection or by some other cheap method, without actual measurement of the characteristic of interest.

3. Measure accurately the maximum ordered observation from the first set, then the second set, ..., the last set.

4. Step (3) is repeated on another n sets of size $1, 2, 3, \cdots, n$, respectively, however the minimum ordered observations are measured instead of the maximum ordered observations.

5. If needed, this process can be replicated k times (cycles).

Clearly, only the two extreme values are used in MERSS, maximum or minimum of sets of varied size, whereas the ranks of all the elements of each set are needed in RSS. Since it is not difficult to identify maximum or minimum units, MERSS is a very useful modification of RSS. It allows for an increase of set size without introducing too many ranking errors.

AL-Saleh et al. (2003a) studied maximum likelihood estimator (MLE) of location parameter for normal distribution based on MERSS. AL-Saleh et al. (2003b) considered the MLE of the mean of exponential distribution using MERSS and they showed that MLE of MERSS is always performed better than simple random sampling (SRS) numerically. Abu-Dayyeh et al. (2009) studied the modified MLE of the mean of the exponential distribution under MERSS. Al-Hadhrami et al. (2012) studied the Bayes estimators of the population mean of the normal distribution using MERSS and investigated its properties. For further introduction of MERSS refer to Chen et al. (2013), Chen et al. (2016) and Chen et al. (2019).

A random variable X is said to have a log-logistic distribution with the scale parameter α and the shape parameter β if its distribution function is given by

$$F(x) = \frac{x^\beta}{x^\beta + \alpha^\beta}, \tag{1}$$

where $x > 0$, $\alpha > 0$ and $\beta > 0$. The probability density function (pdf) corresponding to the distribution function in (1) is then given by

$$f(x) = \frac{\beta\alpha^\beta x^{\beta-1}}{(x^\beta + \alpha^\beta)^2}.$$

We write $LLD(\alpha,\ \beta)$ to denote the distribution as defined in (1). The applications of log-logistic distribution are well known in wealth or income (see Fisk (1961)), hydrology for modelling stream flow rates and precipitation (see Shoukri et al. (1988)) and engineer of survival analysis (see Ashkar et al. (2003)). For further details on the importance and applications of a log-logistic distribution one may refer to Bennett (1983), Ahmad et al. (1988), Robson et al. (1999) and Geskus (2001).

Parameter estimation problems for the log-logistic distribution have been discussed by many authors. Among recent literature, Balakrishnan et al. (1987) studied the best linear unbiased estimator (BLUE) of the scale parameter of a log-logistic distribution under SRS. Chen (2006) discussed about the interval estimation for the shape parameter of the log-logistic distribution under SRS. Lesitha et al. (2013) provided an unbiased estimator and BLUE of the scale parameter of a log-logistic distribution under RSS. Further, inference on the parameters of the log-logistic distribution has been studied by many authors using SRS including Tiku et al. (1992), Gupta et al. (1999), Kus et al. (2006), Abbas et al. (2016) and He et al. (2020).

In this paper, we consider several traditional estimators and ad hoc estimators of the scale and shape parameters α and β from $LLD\,(\alpha,\beta)$ based on MERSS. In Sect. 2, we study an unbiased estimator or modified unbiased estimator and BLUE or modified BLUE of α and β in case when one parameter is known and ad hoc estimators in case when both parameters are unknown. In Sect. 3, we consider the MLEs of the parameters of this distribution. The relative efficiencies of all estimators are simulated and the conclusions will be presented in Sect. 4.

§2 Several types of estimators

In this section, we deal with several types of estimators of α and β of the $LLD\,(\alpha,\beta)$ under MERSS:

(i) An unbiased estimator and BLUE of α defined from $LLD\,(\alpha,\beta)$ in which β is known,

(ii) A modified unbiased estimator and modified BLUE of β when α is known and

(iii) Ad hoc estimators of α and β when α and β are both unknown.

2.1 Unbiased estimator and BLUE of α when β is known

Let $\{x_1,x_2,x_3,\cdots,x_{2n}\}$ be a simple random sample of size $2n$ from (1) in which β is known. Then the pdf of $\frac{x_i}{\alpha}$ is

$$f^1\,(x)=\frac{\beta x^{\beta-1}}{\left(1+x^{\beta}\right)^2}.$$

Let

$$E(\frac{x_i}{\alpha})=\gamma. \tag{2}$$

Then we have

$$E\left(\frac{x_i}{\gamma}\right)=\alpha. \tag{3}$$

Thus the BLUE of α under SRS is given by

$$\hat{\alpha}_{SRS,\ BLUE}=\frac{1}{2n}\sum_{i=1}^{2n}\frac{x_i}{\gamma}. \tag{4}$$

