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Numerical solutions of two-dimensional nonlinear integral equations via Laguerre Wavelet method with convergence analysis

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Abstract. In this paper, the approximate solutions for two different type of two-dimensional nonlinear integral equations: two-dimensional nonlinear Volterra-Fredholm integral equations and the nonlinear mixed Volterra-Fredholm integral equations are obtained using the Laguerre wavelet method. To do this, these two-dimensional nonlinear integral equations are transformed into a system of nonlinear algebraic equations in matrix form. By solving these systems, unknown coefficients are obtained. Also, some theorems are proved for convergence analysis. Some numerical examples are presented and results are compared with the analytical solution to demonstrate the validity and applicability of the proposed method.

§1 Introduction

Various problems in plasma physics, electrical engineering, electromagnetic analysis, the Spatio-temporal development of an epidemic, theory of parabolic initial boundary value problems, physical phenomena [17, 31–36, 40, 41], population dynamics, and Fourier problems (see e.g [7, 10, 11, 38, 42, 43] arise to the two-dimensional nonlinear integral equation.

There are several motivations for studying the numerical solution of two-dimensional integral equations such as triangular functions [3], block-by-block method [23], rationalized Harr function [1, 2] and block pulse functions [24] , Legendre polynomials [37], Tau method [14], reproducing kernel method [9], Hybrid function method [20], hybrid of block-pulse and parabolic functions [25], a new collocation method [26], two-dimensional orthonormal Bernstein polynomials [27], Bernoulli wavelet method [28], piecewise linear functions [29] and two-dimensional delta basis functions [30].

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The purpose of the present paper concerns two topics. One of the our intentions is to obtain the numerical solution of the two-dimensional nonlinear Volterra-Fredholm integral equations

$$u(x,t)-\lambda_1\int_0^t\int_0^x G_1(x,t,y,z,u(y,z))\mathtt{d}y\mathtt{d}z-\lambda_2\int_0^1\int_0^1 G_2(x,t,y,z,u(y,z))\mathtt{d}y\mathtt{d}z=f(x,t) \quad (1)$$

where $u(x,t)$ is an unknown function on some region $\Omega:=[0,1]\times[0,1]$ and

$$X=\{(x,t,y,z,u(y,z))|0\le y\le x\le 1,\quad 0\le z\le t\le 1\}.$$

The functions $f(x,t)$, $G_1(x,t,y,z,u(y,z))$ and $G_2(x,t,y,z,u(y,z))$ are assumed to be given smooth real valued function on Ω and λ_1, λ_2 are given real constants.

The other aim is to attain the numerical solution of the two-dimensional nonlinear mixed Volterra-Fredholm integral equation

$$u(x,t)=f(x,t)+\lambda\int_0^x\int_\Omega G(x,t,y,z,u(y,z))\mathtt{d}z\mathtt{d}y,\qquad (x,t)\in[0,1]\times\Omega \quad (2)$$

where $u(x,t)$ is the unknown in $D=[0,1]\times\Omega$, where Ω is a closed subset of $\mathbb{R}^n$,$n=1,2,3$. The functions $f(x,t)$ and $G(x,t,y,z,u)$ are given functions in D.

In general,it is not easy to derive the analytical solutions to most of the nonlinear mixed Volterra-Fredholm integral equation. Therefore, it is vital to develop some reliable and efficient techniques to solve equation (1) such as two-dimensional triangular functions [18], two-dimensional Legendre wavelets method [4], Adomian decomposition series [19, 44], the trapezoidal Nystrom and Euler Nystrom method [12, 13], He's variational iteration method [46], homotopy perturbation method [45] and two-dimensional block-pulse functions [21].

In this study, two-dimensional Laguerre wavelet is introduced and the numerical solutions for equation (1) and equation (2) are computed by two-dimensional Laguerre wavelet method. Thus, this paper is organized as follows. In Section 2, the basic definition and properties of Laguerre polynomial are described. The two-dimensional Laguerre wavelets have constructed a base on the Laguerre wavelet in Section 3. In Section 4 how the Laguerre wavelet method can be used to reduce equation (1) and Eq.(2) to systems of nonlinear algebraic equations are explained. Convergence analysis is discussed in Section 5. In Section 6, we apply the described method to the solution of the Darboux problem. In Section 7, we present some numerical examples which show the efficiency and accuracy of the proposed method. Finally, we give the main conclusions of this study in Section 8.

§2 Laguerre Polynomials

For any $\alpha>-1$, the Laguerre polynomials $L_k^{(\alpha)}(t)$, $k\ge 0$, are the eigenfunctions of the singular Sturm-Liouville problem in $(0,+\infty)$

$$\left(t^{\alpha+1}e^{-t}\left(L_k^{(\alpha)}(t)\right)'\right)'+kt^\alpha e^{-t}L_k^{(\alpha)}(t)=0$$

They are orthogonal in $(0,+\infty)$ with respect to the weight $w(t)=t^\alpha e^{-t}$ and

$$\int_0^\infty L_k^{(\alpha)}(t)L_m^{(\alpha)}(t)t^\alpha e^{-t}\mathtt{d}t=\Gamma(\alpha+1)\begin{pmatrix}k+\alpha\\ k\end{pmatrix}\delta_{km},\qquad k,m\ge 0.$$

