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Global asymptotical stability in a rational difference equation

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Abstract. In this paper we prove a global attractivity result for the unique positive equilibrium point of a difference equation, which improves and generalizes some known ones in the existing literature. Especially, our results completely solve an open problem and some conjectures proposed in [1, 2, 3, 4].

§1 Introduction

Due to the strong practical application background [1, 3, 8, 12, 13, 17, 20], the subject of the dynamical behaviors of difference equations, including stability, oscillation, boundary value problem, periodic and homoclinic orbits [2, 3, 4, 17-19], has undergone a rapid development in the last three decades. In particular, the study on qualitative properties of rational difference equations (for short, RDEs) has received much attention in the past two decades. For example, we may refer to the monographs [3, 4], the articles [2, 5-7, 9-11, 14-16] and the references cited therein. RDEs may display very complicate dynamical behaviors. One may refer to [4] in which the bifurcation phenomenon of trichotomy of period two was found by a very simple second order RDE. Moreover, we have not found any effective methods to deal with this kind of behavior.

Our main aim in this paper is to investigate the global asymptotic stability of the following difference equation

$$x_{n+1} = \frac{p + qx_n}{1 + rx_{n-k}}, n = 0, 1, \cdots,$$

where $p, q \in [0, \infty)$, $r > 0$, $k \geq 1$ is an integer and initial conditions $x_{-k}, \cdots, x_{-1}, x_0 \in (0, \infty)$.

Our motivation comes from a flourishing stream of the following known work.

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G. Ladas et al [3] considered the following RDE

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-1}},\ n = 0, 1, \cdots \tag{1.1}$$

where

$$a, b, A \in (0,\ \infty) \tag{1.2}$$

and the initial values x_{-1}, x_0 are arbitrary positive numbers.

Eq.(1.1) has a unique positive equilibrium point $\bar{x}$, which is the unique positive root of the equation

$$\bar{x} = \frac{a + b\bar{x}}{A + \bar{x}}, \quad \text{namely,} \quad \bar{x} = \frac{b - A + \sqrt{(b-A)^2 + 4a}}{2}.$$

It can be easily shown that $\bar{x}$ is locally asymptotically stable [2, 3] when (1.2) is satisfied. For the global asymptotic stability of $\bar{x}$, V.L. Kocic and G. Ladas [3] in 1993, G. Ladas [1] in 1994, G. Ladas [2] in 1995 obtained some results, respectively.

Theorem 1.1 [1, 2, 3] Suppose that (1.2) holds, and one of the following conditions is true

(1) $b < A$;

(2) $b \ge A$ and $a < Ab$;

(3) $b \ge A$ and $Ab < a < 2A(b + A)$;

(4) $b \ge \sqrt{(1 + \sqrt{5})/2A}$, $Ab < a$ and $b^2/A < \bar{x} < 2b$.

Then $\bar{x}$ is globally asymptotically stable.

Here, the positive equilibrium point $\bar{x}$ of Eq.(1.1) is said to be globally asymptotically stable if it is both locally asymptotically stable and globally attractive. While, $\bar{x}$ is said to be globally attractive if for arbitrary positive initial values x_{-1}, x_0, the solution $\{x_n\}$ of Eq.(1.1) converges to $\bar{x}$. Clearly, the positive equilibrium $\bar{x}$ of Eq.(1.1) is globally asymptotically stable if and only if $\bar{x}$ is globally attractive because $\bar{x}$ is locally asymptotically stable.

For RDEs, there has been a point of view that the local asymptotical stability of an equilibrium point implies its global asymptotical stability. Especially in Mathematical Biology this has been the case. Hence, many research projects in [3] and Open Problems and Conjectures [3, 4] are given based on this kind of idea. Indeed, most of known results [3, 4] support this kind of point of view. But later V.L. Kocic and G. Ladas' work provided counter examples for this. How about Eq.(1.1)? Computer simulations show that the declaration is also true; that is, the local asymptotical stability of the positive equilibrium point $\bar{x}$ of Eq.(1.1) implies its global asymptotical stability, which is equivalent to its global attractivity as long as (1.2) holds. But, except the partial results mentioned in the above Theorem 1.1, this point of view can not be completely and theoretically proved. So, G. Ladas presented the following conjecture in [1-3] respectively.