Let $\{x_{11},x_{22},x_{33},\cdots,x_{nn},y_{11},y_{12},y_{13},\cdots,y_{1n}\}$ be a moving extremes ranked set sample of size $2n$ from (1) in which β is known. Then $\frac{x_{ii}}{\alpha}$ has the same density as the ith order statistic of an SRS of size i from $f^1(x)$(see David(1981)), i.e. the pdf of $\frac{x_{ii}}{\alpha}$ is

$$f_{ii}^1(x) = \frac{i\beta x^{i\beta-1}}{(1+x^\beta)^{i+1}}.$$

Also $\frac{y_{1i}}{\alpha}$ has the same density as the first order statistic of an SRS of size i from $f^1(x)$(see David(1981)), i.e. the pdf of $\frac{y_{1i}}{\alpha}$ is

$$f_{1i}^1(x) = \frac{i\beta x^{\beta-1}}{(1+x^\beta)^{i+1}}.$$

Let

$$E\left(\frac{x_{ii}}{\alpha}\right) = \gamma_{ii} \tag{5}$$

with

$$Var\left(\frac{x_{ii}}{\alpha}\right) = \sigma_{ii}^2 \tag{6}$$

and

$$E\left(\frac{y_{1i}}{\alpha}\right) = \gamma_{1i} \tag{7}$$

with

$$Var\left(\frac{y_{1i}}{\alpha}\right) = \sigma_{1i}^2. \tag{8}$$

From (5)-(8), it can be seen that

$$E\left(\frac{x_{ii}}{\gamma_{ii}}\right) = \alpha \tag{9}$$

with

$$Var\left(\frac{x_{ii}}{\gamma_{ii}}\right) = \frac{\alpha^2\sigma_{ii}^2}{\gamma_{ii}^2} \tag{10}$$

and

$$E\left(\frac{y_{1i}}{\gamma_{1i}}\right) = \alpha \tag{11}$$

with

$$Var\left(\frac{y_{1i}}{\gamma_{1i}}\right) = \frac{\alpha^2\sigma_{1i}^2}{\gamma_{1i}^2}. \tag{12}$$

Thus an UE of α under MERSS is given by

$$\hat{\alpha}_{MERSS,\ UE} = \frac{1}{2n}\sum_{i=1}^{n}\left(\frac{x_{ii}}{\gamma_{ii}} + \frac{y_{1i}}{\gamma_{1i}}\right). \tag{13}$$

According to the lemma (see Casella et al., 2002, p.338), combine (9)-(11) with (12), we have

$$\hat{\alpha}_{MERSS,\ BLUE} = \frac{1}{2}\left(\sum_{i=1}^{n}\frac{\gamma_{1i}^2\gamma_{ii}^2}{\gamma_{1i}^2\sigma_{ii}^2+\gamma_{ii}^2\sigma_{1i}^2}\right)^{-1}\sum_{i=1}^{n}\frac{\gamma_{1i}\gamma_{ii}\left(x_{ii}\gamma_{1i}+y_{1i}\gamma_{ii}\right)}{\gamma_{1i}^2\sigma_{ii}^2+\gamma_{ii}^2\sigma_{1i}^2}. \tag{14}$$

2.2 Modified unbiased estimator and modified BLUE of β when α is known

Let $\{x_1, x_2, x_3, \cdots, x_{2n}\}$ be a simple random sample of size $2n$ from (1) in which α is known. Then the pdf of $\beta ln\frac{x_i}{\alpha}$ is

$$f^2(x) = \frac{e^x}{(1+e^x)^2}.$$

Note that $E\left(\beta ln\frac{x_i}{\alpha}\right) = 0$. Hence we consider the estimators of β based on order statistics $\beta ln\frac{x_{(1)}}{\alpha} \le \beta ln\frac{x_{(2)}}{\alpha} \cdots \le \beta ln\frac{x_{(2n)}}{\alpha}$.

Let

$$E\left(\beta ln\frac{x_{(i)}}{\alpha}\right) = \xi_{(i)}, \tag{15}$$

then it can be seen that

$$E\left(\frac{ln\frac{x_{(i)}}{\alpha}}{\xi_{(i)}}\right)=\frac{1}{\beta}.$$

Thus the BLUE of $\frac{1}{\beta}$ under SRS is given by

$$\frac{1}{2n}\sum_{i=1}^{2n}\frac{ln\frac{x_{(i)}}{\alpha}}{\xi_{(i)}}.$$

Then we suggest the following estimator of β

$$\hat{\beta}_{SRS,\ MBLUE}=2n\left(\sum_{i=1}^{2n}\frac{ln\frac{x_{(i)}}{\alpha}}{\xi_{(i)}}\right)^{-1}, \tag{16}$$

which will be called the modified unbiased estimator of β.