The Laguerre polynomial satisfy the recursion relation

$$L_{k+1}^{(\alpha)}(t) = (2k+\alpha+1-t)L_k^{(\alpha)}(t) - (k+\alpha)L_{k-1}^{(\alpha)}(t),$$

where $L_0^{(\alpha)}(t) = 1$ and $L_1^{(\alpha)}(t) = \alpha+1-t$. In the particular case $\alpha = 0$, the polynomial $L_k(t) = L_k^{(0)}(t)$ satisfy $L_k(0) = 1$ and are orthonormal in $(0, +\infty)$ [5]. There is well-known a classical global uniform estimates given by [22, 39]

$$|L_n^{\alpha}(t)| \leq \frac{(\alpha+1)n}{n!} e^{\frac{t}{2}}, \qquad \alpha \geq 0, \quad t \geq 0, \quad n = 0, 1, 2, ... \tag{3}$$

In this paper, we assume that $\alpha = 0$, therefore

$$|L_n(t)| \leq \frac{1}{(n-1)!} e^{\frac{t}{2}}, \qquad t \geq 0, \quad n = 0, 1, 2, ... \tag{4}$$

§3 Wavelets and Laguerre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet $\psi(t)$. They are defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\Big(\frac{t-b}{a}\Big), \qquad a, b \in \mathbb{R},$$

where a is dilation parameter and b is a translation parameter.

The Laguerre wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments, defined on interval $[0, 1)$ by:

$$\psi_{n,m}(t) = \begin{cases} \frac{2^{\frac{k}{2}}}{m!} L_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ \\ 0, & \text{elsewhere.} \end{cases} \tag{5}$$

where $k \in \mathbb{Z}^+$, $n = 1, 2, 3, ..., 2^{k-1}$ and $m = 0, 1, ..., M-1$ is the order of the Laguerre polynomials and M is a fixed positive integer [15].

The two-dimensional Laguerre wavelets are defined as

$$\psi_{n,i,l,j}(x,t) = \begin{cases} \frac{2^{\frac{k_1+k_2}{2}}}{i!j!} \; L_i(2^{k_1}x - 2n + 1) \; L_j(2^{k_2}t - 2l + 1), & \frac{n-1}{2^{k_1-1}} \leq x < \frac{n}{2^{k_1-1}}, \; \frac{l-1}{2^{k_2-1}} \leq t < \frac{l}{2^{k_2-1}}, \\ \\ 0, & \text{elsewhere.} \end{cases} \tag{6}$$

where $n = 1, 2, ..., 2^{k_1-1}$, $l = 1, 2, ..., 2^{k_2-1}$, k_1 and k_2 are any positive integers, i and j are the order of the Laguerre polynomials .

3.1 Function approximation by Laguerre wavelets

A function $u(x, t)$ defined over $[0, 1) \times [0, 1)$ can be expanded in terms of Laguerre wavelets as

$$u(x,t) = \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} c_{n,i,l,j} \psi_{n,i,l,j}(x,t). \tag{7}$$

If the infinite series in equation (7) is truncated, then it can be written as:

$$u_{k,M}(x,t)=\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)=C^T\ \Psi(x,t), \tag{8}$$

where $\Psi(x,t)$ is $(2^{k_1-1}2^{k_2-1}M_1M_2\ \times\ 1)$ matrix, given by

$$\begin{aligned}\Psi(x,t)=&[\Psi_{1,0,1,0}(x,t),\Psi_{1,0,1,1}(x,t),...,\Psi_{1,0,1,M_2-1}(x,t),...,\Psi_{1,0,2^{k_2-1},M_2-1}(x,t),...,\\&\Psi_{2^{k_1-1},M_1-1,2^{k_2-1},M_2-1}(x,t)].\end{aligned}$$

Also, C is $(2^{k_1-1}2^{k_2-1}M_1M_2\ \times\ 1)$ matrix whose elements can be calculated from the formula

$$c_{n,i,l,j}=\int_0^1\int_0^1\psi_{n,i}(x)\psi_{l,j}(t)u(x,t)\mathtt{d}x\mathtt{d}t,$$

and

$$C\ =\ [c_{1,0,1,0},c_{1,0,1,1},...,c_{1,0,1,M_2-1},...,c_{1,0,2^{k_2-1},M_2-1},...,c_{2^{k_1-1},M_1-1,2^{k_2-1},M_2-1}]^T$$

§4 Two-dimensional Nonlinear Integral Equations

In this section, the numerical solutions of two different kind of two-dimensional nonlinear integral equation are obtained by the two-dimensional Laguerre wavelets method.

4.1 Two-dimensional Nonlinear Volterra-Fredholm Integral Equation-s

Consider the following nonlinear integral equation

$$u(x,t)-\lambda_1\int_0^t\int_0^x G_1(x,t,y,z,u(y,z))\mathtt{d}y\mathtt{d}z-\lambda_2\int_0^1\int_0^1 G_2(x,t,y,z,u(y,z))\mathtt{d}y\mathtt{d}z=f(x,t) \tag{9}$$

where $u(x,t)$ is an unknown function on some region $\Omega:=[0,1]\times[0,1]$ and $X=\{(x,t,y,z,u(y,z))|\ 0\le y\le x\le 1, 0\le z\le t\le 1\}$. The functions $f(x,t)$, $G_1(x,t,y,z,u(y,z))$ and $G_2(x,t,y,z,u(y,z))$ are assumed to be given smooth real valued function on Ω and λ_1, λ_2 are given real constants.