Conjecture 1.1 [3, Conjecture 6.1.1, P_{154}] Assume that (1.2) holds. Then the positive equilibrium point $\bar{x}$ of Eq.(1.1) is globally asymptotically stable.

Many researchers have paid attention to this conjecture. In 2000, Ou et al [5] obtained the following results.

Theorem 1.2 [5, Theorem 1, P_{33-34}] Assume that $a, b, A \in (0,\ \infty)$ and $b = A$ or

$$b > A \text{ with } Ab < a < 2A(b + A) + 4A^3/(b - A) + 2A^2\sqrt{b^4 - A^4}/(b - A)^2, \tag{1.3}$$

then $\bar{x}$ of Eq.(1.1) is a global attractor of all positive solutions.

Although Theorem 1.2 does not solve the Conjecture 1.1 completely, it is obvious that the result of Theorem 1.2 is better than (3) of Theorem 1.1. In 2002, by further analysis of semi-cycle of Eq.(1.1), Li, et al [6] obtained the following three results.

Theorem 1.3 [6, Theorem 1, P_{10}] Suppose that (1.2) holds and that $b = A$ and $Ab < a$ or $b > A$ and $Ab < a \leq 2Ab + (\frac{2Ab}{b-A})^2$ is true. Then every positive solution $\{x_n\}$ of Eq.(1.1) tends to a finite limit as $n \to \infty$.

Theorem 1.4 [6, Theorem 2, P_{10}] Suppose that (1.2) and $a(b - A) + b^2 \bar{<} 2A^2\sqrt{a + A^2}$ are valid. Then $\bar{x}$ is a global attractor of any positive solution $\{x_n\}$ of Eq.(1.1).

Theorem 1.5 [6, Theorem 3, P_{10}] Suppose that (1.2) and $r < 4A^3$ hold, where $r = \bar{x}(A + \bar{x})^2$. Then $\bar{x}$ is a global attractor of any positive solution $\{x_n\}$ of Eq.(1.1).

In Theorem 1.5, $r < 4A^3$ is equivalent to $\bar{x} < A$, i.e., $\sqrt{(b - A)^2 + 4a} < 3A - b$. These results in Theorems 1.3–1.5 partly include and improve the corresponding ones in Theorem 1.1. To see this, refer to [6, Remarks 1, 2, 3]; but they do not contain each other.

In 2003, by studying the global attractivity of a general difference equation, Li, et al [7] showed the next global attractivity theorem.

Theorem 1.6 [7, Theorem 2, P_{272}] Suppose that (1.2) holds and that $a < A(2A - b)$. Then the positive equilibrium point of Eq.(1.1) is globally attractive.

This result is obviously different from the known ones. In the same year, by studying the global attractivity of a general difference equation of non-increasing nonlinearities, H. A. El-Morshedy derived the following result for the global attractivity of Eq.(1.1).

Theorem 1.7 [8, Theorem 3.2, P_{757}] Assume that $b \geq A, Ab < a$ and

$$(b - a)\bar{x} < b^2 + A^2.$$

Then the unique positive equilibrium of Eq.(1.1) is globally asymptotically stable.

In the above Theorem 1.7, $(b - a)\bar{x} < b^2 + A^2$ is equivalent to $a \leq 2Ab + (\frac{2Ab}{b-A})^2$, which has actually been formulated in Theorem 1.3. Note that Theorems 1.1 to 1.7 only partially answer the above Conjecture 1.1. Hence, it is worth further considering the Conjecture 1.1.

For the special case of Eq.(1.1) with the form [4, P_{79}]

$$x_{n+1} = \frac{p + qx_n}{1 + x_{n-1}},\ n = 0, 1, \cdots \tag{1.4}$$

where $p,\ q \in (0,\ \infty)$ and the initial values x_{-1}, x_0 are arbitrary positive numbers, the following question is presented.