Let $\{x_{22},x_{33},x_{44},\cdots,x_{n+1n+1},y_{12},y_{13},y_{14},\cdots,y_{1n+1}\}$ be a moving extremes ranked set sample of size $2n$ from $LLD(\alpha,\beta)$ with α is known. Then $\beta ln\frac{x_{ii}}{\alpha}$ has the same density as the ith order statistic of an SRS of size i from $f^2(x)$, i.e. the pdf of $\beta ln\frac{x_{ii}}{\alpha}$ is

$$f_{ii}^2\left(x\right)=\frac{ie^{ix}}{\left(1+e^x\right)^{i+1}}.$$

Also $\beta ln\frac{y_{1i}}{\alpha}$ has the same density as the first order statistic of an SRS of size i from $f^2(x)$, i.e. the pdf of $\beta ln\frac{y_{1i}}{\alpha}$ is

$$f_{1i}^2\left(x\right)=\frac{ie^{x}}{\left(1+e^x\right)^{i+1}}.$$

Let

$$E\left(\beta ln\frac{x_{ii}}{\alpha}\right)=\xi_{ii} \tag{17}$$

with

$$Var\left(\beta ln\frac{x_{ii}}{\alpha}\right)=\delta_{ii}^2 \tag{18}$$

and

$$E\left(\beta ln\frac{y_{1i}}{\alpha}\right)=\xi_{1i} \tag{19}$$

with

$$Var\left(\beta ln\frac{y_{1i}}{\alpha}\right)=\delta_{1i}^2, \tag{20}$$

Then we have

$$E\left(\frac{ln\frac{x_{ii}}{\alpha}}{\xi_{ii}}\right)=\frac{1}{\beta} \tag{21}$$

with

$$Var\left(\frac{ln\frac{x_{ii}}{\alpha}}{\xi_{ii}}\right)=\frac{\delta_{ii}^2}{\xi_{ii}^2\beta^2} \tag{22}$$

and

$$E\left(\frac{ln\frac{y_{1i}}{\alpha}}{\xi_{1i}}\right)=\frac{1}{\beta} \tag{23}$$

with

$$Var\left(\frac{ln\frac{y_{1i}}{\alpha}}{\xi_{1i}}\right)=\frac{\delta_{1i}^2}{\xi_{1i}^2\beta^2}. \tag{24}$$

Thus an unbiased estimator of $\frac{1}{\beta}$ under MERSS is given by

$$\frac{1}{2n}\sum_{i=2}^{n+1}\left(\frac{ln\frac{y_{1i}}{\alpha}}{\xi_{1i}}+\frac{ln\frac{x_{ii}}{\alpha}}{\xi_{ii}}\right).$$

Then we suggest the following estimator of β

$$\hat{\beta}_{MERSS,\ MUE}=2n\left(\sum_{i=2}^{n+1}\frac{ln\frac{y_{1i}}{\alpha}}{\xi_{1i}}+\frac{ln\frac{x_{ii}}{\alpha}}{\xi_{ii}}\right)^{-1}, \tag{25}$$

which will be called the modified unbiased estimator of β. According to the lemma (see Casella et al., 2002, p.338), combine(21)-(23) with (24), we can obtain the BLUE of $\frac{1}{\beta}$ under MERSS

$$-\frac{1}{2}\sum_{i=2}^{n+1}\frac{\left(\xi_{ii}ln\frac{y_{1i}}{\alpha}+\xi_{1i}ln\frac{x_{ii}}{\alpha}\right)\xi_{1i}\xi_{ii}}{\xi_{1i}^2\delta_{ii}^2+\xi_{ii}^2\delta_{1i}^2}\left(\sum_{i=2}^{n+1}\frac{\xi_{1i}^2\xi_{ii}^2}{\xi_{1i}^2\delta_{ii}^2+\xi_{ii}^2\delta_{1i}^2}\right)^{-1}.$$

Then we suggest the following estimator of β

$$\hat{\beta}_{MERSS,\ MBLUE}=-2\sum_{i=2}^{n+1}\frac{\xi_{1i}^2\xi_{ii}^2}{\xi_{1i}^2\delta_{ii}^2+\xi_{ii}^2\delta_{1i}^2}\left[\sum_{i=2}^{n+1}\frac{\left(\xi_{ii}ln\frac{y_{1i}}{\alpha}+\xi_{1i}ln\frac{x_{ii}}{\alpha}\right)\xi_{1i}\xi_{ii}}{\xi_{1i}^2\delta_{ii}^2+\xi_{ii}^2\delta_{1i}^2}\right]^{-1}, \tag{26}$$

which will be called the modified BLUE of β.

2.3 Ad hoc estimators of α and β

If $Z=lnX$, then the pdf of Z

$$f_3(z)=\frac{e^{-((z-\theta)/\lambda)}}{\lambda\left[1+e^{-((z-\theta)/\lambda)}\right]^2}, \tag{27}$$

where $\theta=ln\alpha$ and $\lambda=\frac{1}{\beta}$.