For solving the above problem (9), by attention to equation (8) we first expand $u(x,t)$ by the two-dimensional Laguerre wavelets as

$$u_{k,M}(x,t)=\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)=C^T\ \Psi(x,t) \tag{10}$$

where the coefficients $c_{n,i,l,j}$ are unknown. Then from equations (9) and (10) we have

$$\begin{aligned}&u_{k,M}(x,t)-\lambda_1\int_0^t\int_0^x G_1(x,t,y,z,u_{k,M}(y,z))dydz-\lambda_2\int_0^1\int_0^1 G_2(x,t,y,z,u_{k,M}(y,z)dydz\\&=f(x,t)\end{aligned} \tag{11}$$

Let (x_i,t_j) be the set of $2^{k_1-1}M_1\times 2^{k_2-1}M_2$ zero point of the shifted Chebyshev polynomial in $[0,1]$. Now, we collocate equation (11) at (x_i,t_j) as

$$u_{k,M}(x_i,t_j)-\lambda_1\int_0^{t_i}\int_0^{x_j}G_1(x_i,t_j,y,z,u_{k,M}(y,z))dydz$$
$$+\lambda_2\int_0^1\int_0^1 G_2(x_i,t_j,y,z,u_{k,M}(y,z)dydz=f(x_i,t_j) \tag{12}$$

Gauss quadrature formulas will be used to compute the integral terms in equation (12). For this purpose, we transfer the y-intervals and the z-intervals into $[-1,1]$ by means of the transformations

$$\begin{aligned}
\tau_1 &= \frac{2}{x_i}y-1, \quad \Rightarrow y=\frac{x_i}{2}(\tau_1+1), \quad y\in[0,x_i],\\
\tau_2 &= \frac{2}{t_j}z-1, \quad \Rightarrow y=\frac{t_j}{2}(\tau_2+1), \quad z\in[0,t_j],\\
\eta_1 &= 2y-1, \quad \Rightarrow y=\frac{1}{2}(\eta_1+1), \quad y\in[0,1],\\
\eta_2 &= 2z-1, \quad \Rightarrow y=\frac{1}{2}(\eta_2+1), \quad z\in[0,1].
\end{aligned}$$

So equation (12) converts to

$$u_{k,M}(x_i,t_j)-\frac{\lambda_1x_it_j}{4}\int_{-1}^1\int_{-1}^1 G_1(x_i,t_j,\frac{x_i}{2}(\tau_1+1),\frac{t_j}{2}(\tau_2+1),u_{k,M}(\frac{x_i}{2}(\tau_1+1),\frac{t_j}{2}(\tau_2+1)))d\tau_1$$
$$d\tau_2-\frac{\lambda_2}{4}\int_{-1}^1\int_{-1}^1 G_2(x_i,t_j,\frac{1}{2}(\eta_1+1),\frac{T_2}{2}(\eta_2+1),u_{k,M}(\frac{T_1}{2}(\eta_1+1),\frac{1}{2}(\eta_2+1))d\eta_1d\eta_2=f(x_i,t_j)$$

Using the Gauss quadrature formula, we estimate the integrals and gets

$$u_{k,M}(x_i,t_j)-\frac{\lambda_1x_it_j}{4}\sum_{p=1}^{r_1}\sum_{q=1}^{r_2}\omega_p\omega_qG_1(x_i,t_j,\frac{x_i}{2}(\tau_q+1),\frac{t_j}{2}(\tau_p+1),u_{k,M}(\frac{x_i}{2}(\tau_q+1),\frac{t_j}{2}(\tau_p+1)))$$
$$-\frac{\lambda_2}{4}\sum_{p=1}^{r_1}\sum_{q=1}^{r_2}\omega_p\omega_qG_2(x_i,t_j,\frac{1}{2}(\eta_q+1),\frac{1}{2}(\eta_p+1),u_{k,M}(\frac{1}{2}(\eta_q+1),\frac{1}{2}(\eta_p+1))=f(x_i,t_j)$$

where τ_p , η_p and τ_q, η_q are zeros of Legendre polynomials of degrees r_1 and r_2, respectively, and ω_p and ω_q are the corresponding weights. Equation (**??**) gives $2^{k_1-1}M_1\times 2^{k_2-1}M_2$ nonlinear equation which can be solved using Newton's iterative method. The initial values required to start Newton's iterative method can be chosen by using the physical behavior of the given integral equations.

4.2 The Nonlinear Mixed Volterra-Fredholm Integral Equation

Consider the nonlinear mixed Volterra-Fredholm integral equation

$$u(x,t)=f(x,t)+\lambda\int_0^x\int_\Omega G(x,t,y,z,u(y,z))\mathtt{d}z\mathtt{d}y, \qquad (x,t)\in[0,1]\times\Omega \tag{13}$$

where $u(x,t)$ is the unknown in $D=[0,1]\times\Omega$, where Ω is a closed subset of $\mathbb{R}^n$,$n=1,2,3$. The functions $f(x,t)$ and $G(x,t,y,z,u)$ are given functions in D.