Conjecture 1.2 [4, Conjecture 6.10.1, P_{124}] Assume $p,\ q \in (0,\ \infty)$. Show that every positive solution of Eq. (1.4) has a finite limit.

In 2007, Nussbaum considered this conjecture and derived the following conclusion.

Theorem 1.8 [10, Theorem 6.1, P_{1083}] Assume either that $0 < q \leq 1$ and $p > 0$ or that

$$q > 1 \text{ and } 0 < p \leq 2q + \frac{4q^2}{(q-1)^2}.$$

Then if $x_{-1} > 0,\ x_0 > 0$ and $x_n,\ n \geq 1$, is defined by (1.4),

$$\lim_{n\to\infty} x_n = \bar{x} = (q - 1 + \sqrt{(q-1)^2 + 4p})/2.$$

Noticing $2 < \frac{4q^2}{(q-1)^2}$ for $q > 1$, Theorem 1.8 evidently improves Theorem 1.7. However, Theorem 1.8 does not entirely solve the Conjecture 1.2, either.

V. L. Kocic and G. Ladas again considered in [3] the generalization of Eq.(1.1), i.e., $(k+1)-th$ order rational difference equation

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-k}},\ n = 0, 1, \cdots, \tag{1.5}$$

where

$$a, b \in [0,\ , \infty), A \in (0,\ \infty), k \in \{1, 2, \cdots\} \tag{1.6}$$

and the initial values $x_{-k}, \cdots, x_{-1}, x_0$ are arbitrary positive numbers. Eq.(1.5) has a positive equilibrium $\bar{x}$, which is the same as Eq.(1.1), provided that

$$\text{either} \quad a > 0 \quad \text{or} \quad a = 0 \quad \text{and} \quad b > A. \tag{1.7}$$

The following results are obtained.

Theorem 1.9 [3, Theorem 3.4.3, P_{71}] Assume that (1.6) and (1.7) hold. Then the positive equilibrium $\bar{x}$ of Eq.(1.5) is a global attractor of all positive solutions provided that one of the following six conditions is satisfied:
(a) $a > 0$ and $A > b > 0$;
(b) $a > 0$ and $b = 0$;
(c) $b > 0$, $k \geq 2$, $Ab < a$, and $\bar{x}k \leq A$;
(d) $b > 0$, $k \geq 2$, $a \leq Ab$, and $\bar{x}(k-1) \leq A$;
(e) $b > 0$, $k = 1$, $a \leq Ab$;
(f) $b > 0$, $k = 1$, and $Ab \leq a \leq 2Ab + 2A^2$.

For related work associated with Eq.(1.5), refer also to [21, 22]. If Conjecture 1.1 is true, then, as a special case of Eq.(1.5), i.e, Eq.(1.5) with $k = 1$, the holding condition (1.2) for the attractivity of Eq.(1.1) should be included in Theorem 1.10. Obviously, it is not so. Therefore, it is also worthy for further studying the attractivity of Eq.(1.5). These problems motivate us to investigate in this paper the global stability of the following difference equation

$$x_{n+1} = \frac{p + qx_n}{1 + rx_{n-k}}, n = 0, 1, \cdots, \tag{1.8}$$

where $p, q \in [0, \infty)$, $r \in (0,\ \infty)$, $k \geq 1$ is a positive integer and the initial conditions $x_{-k}, \cdots, x_{-1}, x_0 \in (0, \infty)$. To avoid the trivial case, we suppose that $p + q > 0$.

Eq.(1.8) has a unique nonnegative equilibrium point, still denoted by $\bar{x}$, i.e.,

$$\overline{x} = \frac{q - 1 + \sqrt{(q-1)^2 + 4pr}}{2r}.$$

When $p = 0$ and $q \in (0,\ 1]$, $\bar{x} = 0$. At this time, it is easy to see from Eq.(1.8) that $x_{n+1} < qx_n$ and so x_n eventually monotonically approaches $\bar{x}$. Hence, in the sequel one will only consider the behavior of positive equilibrium point of Eq.(1.8), namely, one will assume that $p > 0$ or $q \in (1,\ \infty)$. Our main result in this paper is the following.