Let $\{z_1,z_2,z_3,\cdots,z_{2n}\}$ be a simple random sample of size $2n$ from (27) and

$$U_{(i)}=\frac{z_{(i)}-\theta}{\lambda},$$

where $z_{(1)}\le z_{(2)}\le\cdots\le z_{(2n)}$. Denote $a_i=E\left(U_{(i)}\right)$ and $v_{ij}=\text{cov}\left(U_{(i)},U_{(j)}\right)$. The most well known estimators of location parameter θ and scale parameter λ using the order statistics, are the BLUEs (Arnold et al., 1992 and Balakrishnan et al., 1992) which can be written as

$$\hat{\theta}_{SRS}=-a'CW$$

and

$$\hat{\lambda}_{SRS}=l_{2n}'CW,$$

where $a=\begin{pmatrix}a_1\\ \vdots\\ a_{2n}\end{pmatrix}$, $l_{2n}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$, $V=(v_{ij})_{2n\times 2n}$, $W=\begin{pmatrix}z_{(1)}\\ \vdots\\ z_{(2n)}\end{pmatrix}$,

$C=\frac{V^{-1}\left(l_{2n}a'-al_{2n}'\right)V^{-1}}{d}$ and $d=\left(l_{2n}'V^{-1}l_{2n}\right)\left(a'V^{-1}a\right)-\left(l_{2n}'V^{-1}a\right)^2$.

Then we suggest the following estimators of α and β

$$\tilde{\alpha}_{SRS,\ MBLUE}=e^{-a'CW} \tag{28}$$

and

$$\tilde{\beta}_{SRS,\ MBLUE}=\frac{1}{l_{2n}'CW} \tag{29}$$

which will be called the modified BLUE of α and β, respectively.

Let $\{z_{11},z_{22},z_{33},\cdots,z_{nn},w_{11},w_{12},w_{13},\cdots,w_{1n}\}$ be a moving extremes ranked set sample of size $2n$ from (27). Denote its order statistics as $z_{(1)}^*\le z_{(2)}^*\le...\le z_{(2n)}^*$. The following ad hoc estimators of α and β are the same as the estimators under SRS, expect that the statistics under SRS are replaced with their counterparts under MERSS. Denote

$$U^*_{(i)} = \frac{z^*_{(i)} - \theta}{\lambda},$$

$a^*_i = E\left(U^*_{(i)}\right)$ and $v^*_{ij} = \text{cov}\left(U^*_{(i)}, U^*_{(j)}\right)$.

Then we can obtain ad hoc estimators of α and β

$$\tilde{\alpha}_{MERSS,\ AHE} = e^{-a^{*'} C^* W^*} \tag{30}$$

and

$$\tilde{\beta}_{MERSS,\ AHE} = \frac{1}{l'_{2n} C^* W^*}, \tag{31}$$

where $a^* = \begin{pmatrix} a^*_1 \\ \vdots \\ a^*_{2n} \end{pmatrix}$, $V^* = \left(v^*_{ij}\right)_{2n \times 2n}$, $W^* = \begin{pmatrix} z^*_{(1)} \\ \vdots \\ z^*_{(2n)} \end{pmatrix}$, $C^* = \dfrac{V^{*-1}\left(l_{2n} a^{*'} - a^* l'_{2n}\right) V^{*-1}}{d}$

and $d^* = \left(l'_{2n} V^{*-1} l_{2n}\right)\left(a^{*'} V^{*-1} a^*\right) - \left(l'_{2n} V^{*-1} a^*\right)^2$.

§3 MLEs

In this section, we consider MLEs of the parameters of $LLD(\alpha, \beta)$ under MERSS. Under some regularity conditions, the asymptotic efficiency of the MLEs can be obtained from the inverse of the Fisher information matrix.

The fisher information matrix for α and β under SRS

$$I_{SRS}(\alpha, \beta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix} = \begin{pmatrix} \frac{2n\beta^2}{3\alpha^2} & 0 \\ 0 & \frac{2n(3+\pi^2)}{9\beta^2} \end{pmatrix} \tag{32}$$

is given by Reath et al. (2018).