Now, consider equation (13) with $\Omega=[0,1]$. We solve equation (13) by the Laguerre wavelet. According to the process described in Section 4.1, we consider $u(x,t)$ in equation (13) is approximated by two-dimensional

$$u_{k,M}(x,t)=\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)=C^T\ \Psi(x,t) \tag{14}$$

Substituting equation (14) into equation (13) gives

$$u_{k,M}(x,t)=f(x,t)+\lambda\int_0^x\int_0^1 G(x,t,y,z,u_{k,M}(y,z))\mathtt{d}z\mathtt{d}y \tag{15}$$

Collocation equation (15) at $2^{k_1-1}M_1\times 2^{k_2-1}M_2$ point (x_i,t_j), gives

$$u_{k,M}(x_i,t_j)=f(x_i,t_j)+\lambda\int_0^{x_i}\int_0^1 G(x_i,t_j,y,z,u_{k,M}(y,z))\mathtt{d}z\mathtt{d}y \tag{16}$$

where x_i and t_j are zeros of the shifted Chebyshev polynomial in $[0,1]$. We transform the integrals over $[0,x_i]$, $[0,1]$ into the integral over $[-1,1]$. For this purpose, linear transformation must be applied with the following form

$$\begin{aligned}\tau &= \frac{2}{x_i}y-1, \quad \Rightarrow y=\frac{x_i}{2}(\tau+1), \quad y\in[0,x_i],\\ \alpha &= 2z-1, \quad \Rightarrow z=\frac{1}{2}(\alpha+1), \quad z\in[0,1],\end{aligned}$$

Let

$$H(x_i,t_j,\tau,\alpha)=G\Big(x_i,t_j,\frac{x_i}{2}(\tau+1),\frac{1}{2}(\alpha+1),u_{k,M}\Big(\frac{x_i}{2}(\tau+1),\frac{1}{2}(\alpha+1)\Big)\Big)$$

Equation (16) may then be restated as

$$u_{k,M}(x_i,t_j)=f(x_i,t_j)+\frac{\lambda x_i}{4}\int_{-1}^1\int_{-1}^1 H(x_i,t_j,\tau,\alpha)\mathtt{d}\alpha\mathtt{d}\tau$$

Using the Gauss quadrature formula relative to the quadrature weights ω_q and ω_p, we estimate the integrals and gets

$$u_{k,M}(x_i,t_j)-\frac{\lambda x_i}{4}\sum_{p=1}^{r_1}\sum_{q=1}^{r_2}\omega_p\omega_q H\Big(x_i,t_j,\tau_q,\alpha_p\Big)=f(x_i,t_j) \tag{17}$$

where τ_q and α_p are zeros of Legendre polynomial of degrees r_1 and r_2, respectively. By solving the nonlinear system (17), we can find the unknown coefficients $c_{n,i,l,j}$ and then we have the approximate solution of equation (13).

§5 Convergence Analysis

Theorem 5.1. *If $u(x,t)$ defined on $[0,1)\times[0,1)$ and $|u_{k,M}(x,t)|\le\mathcal{M}$, then the Laguerre wavelets expansion of $u_{k,M}(x,t)$ defined in equation (8) converges uniformly and also*

$$|c_{n,i,l,j}|\le\frac{4\mathcal{M}(\sqrt{e}-1)^2}{2^{\frac{k_1+k_2}{2}}i!(i-1)!\ j!(j-1)!}$$

Proof. *The function $u_{k,M}(x,t)\in[0,1)\times[0,1)$ can be expressed by the two-dimensional Laguerre* *wavelets as*

$$\begin{aligned} u_{k,M}(x,t) &= \sum_{n=1}^{2^{k_1-1}} \sum_{i=0}^{M_1-1} \sum_{l=1}^{2^{k_2-1}} \sum_{j=0}^{M_2-1} c_{n,i,l,j}\psi_{n,i,l,j}(x,t) \\ &= \sum_{n=1}^{2^{k_1-1}} \sum_{i=0}^{M_1-1} \sum_{l=1}^{2^{k_2-1}} \sum_{j=0}^{M_2-1} c_{n,i,l,j}\psi_{n,i}(x)\psi_{l,j}(t) \end{aligned}$$

where the coefficients $c_{n,i,l,j}$ *can be determined as*

$$\begin{aligned} c_{n,i,l,j} &= \langle\langle u_{k,M}(x,t), \psi_{n,i}(x)\rangle, \psi_{l,j}(t)\rangle \\ &= \int_0^1 \left(\int_0^1 u_{k,M}(x,t)\psi_{n,i}(x)\,dx\right)\psi_{l,j}(t)\,dt \\ &= \int_{I_{l,k_2}} \left(\int_{n,k_1} u_{k,M}(x,t)\psi_{n,i}(x)\,dx\right)\psi_{l,j}(t)\,dt \\ &= \frac{2^{\frac{k_1+k_2}{2}}}{i!j!}\int_{I_{l,k_2}} \left(\int_{n,k_1} u_{k,M}(x,t)L_i(2^{k_1}x-2n+1)\,dx\right)L_j(2^{k_2}t-2l+1)\,dt, \end{aligned}$$

where $I_{n,k_1} = \left[\frac{n-1}{2^{k_1-1}}, \frac{n}{2^{k_1-1}}\right)$ *and* $I_{l,k_2} = \left[\frac{l-1}{2^{k_2-1}}, \frac{l}{2^{k_2-1}}\right)$.