Theorem XY Assume that $p > 0$ or $q \in (1,\ \infty)$, $r \in (0,\ \infty)$ and $k \geq 1$ is a positive integer. Then the unique positive equilibrium $\bar{x}$ of Eq.(1.8) is a global attractor of all of its positive solutions.

The proof of Theorem XY will be given in Section 3. It is easy to see from Theorem XY that

Conjectures 1.1 and 1.2 are true. Up to here, the above two conjectures have been completely solved. Furthermore, our results include and improve the corresponding ones in [1-10]. In addition, our results also include and improve the corresponding ones in [14, 15], completely solve the conjecture in [3, P_{76}] and answer the open problem in [4, P_{129}] stated in Section 4.

2. Several Key Lemmas

For readers'convenience, we present here some known lemmas used in the sequel.

Lemma 2.1 [3, Theorem 2.3.1, P_{40}] Consider the difference equation

$$x_{n+1} = x_n f(x_n, x_{n-k_1}, \cdots, x_{n-k_r}), \tag{2.1}$$

where $k_1, k_2, \cdots, k_r$ are positive integers. Denote by k the maximum of $k_1, k_2, \cdots, k_r$. Also, assume that the function f satisfies the following hypotheses:

(H1) $f \in C[(0,\infty) \times [0,\infty)^r,\ (0,\infty)]$ and $g \in C[[0,\infty)^{r+1},\ (0,\infty)]$,
where $g(u_0, u_1, \cdots, u_r) = u_0 f(u_0, u_1, \cdots, u_r)$ for $u_0 \in (0,\infty)$
and $u_1, \cdots, u_r \in [0,\infty)$, $g(0, u_1, \cdots, u_r) = \lim_{u_0 \to 0^+} g(u_0, u_1, \cdots, u_r)$;

(H2) $f(u_0, u_1, \cdots, u_r)$ is nonincreasing in $u_1, \cdots, u_r$;

(H3) The equation $f(x, x, \cdots, x) = 1$ has a unique positive solution $\overline{x}$;

(H4) Either the function $f(u_0, u_1, \cdots, u_r)$ does not depend on u_0 or for every $x > 0$ and $u \geq 0$,

$$[f(x, u, \cdots, u) - f(\overline{x}, u, \cdots, u)](x - \overline{x}) \leq 0$$

with

$$[f(x, \overline{x}, \cdots, \overline{x}) - f(\overline{x}, \overline{x}, \cdots, \overline{x})](x - \overline{x}) < 0 \quad for \quad x \neq \overline{x}.$$

Define a new function F given by

$$F(x) = \begin{cases} max_{x \leq y \leq \overline{x}} G(x, y) & for \quad 0 \leq x \leq \overline{x} \\ min_{\overline{x} \leq y \leq x} G(x, y) & for \quad x > \overline{x} \end{cases} \tag{2.2}$$

where

$$G(x, y) = y f(y, x, \cdots, x) f(\overline{x}, \overline{x}, \cdots, \overline{x}, y)[f(\overline{x}, x, \cdots, x)]^{k-1}. \tag{2.3}$$

Then

(a) $F \in C[(0,\infty), (0,\infty)]$ and F is nonincreasing in $[0,\infty)$;

(b) Assume that the function F has no periodic points of prime period 2. Then $\overline{x}$ is a global attractor of all positive solutions of Eq.(2.1).

Lemma 2.2 [3, Lemma 1.6.3 (a) and (d)] Let $F \in [[0,\infty), (0,\infty)]$ be a nonincreasing function and let $\overline{x}$ denote the unique fixed point of F, then the following statements are equivalent:

(a) $\overline{x}$ is the only fixed point of F^2 in $(0,\infty)$;

(b) $\overline{x}$ is a global attractor of all positive solutions of the difference equation

$$x_{n+1} = F(x_n), \quad n = 0, 1, \cdots \tag{2.4}$$

with $x_0 \in [0,\infty)$.