Let $\{x_{11}, x_{22}, x_{33}, \cdots, x_{nn}, y_{11}, y_{12}, y_{13}, \cdots, y_{1n}\}$ be a moving extremes ranked set sample of size $2n$ from $LLD(\alpha, \beta)$, then the pdfs of x_{ii} and $y_{1i} (i = 1, 2, ..., m)$ are respectively

$$f_{ii}(x) = \frac{i\beta \left(\frac{x}{\alpha}\right)^{i\beta - 1}}{\alpha \left[1 + \left(\frac{x_{ii}}{\alpha}\right)^{\beta}\right]^{i+1}}$$

and

$$f_{1i}(x) = \frac{i\beta \left(\frac{x}{\alpha}\right)^{\beta - 1}}{\alpha \left[1 + \left(\frac{x}{\alpha}\right)^{\beta}\right]^{i+1}}.$$

In order to get the MLEs, we start with the likelihood function

$$\begin{aligned} L_{MERSS}(\alpha, \beta) &= \prod_{i=1}^{n} f_{ii}(x_{ii}) f_{1i}(y_{1i}) \\ &= \prod_{i=1}^{n} \frac{i\beta \left(\frac{x_{ii}}{\alpha}\right)^{i\beta - 1}}{\alpha \left[1 + \left(\frac{x_{ii}}{\alpha}\right)^{\beta}\right]^{i+1}} \frac{i\beta \left(\frac{y_{1i}}{\alpha}\right)^{\beta - 1}}{\alpha \left[1 + \left(\frac{y_{1i}}{\alpha}\right)^{\beta}\right]^{i+1}}. \end{aligned}$$

The log-likelihood function is

$$lnL_{MERSS} = C + 2nln\beta - 2nln\alpha + \sum_{i=1}^{n}(i\beta-1)ln\frac{x_{ii}}{\alpha} + \sum_{i=1}^{n}(\beta-1)ln\frac{y_{1i}}{\alpha}$$
$$-\sum_{i=1}^{n}(i+1)ln\left[1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}\right] - \sum_{i=1}^{n}(i+1)ln\left[1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}\right],$$

where C is a constant. Then we have

$$\frac{\partial lnL_{MERSS}}{\partial\alpha} = -\frac{\beta}{\alpha}\left[\frac{n(n+3)}{2} - \sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}}{1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}} - \sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}}{1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}}\right] \tag{33}$$

and

$$\frac{\partial lnL_{MERSS}}{\partial\beta} = \frac{2n}{\beta} + \sum_{i=1}^{n}iln\frac{x_{ii}}{\alpha} + \sum_{i=1}^{n}ln\frac{y_{1i}}{\alpha} - \sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}ln\frac{x_{ii}}{\alpha}}{1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}} - \sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}ln\frac{y_{1i}}{\alpha}}{1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}}. \tag{34}$$

The second-order derivative of α and β for the lnL_{MERSS} are computed as

$$\frac{\partial^2 lnL_{MERSS}}{\partial\alpha^2} = -\frac{\beta}{\alpha^2}\left[\frac{n(n+3)}{2} - \sum_{i=1}^{n}\frac{(i+1)(\frac{x_{ii}}{\alpha})^{\beta}}{1+(\frac{x_{ii}}{\alpha})^{\beta}} - \sum_{i=1}^{n}\frac{(i+1)(\frac{y_{1i}}{\alpha})^{\beta}}{1+(\frac{y_{1i}}{\alpha})^{\beta}}\right]$$
$$-\frac{\beta^2}{\alpha^2}\left[\sum_{i=1}^{n}\frac{(i+1)(\frac{x_{ii}}{\alpha})^{\beta}}{\left(1+(\frac{x_{ii}}{\alpha})^{\beta}\right)^2} + \sum_{i=1}^{n}\frac{(i+1)(\frac{y_{1i}}{\alpha})^{\beta}}{\left(1+(\frac{y_{1i}}{\alpha})^{\beta}\right)^2}\right], \tag{35}$$

$$\frac{\partial^2 L_{MERSS}}{\partial\beta^2} = -\frac{2n}{\beta^2} - \sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}\left(ln\frac{x_{ii}}{\alpha}\right)^2}{\left[1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}\right]^2} - \sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}\left(ln\frac{y_{1i}}{\alpha}\right)^2}{\left[1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}\right]^2} \tag{36}$$

and

$$\frac{\partial^2 lnL_{MERSS}}{\partial\alpha\partial\beta} = -\frac{n(n+3)}{2\alpha} + \frac{1}{\alpha}\sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}}{1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}} + \frac{1}{\alpha}\sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}}{1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}}$$
$$+\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}ln\frac{x_{ii}}{\alpha}}{\left(1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}\right)^2} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}ln\frac{y_{1i}}{\alpha}}{\left(1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}\right)^2} \tag{37}$$