Now by change of variable $u = 2^{k_1}x - 2n + 1$, *we obtain :*

$$c_{n,i,l,j} = \frac{2^{\frac{k_1+k_2}{2}}}{2^{k_1}i!j!}\int_{I_{l,k_2}} \left(\int_0^1 u_{k,M}(\frac{u+2n-1}{2^{k_1}}, t)L_i(u)\,du\right)L_j(2^{k_2}t-2l+1)\,dt,$$

Similarly, changing the variable for t *as* $v = 2^{k_2}t - 2l + 1$, *we get :*

$$c_{n,i,l,j} = \frac{1}{2^{\frac{k_1+k_2}{2}}\,i!j!}\int_0^1 \left(\int_0^1 u_{k,M}(\frac{u+2n-1}{2^{k_1}}, \frac{v+2n-1}{2^{k_2}})L_i(u)\,du\right)L_j(v)\,dv,$$

Now by equation (4), we observe that

$$\begin{aligned} |c_{n,i,l,j}| &\le \frac{1}{2^{\frac{k_1+k_2}{2}}\,i!j!}\int_0^1 \left(\int_0^1 \left|u_{k,M}(\frac{u+2n-1}{2^{k_1}}, \frac{v+2n-1}{2^{k_2}})\right|\,|L_i(u)|\,du\right)|L_j(v)|\,dv \\ &= \frac{\mathcal{M}}{2^{\frac{k_1+k_2}{2}}\,i!j!}\left(\int_0^1 |L_i(u)|\,du\right)\left(\int_0^1 |L_j(v)|\,dv\right) \\ &= \frac{4\mathcal{M}(\sqrt{e}-1)^2}{2^{\frac{k_1+k_2}{2}}\,i!(i-1)!\;j!(j-1)!} \end{aligned}$$

This means that the series $\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j}$ *is absolutely convergent and hence the series*

$$\sum_{n=1}^{2^{k_1-1}} \sum_{i=0}^{M_1-1} \sum_{l=1}^{2^{k_2-1}} \sum_{j=0}^{M_2-1} c_{n,i,l,j}\psi_{n,i,l,j}(x,t),$$

is uniformly convergent. □

Theorem 5.2. *Let* $u_{k,M}(x,t) = \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1} c_{n,i,l,j}\psi_{n,i,l,j}(x,t)$ *be the truncated series, then the truncated error* $E_{n,i,l,j}(x,t)$ *can be defined as*

$$\|E_{n,i,l,j}(x,t)\|_2^2 \le \sum_{n=2^{k_1-1}+1}^{\infty}\sum_{i=M_1}^{\infty}\sum_{l=2^{k_2-1}+1}^{\infty}\sum_{j=M_2}^{\infty}\left(\frac{4\mathcal{M}(\sqrt{e}-1)^2}{2^{\frac{k_1+k_2}{2}}\,i!(i-1)!\;j!(j-1)!}\right)^2.$$

Proof. *Any function $u(x,t)$ can be expressed by the Laguerre wavelet as*

$$u(x,t)=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=0}^{\infty}c_{n,i,l,j}\psi_{n,i,l,j}(x,t).$$

If

$$u_{k,M}(x,t)=\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t),$$

be the truncated series, then the truncated error term can be calculated as

$$E_{n,i,l,j}(x,t)=u(x,t)-u_{k,M}(x,t)=\sum_{n=2^{k_1-1}+1}^{\infty}\sum_{i=M_1}^{\infty}\sum_{l=2^{k_2-1}+1}^{\infty}\sum_{j=M_2}^{\infty}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)$$

Therefore,

$$\begin{aligned}\|E_{n,i,l,j}(x,t)\|_2^2 &= \|\sum_{n=2^{k_1-1}+1}^{\infty}\sum_{i=M_1}^{\infty}\sum_{l=2^{k_2-1}+1}^{\infty}\sum_{j=M_2}^{\infty}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)\|_2^2\\ &= \int_0^1\int_0^1\left|\sum_{n=2^{k_1-1}+1}^{\infty}\sum_{i=M_1}^{\infty}\sum_{l=2^{k_2-1}+1}^{\infty}\sum_{j=M_2}^{\infty}c_{n,i,l,j}\psi_{n,i}(x)\psi_{l,j}(t)\right|^2 dx\,dt\\ &\le \sum_{n=2^{k_1-1}+1}^{\infty}\sum_{i=M_1}^{\infty}\sum_{l=2^{k_2-1}+1}^{\infty}\sum_{j=M_2}^{\infty}|c_{n,i,l,j}|^2\int_0^1\int_0^1|\psi_{n,i}(x)\psi_{l,j}(t)|^2 dx\,dt\\ &= \sum_{n=2^{k_1-1}+1}^{\infty}\sum_{i=M_1}^{\infty}\sum_{l=2^{k_2-1}+1}^{\infty}\sum_{j=M_2}^{\infty}\left(\frac{4\mathcal{M}(\sqrt{e}-1)^2}{2^{\frac{k_1+k_2}{2}}i!(i-1)!\ j!(j-1)!}\right)^2.\end{aligned}$$

That is establishing the claim. □

§6 Application on the Darboux problem

Consider the Darboux problem

$$\frac{\partial^2 u}{\partial x\partial t}=H(x,t,u)$$

with initial condition

$$u(x,0)=f_1(x),\qquad u(0,t)=f_2(t),\quad (x,t)\in\Omega,$$

where f_1 and f_2 are given continuous functions on $\Omega=[0,1]\times[0,1]$ with $f_1(0)=f_2(0)$. In [6], it has been shown that this problem is equivalent to

$$u(x,t)=f(x,t)+\int_0^t\int_0^x K(y,z,u(y,z))\mathrm{d}y\mathrm{d}z \tag{18}$$

where $f(x,t)=f_1(x)+f_2(t)-f_1(0)$. An equation of the form (18) will be considered in Examples 7.1 and 7.2.

§7 Numerical experiments

We propose some examples to approximate the solution of two-dimensional nonlinear Volterra-Fredholm integral equations and mixed nonlinear Volterra-Fredholm two-dimensional integral

Table 1. Numerical results for Example 7.1.

$(x,y)=(\frac{1}{2^\ell},\frac{1}{2^\ell})$	Presented method with $M_1=M_2=4$	Method in [37]with M=3	Method in [8] with $m=16$
$\ell=2$	0	1.7×10^{-4}	1.40×10^{-4}
$\ell=3$	2.22045×10^{-16}	1.3×10^{-5}	2.18×10^{-6}
$\ell=4$	2.22045×10^{-16}	3.5×10^{-5}	3.59×10^{-8}
$\ell=5$	0	3.2×10^{-5}	5.21×10^{-10}
$\ell=6$	8.88178×10^{-16}	1.9×10^{-5}	4.91×10^{-11}

equations using the Laguerre wavelet method. We consider the absolute error between the exact solution and the present solution defined as

$$E(x,t)=|u(x,t)-u_{k,M}(x,t)|,\qquad (x,t)\in[0,1]\times[0,1],$$

to illustrate the performance of the method. The computations associated with the examples were performed using Mathematica 10 software on a PC. Newton's iteration method is used to solve the nonlinear systems and we solved this system using the Mathematica function FindRoot, which uses Newton's method as the default method. The initial values required to start Newton's iterative method can be chosen by using the physical behavior of the given integral equations.

Example 7.1. *(See [37]) Consider the following two-dimensional nonlinear Volterra integral equation of second kind*

$$u(x,t)=f(x,t)+\int_0^t\int_0^x u^2(y,z)\,dy\,dz \tag{19}$$

where

$$f(x,y)=x^2+t^2-\frac{1}{45}xt(9x^4+10x^2t^2+9t^4)$$

The exact solution of equation (19) is given by $u(x,t)=x^2+t^2$. By applying the method discussed in detail in Section 4.1, this problem has been solved by Laguerre polynomial for $M_1=M_2=4$, $k_1=k_2=1$. Table 1 and Figure 1 show the approximate solution obtained by Laguerre wavelets method.

It is evident from the Table 1, that the numerical solution converge to the exact solution. It is also concluded that the proposed method is very efficient for numerical solution of these problems.

Example 7.2. *Consider the following two-dimensional nonlinear Volterra integral equation*

$$u(x,t)=f(x,t)+\int_0^t\int_0^x (xy^2+\cos z)u^2(y,z)\,dt\,dz,\qquad (x,t)\in[0,1]\times[0,1]$$

where

$$f(x,t)=x\sin t\Big(1-\frac{1}{9}x^2\sin^2 t\Big)+\frac{1}{10}x^6\Big(\frac{1}{2}\sin 2t\Big)$$

and the exact solution is $u(x,t)=x\sin t$. Table 2 and Figure 2 show the numerical results.

Example 7.3. *Consider the following two-dimensional linear Volterra-Fredholm integral equa-*

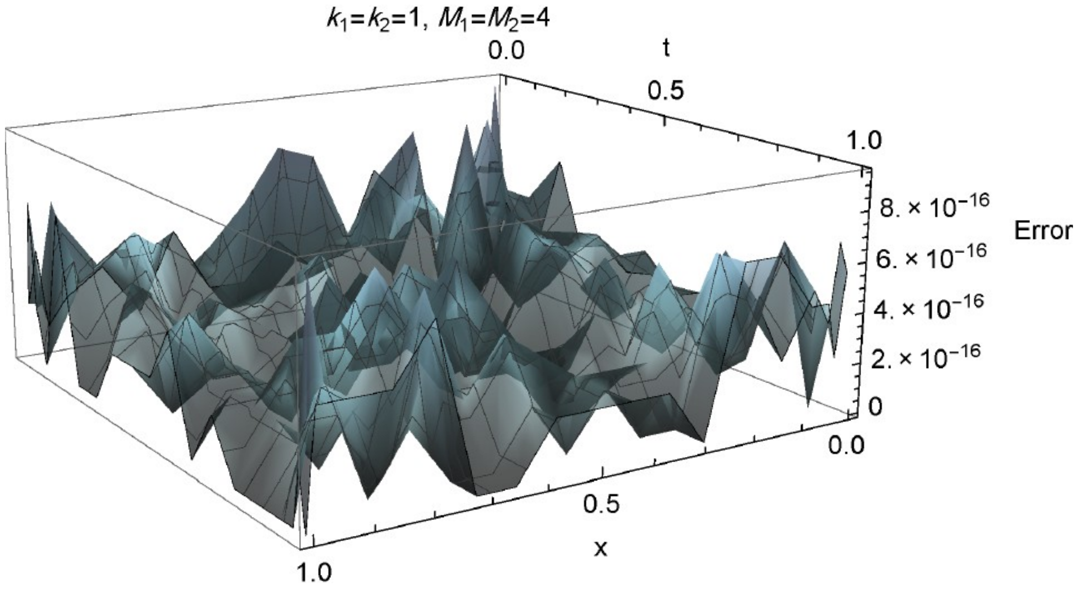

Figure 1. The error function graph of Example 7.1.