respectively. Thus we have

$$I_{11}^{*} = -E\left(\frac{\partial^2 lnL_{MERSS}}{\partial\alpha^2}\right)$$
$$= \frac{n(n+3)\beta}{2\alpha^2} - \frac{\beta^2}{\alpha^3}\sum_{i=1}^{n}\int_0^{\infty}\frac{i(i+1)\left(\frac{x}{\alpha}\right)^{i\beta+\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{i+2}}dx - \frac{\beta^2}{\alpha^3}\sum_{i=1}^{n}\int_0^{\infty}\frac{i(i+1)\left(\frac{x}{\alpha}\right)^{2\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{i+2}}dx$$
$$+\frac{\beta^3}{\alpha^3}\sum_{i=1}^{n}\int_0^{\infty}\frac{i(i+1)\left(\frac{x}{\alpha}\right)^{i\beta+\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{i+3}}dx + \frac{\beta^3}{\alpha^3}\sum_{i=1}^{n}\int_0^{\infty}\frac{i(i+1)\left(\frac{x}{\alpha}\right)^{2\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{i+3}}dx \tag{38}$$
$$= \frac{n(n+3)\beta}{2\alpha^2} - \frac{\beta}{\alpha^2}\sum_{i=1}^{n}\int_0^{\infty}\frac{i(i+1)t^i}{(1+t)^{i+2}}dt - \frac{\beta}{\alpha^2}\sum_{i=1}^{n}\int_0^{\infty}\frac{i(i+1)t}{(1+t)^{i+2}}dt$$
$$+\frac{\beta^2}{\alpha^2}\sum_{i=1}^{n}\int_0^{\infty}\frac{i(i+1)t^i}{(1+t)^{i+3}}dt + \frac{\beta^2}{\alpha^2}\sum_{i=1}^{n}\int_0^{\infty}\frac{i(i+1)t}{(1+t)^{i+3}}dt$$

$$
\begin{aligned}
&= \frac{n(n+3)\beta}{2\alpha^2} - \frac{n(n+1)\beta}{2\alpha^2} - \frac{n\beta}{\alpha^2} + \frac{2\beta^2}{\alpha^2}\sum_{i=1}^{n}(1-\frac{2}{i+2}) \\
&= \frac{2\beta^2}{\alpha^2}\sum_{i=1}^{n}(1-\frac{2}{i+2}),
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
I_{22}^{*} &= E\left\{\frac{2n}{\beta^2} + \sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}\left(ln\frac{x_{ii}}{\alpha}\right)^2}{\left[1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}\right]^2} + \sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}\left(ln\frac{y_{1i}}{\alpha}\right)^2}{\left[1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}\right]^2}\right\} \\
&= \frac{2n}{\beta^2} + \frac{1}{\beta^2}E\left[\sum_{i=1}^{n}\frac{(i+1)h_{ii}ln^2h_{ii}}{(1+h_{ii})^2} + \sum_{i=1}^{n}\frac{(i+1)s_{1i}ln^2s_{1i}}{(1+s_{1i})^2}\right]
\end{aligned} \tag{40}
$$

and

$$
\begin{aligned}
I_{12}^{*} &= -E\left[-\frac{n(n+3)}{2\alpha} + \frac{1}{\alpha}\sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}}{1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}} + \frac{1}{\alpha}\sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}}{1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}}\right. \\
&\left. +\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}ln\frac{x_{ii}}{\alpha}}{\left(1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}\right)^2} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}ln\frac{y_{1i}}{\alpha}}{\left(1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}\right)^2}\right] \\
&= \frac{n(n+3)}{2\alpha} - \frac{n(n+1)}{2\alpha} - \frac{n}{\alpha} - \frac{\beta}{\alpha}E\left[\sum_{i=1}^{n}\frac{(i+1)\left(\frac{x_{ii}}{\alpha}\right)^{\beta}ln\frac{x_{ii}}{\alpha}}{\left(1+\left(\frac{x_{ii}}{\alpha}\right)^{\beta}\right)^2} + \sum_{i=1}^{n}\frac{(i+1)\left(\frac{y_{1i}}{\alpha}\right)^{\beta}ln\frac{y_{1i}}{\alpha}}{\left(1+\left(\frac{y_{1i}}{\alpha}\right)^{\beta}\right)^2}\right] \\
&= -\frac{1}{\alpha}E\left[\sum_{i=1}^{n}\frac{(i+1)h_{ii}lnh_{ii}}{(1+h_{ii})^2} + \sum_{i=1}^{n}\frac{(i+1)s_{1i}lns_{1i}}{(1+s_{1i})^2}\right],
\end{aligned} \tag{41}
$$

where s_{1i} is 1th order statistics and h_{ii} is maximum order statistics of a moving extremes ranked set sample from $LLD(1,1)$. Combining (38), (39) with (40), we have

$$
I_{MERSS}(\alpha,\beta) = \begin{pmatrix} I_{11}^{*} & I_{12}^{*} \\ I_{12}^{*} & I_{22}^{*} \end{pmatrix} \tag{42}
$$

§4 Numerical comparison

In this section, we will compare the relative efficiencies of the above estimators in Sect.2 and Sect.3.

The efficiency of $\hat{\alpha}_{MERSS,UE}$ with respect to (w.r.t.) $\hat{\alpha}_{SRS,\ BLUE}$ is

$$
eff^1 = \frac{MSE\left(\hat{\alpha}_{SRS,\ BLUE}\right)}{MSE\left(\hat{\alpha}_{MERSS,\ UE}\right)}, \tag{43}
$$

where MSE is an abbreviation of the mean square error. Similarly, denote $\hat{\alpha}_{MERSS,\ MBLUE}$ w.r.t. $\hat{\alpha}_{SRS,\ BLUE}$, $\hat{\beta}_{MERSS,\ MUE}$ w.r.t. $\hat{\beta}_{SRS,\ MBLUE}$, $\hat{\beta}_{MERSS,\ MBLUE}$ w.r.t. $\hat{\beta}_{SRS,\ MBLUE}$, $\tilde{\alpha}_{MERSS,\ AHE}$ w.r.t. $\tilde{\alpha}_{SRS,\ MBLUE}$ and $\tilde{\beta}_{MERSS,\ AHE}$ w.r.t. $\tilde{\beta}_{SRS,\ MBLUE}$ as eff^2, eff^3, eff^4, eff^5 and eff^6, respectively. It can be seen that eff^i (i=3,4) are free of α.

The asymptotic efficiencies $\hat{\alpha}_{MERSS,\ MLE}$ w.r.t. $\hat{\alpha}_{SRS,\ MLE}$, $\hat{\beta}_{MERSS,\ MLE}$ w.r.t. $\hat{\beta}_{SRS,\ MLE}$

and $\left(\hat{\alpha}_{MERSS,\ MLE},\ \hat{\beta}_{MERSS,\ MLE}\right)$ w.r.t. $\left(\hat{\alpha}_{SRS,\ MLE},\ \hat{\beta}_{SRS,\ MLE}\right)$ are respectively

$$aeff^7 = \frac{I_{11}^*}{I_{11}} = \frac{3}{n}\sum_{i=1}^{n}\left(1-\frac{2}{i+2}\right), \tag{44}$$

$$aeff^8 = \frac{I_{22}^*}{I_{22}} = \frac{9}{2n\left(3+\pi^2\right)}\left[2n + E\left(\sum_{i=1}^{n}\frac{(i+1)h_{ii}ln^2h_{ii}}{\left(1+h_{ii}\right)^2} + \sum_{i=1}^{n}\frac{(i+1)s_{1i}ln^2s_{1i}}{\left(1+s_{1i}\right)^2}\right)\right] \tag{45}$$

and

$$aeff^9 = \frac{|I_{MERSS}(\alpha,\beta)|}{|I_{SRS}(\alpha,\beta)|}. \tag{46}$$

It can be seen that $aeff^i$ (i=7, 8, 9) are free of α and β and $aeff^7 > 1$ for $n > 2$.

From Tables 1-4, we conclude the following:

(1) $eff^1 > 1$, which means $\hat{\alpha}_{MERSS,\ UE}$ is more efficient $\hat{\alpha}_{SRS,\ BLUE}$.

(2) $eff^2 > 1$, which means $\hat{\alpha}_{MERSS,\ BLUE}$ is more efficient $\hat{\alpha}_{SRS,\ BLUE}$.

(3) Comparing eff^1 with eff^2, we conclude that $\hat{\alpha}_{MERSS,\ BLUE}$ is more efficient than $\hat{\alpha}_{MERSS,\ UE}$.

(4) $eff^3 > 1$, which means $\hat{\beta}_{MERSS,\ MUE}$ is more efficient $\hat{\beta}_{SRS,\ MBLUE}$.

(5) $eff^4 > 1$, which means $\hat{\beta}_{MERSS,\ MBLUE}$ is more efficient $\hat{\beta}_{SRS,\ MBLUE}$.

(6) Comparing eff^3 with eff^4, we conclude that $\hat{\beta}_{MERSS,\ MBLUE}$ is more efficient than $\hat{\beta}_{MERSS,\ MUE}$.

(7) $eff^5 > 1$, which means $\tilde{\alpha}_{MERSS,\ AHE}$ is more efficient $\tilde{\alpha}_{MERSS,\ MBLUE}$.

(8) $eff^6 > 1$, which means $\tilde{\beta}_{MERSS,\ AHE}$ is more efficient $\tilde{\beta}_{MERSS,\ MBLUE}$.

(9) In conclusion, the estimators of α and β under MERSS are more efficient than that of α and β under SRS in Sect.2.

From Tables 5-7, we conclude the following:

(10) $aeff^i(i = 7, 8, 9) > 1$ and they are increase as n increase.

(11) $aeff^7 > 1$, which means the MLE of α under MERSS is more efficient than the MLE of α under SRS.

(12) $aeff^8 > 1$, which means the MLE of β under MERSS is more efficient than the MLE of β under SRS.

(13) $aeff^9 > 1$, which means the MLEs of α and β under MERSS is more efficient than the MLE of α and β under SRS.