Table 2. Numerical results for example 7.2.

$(x, y) = (\frac{1}{2^\ell}, \frac{1}{2^\ell})$	Presented method with $M_1 = M_2 = 4$	Presented method with $M_1 = M_2 = 6$	Method in [9] with $N = 30$
$\ell = 1$	7.7782×10^{-5}	1.6191×10^{-7}	6.1×10^{-5}
$\ell = 2$	1.7623×10^{-5}	7.5459×10^{-8}	1.2×10^{-4}
$\ell = 3$	1.6143×10^{-5}	1.3338×10^{-8}	7.1×10^{-5}
$\ell = 4$	3.5218×10^{-6}	1.7705×10^{-8}	5.3×10^{-5}
$\ell = 5$	5.8934×10^{-7}	4.6972×10^{-9}	5.9×10^{-5}
$\ell = 6$	1.0493×10^{-6}	2.9310×10^{-10}	7.4×10^{-4}

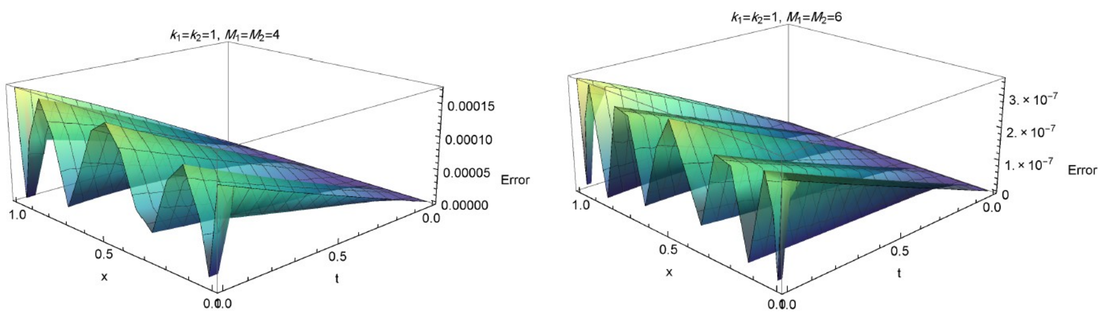

Figure 2. Graph of the Laguerre wavelets approximation error of Example 7.2.

Table 3. Numerical results for example 7.3.

(x, y)	Presented method with $M_1 = M_2 = 4$	Presented method with $M_1 = M_2 = 6$	Method in [9] with $N = 30$
$(0.1, 0.1)$	4.3308×10^{-6}	1.1695×10^{-7}	8.5538×10^{-5}
$(0.2, 0.2)$	2.0279×10^{-5}	3.0373×10^{-7}	6.5403×10^{-4}
$(0.3, 0.3)$	2.3651×10^{-5}	2.0071×10^{-6}	7.2516×10^{-5}
$(0.4, 0.4)$	1.2687×10^{-5}	7.1645×10^{-5}	1.0695×10^{-5}
$(0.5, 0.5)$	4.2236×10^{-5}	1.7806×10^{-5}	3.5914×10^{-5}
$(0.6, 0.6)$	2.0285×10^{-4}	3.4134×10^{-4}	1.2525×10^{-4}
$(0.7, 0.7)$	5.3816×10^{-4}	5.1831×10^{-4}	1.0928×10^{-4}
$(0.8, 0.8)$	8.9278×10^{-4}	5.8443×10^{-4}	1.2101×10^{-4}
$(0.9, 0.9)$	6.8220×10^{-4}	2.8651×10^{-4}	1.6056×10^{-4}

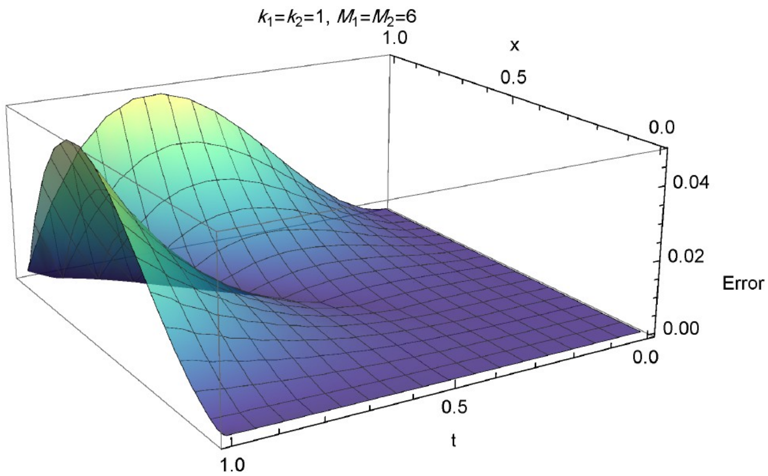

Figure 3. Graph of the Laguerre wavelets approximation error of Example 7.3.

tion:

$$u(x,t) = f(x,t) - \int_0^t \int_0^1 t^2 e^{-z} u(y,z)\, dt\, dz, \qquad t \in [0,1]$$

where

$$f(x,t) = x^2 e^t + \frac{x^3 t^2}{3}$$

and the exact solution is $u(x,t) = x^2 e^t$. Table 3 shows the numerical results.