(14) In conclusion, the MLEs of α and β under MERSS are more efficient than that of α and β under SRS in Sect.3.

(15) In conclusion, the MERSS is more efficient than SRS in estimating the scale and shape parameters of the log-logistic distribution.

Table 1. The efficiency of $\hat{\alpha}_{MERSS,\ UE}$ w.r.t. $\hat{\alpha}_{SRS,\ BLUE}$ and $\hat{\alpha}_{MERSS,\ BLUE}$ w.r.t. $\hat{\alpha}_{SRS,\ BLUE}$.

$(\alpha,\ \beta)$	n	eff^1	eff^2
(1, 3)	6	1.71345	1.84017
	7	1.75349	1.93052
	8	1.65878	1.75611
(2, 3)	6	1.82032	1.84635
	7	1.53121	1.63641
	8	1.70422	1.74200
(3, 2)	6	4.08173	4.91487
	7	2.66697	2.72753
	8	1.71008	1.73732

Table 2. The efficiency of $\hat{\beta}_{MERSS,\ MUE}$ w.r.t.$\hat{\beta}_{SRS,\ MBLUE}$ and $\hat{\beta}_{MERSS,\ MBLUE}$ w.r.t.$\hat{\beta}_{SRS,\ MBLUE}$.

β	n	eff^3	eff^4
3	6	1.18180	2.91918
	7	1.61235	3.65413
	8	1.97523	3.91049
4	6	1.14583	2.78112
	7	1.62455	3.25586
	8	1.90814	3.66518
5	6	1.20770	2.83044
	7	1.46545	3.18749
	8	1.84847	3.56819

Table 3. The efficiency of $\tilde{\alpha}_{MERSS,\ AHE}$ w.r.t. $\tilde{\alpha}_{SRS,\ MBLUE}$.

$(\alpha,\ \beta)$	n	eff^5
(1, 3)	6	1.68089
	7	1.73293
	8	1.95351
(3, 2)	6	7.73763
	7	8.79398
	8	11.44288
(3, 3)	6	3.08259
	7	2.56829
	8	3.77084

Table 4. The efficiency of $\tilde{\beta}_{MERSS,\ AHE}$ w.r.t. $\tilde{\beta}_{SRS,\ MBLUE}$.

$(\alpha,\ \beta)$	n	eff^6
(1, 3)	6	1.27842
	7	1.68150
	8	1.83427
(3, 2)	6	3.08428
	7	3.64138
	8	2.94402
(3, 3)	6	4.55372
	7	2.90449
	8	3.53873

Table 5. The asymptotic efficiency $\hat{\alpha}_{MERSS,\ MLE}$ w.r.t. $\hat{\alpha}_{SRS,\ MLE}$.

n	$\frac{\alpha^2}{\beta^2}I_{11}$	$\frac{\alpha^2}{\beta^2}I_{11}^*$	$aeff^7$
2	1.3333	1.6667	1.25000
3	2.0000	2.8667	1.43335
4	2.6667	4.2000	1.57498
5	3.3333	5.6286	1.68855
6	4.0000	7.1286	1.78215
7	4.6667	8.6841	1.86087
8	5.3333	10.2841	1.92828
9	6.0000	11.9205	1.98675

Table 6. The asymptotic efficiency $\hat{\beta}_{MERSS,\ MLE}$ w.r.t. $\hat{\beta}_{SRS,\ MLE}$.

n	$\beta^2 I_{22}$	$\beta^2 I_{22}^*$	$aeff^8$
2	5.7198	6.1550	1.07608
3	8.5797	10.1005	1.17725
4	11.4396	14.8545	1.29851
5	14.2996	20.3281	1.42187
6	17.1595	26.7176	1.55702
7	20.0194	33.8705	1.69183
8	22.8793	41.7207	1.82351
9	25.7392	50.5267	1.96302

Table 7. The asymptotic efficiency $\left(\hat{\alpha}_{MERSS,\ MLE},\ \hat{\beta}_{MERSS,\ MLE}\right)$ w.r.t. $\left(\hat{\alpha}_{SRS,\ MLE},\ \hat{\beta}_{SRS,\ MLE.}\right)$

n	$\alpha^2 \lvert I_{SRS}(\alpha,\ \beta)\rvert$	$\alpha^2 \lvert I_{MERSS}(\alpha,\ \beta)\rvert$	$aeff^9$
2	7.6264	10.2586	1.34514
3	17.1595	28.9966	1.68982
4	30.5057	62.1831	2.03841
5	47.6652	114.5466	2.40315
6	68.6378	190.3164	2.77276
7	93.4237	294.2395	3.14951
8	122.0229	429.8509	3.52271
9	154.4352	601.5848	3.89539

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Department of Mathematics and Statistics, Jishou University, Jishou 416000, China.
Email: chenwangxue2017@163.com