Example 7.4. *Consider the nonlinear mixed integral equation by*

$$u(x,t) = f(x,t) + \int_0^t \int_0^1 \frac{x(1-y^2)}{(1+t)(1+z^2)} (1 - e^{-u(y,z)})\, dy\, dz \tag{20}$$

where

$$f(x,t) = -Ln\Big(1 + \frac{xt}{1+t^2}\Big) + \frac{xt^2}{8(1+t)(1+t^2)}$$

Table 4. Numerical results for example 7.4.

$(x,y)=(\frac{1}{2^\ell},\frac{1}{2^\ell})$	$M_1=M_2=4$	$M_1=M_2=6$
$\ell=1$	3.9208×10^{-4}	1.2907×10^{-3}
$\ell=2$	5.3914×10^{-4}	1.1706×10^{-4}
$\ell=3$	4.8309×10^{-4}	2.7803×10^{-5}
$\ell=4$	2.0072×10^{-4}	5.6331×10^{-6}
$\ell=5$	3.5691×10^{-4}	1.9092×10^{-6}
$\ell=6$	2.4496×10^{-4}	4.4770×10^{-6}

Table 5. Numerical results for example 7.5.

(x,y)	Presented method with $M_1=M_2=4$	Presented method with $M_1=M_2=6$	Method in [14] with $N=14$
$(0.0,0.0)$	8.6676×10^{-4}	1.7271×10^{-6}	6.45×10^{-6}
$(0.2,0.2)$	4.4562×10^{-4}	2.4632×10^{-6}	1.26×10^{-5}
$(0.4,0.4)$	1.9565×10^{-3}	3.5691×10^{-6}	6.20×10^{-5}
$(0.6,0.6)$	2.4501×10^{-4}	3.7141×10^{-6}	3.18×10^{-4}
$(0.8,0.8)$	1.9121×10^{-3}	3.7411×10^{-6}	6.88×10^{-4}

which has the exact solution $u(x,t)=-Ln\Big(1+\frac{xt}{1+t^2}\Big)$. *The numerical results are shown in Table 4.*

Example 7.5. *(See [14]) Consider the following 2D nonlinear Volterra-Fredholm integral equation*

$$u(x,t)=f(x,t)+16\int_0^1\int_0^1 e^{x+t+y+z}u^3(y,z)dydz+\int_0^t\int_0^x u(y,z)dydz$$

where

$$f(x,t)=2e^{x+t+4}-e^{x+t+8}-e^{x+t}+e^x+e^t-1,$$

whose exact solution is $u(x,t)=e^{x+t}$. *The numerical results using presented method are shown in Table 5 and Figures 4.*

§8 Conclusion

In this study, a two-dimensional wavelet method based on Laguerre polynomial for two-dimensional nonlinear Volterra-Fredholm integral equations and the nonlinear mixed Volterra-Fredholm integral equations is presented and some theorems are proved for convergence analysis. Moreover, the numerical results and absolute errors are presented. As a practical example, the Darboux problem is transformed into a two-dimensional nonlinear Volterra-Fredholm integral equation and the solution of the Darboux problem is obtained by the two-dimensional Laguerre wavelets method. It has been shown that the obtained results are in excellent agreement with the exact solution. Since this method is very powerful and efficient makes it necessary to investigate a method for solution of such equations and we hope that this work is a step in this direction.

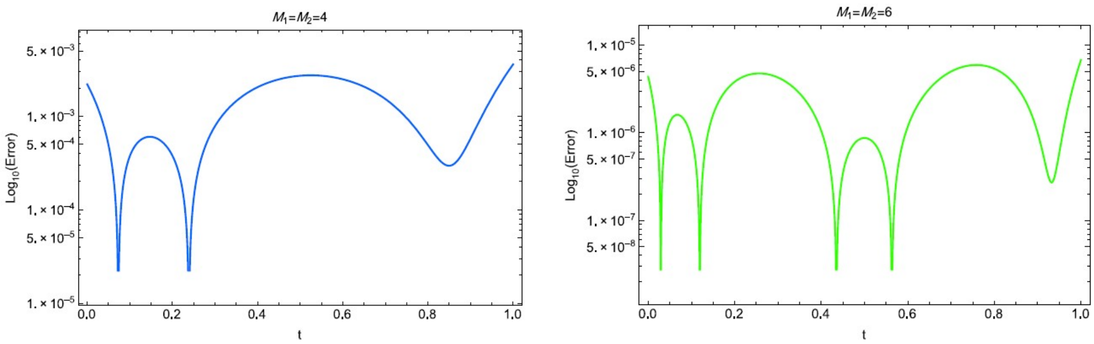

Figure 4. The error function graph for Example 7.5.

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