Lemma 2.3 [11] Consider the difference Eq.(2.4), where F is a decreasing function which maps some interval I into itself. Assume that F has negative Schwarzian derivative

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}(\frac{F''(x)}{F'(x)})^2 = [\frac{F''(x)}{F'(x)}]' - \frac{1}{2}(\frac{F''(x)}{F'(x)})^2 < 0$$

everywhere on I, except for point x, where $F'(x) = 0$. Then the positive equilibrium $\overline{x}$ of Eq.(2.4) is globally attractor of all positive solutions of Eq.(2.4).

3. Proof of Main Result

In this section we shall give the proof of our main result in this paper.

Proof of Theorem XY We will mainly utilize Lemma 2.1 to prove our results. Eq.(1.8) can be rewritten into

$$x_{n+1} = x_n \frac{\frac{p}{x_n} + q}{1 + rx_{n-k}}. \tag{3.1}$$

Set

$$f(u_0, u_1, \cdots, u_k) = \frac{\frac{p}{u_0} + q}{1 + ru_k}.$$

It is easy to verify that the function f satisfies the hypotheses (H1)-(H4) of Lemma 2.1. The function G defined by (2.3) takes the form $G(x,y) = \frac{p+qy}{1+ry}(\frac{1+r\bar{x}}{1+rx})^k$. Moreover, $\frac{\partial G(x,y)}{\partial y} = \frac{q-pr}{(1+ry)^2}(\frac{1+r\bar{x}}{1+rx})^k$. In order to apply Lemma 2.1, one has to calculate the function F defined by (2.2). There are two cases to be considered.

Case(I) $0 \leq q \leq pr$.

In this case, the function F of (2.2) is given by

$$F(x) = A\frac{p + qx}{(1 + rx)^{k+1}}, x \in (0, \infty), \quad \text{where} \quad A = (1 + r\overline{x})^k.$$

We now show that the function F has no periodic points of prime period 2.

Notice that $F'(x) = -A\frac{kqrx+(k+1)pr-q}{(1+rx)^{k+2}} < 0$. Take $I = [0, pA]$. For any given $x \in I$, one has $0 \leq F(x) \leq F(0) =: \lim_{x \to 0^+} F(x) = pA$. So, $F(I) \subset I$. Again,

$$F''(x) = A(k + 1)r\frac{(k + 2)pr - 2q + kqrx}{(1 + rx)^{k+3}} > 0.$$

So, one has

$$\frac{F''(x)}{F'(x)} = -(k + 1)r\frac{(k + 2)pr - 2q + kqrx}{(1 + rx)(kqrx + (k + 1)pr - q)}$$

and hence

$$(\frac{F''(x)}{F'(x)})' = \frac{(k + 1)r\triangle}{(1 + rx)^2(kqrx + (k + 1)pr - q)^2},$$

where $\triangle$ takes this form

$$\begin{aligned} \triangle &= -kqr(1 + rx)(kqrx + (k + 1)pr - q) \\ &\quad +[(k + 2)pr - 2q + kqrx][2kqr^2x + ((k + 1)pr - q)r + kqr] \\ &= kqr[kqr^2x^2 + 2((k + 1)pr - q)rx + pr - q] + ((k + 1)pr - q)((k + 2)pr - 2q)r > 0. \end{aligned}$$

Accordingly,

$$SF(x) = (\frac{F''(x)}{F'(x)})' - \frac{1}{2}[\frac{F''(x)}{F'(x)}]^2 = \frac{(k + 1)r\Gamma}{2[(1 + rx)(kqrx + (k + 1)pr - q)]^2},$$

where Γ has the following expression

$$\begin{aligned} \Gamma &= 2\triangle - (k + 1)r[((k + 2)pr - 2q + kqrx]^2 \\ &= -(k - 1)k^2q^2r^3x^2 - 2k^2qr^2[(k + 1)pr - 2q]x - kr[(k + 2)pr - 2q][(k + 1)pr - q] < 0. \end{aligned}$$

Therefore, $SF(x) < 0$. By Lemma 2.3, $\overline{x}$ is a global attractor of all positive solutions of Eq.(2.4). Thereout, according to Lemma 2.2, $\overline{x}$ is the only fixed point of F^2 in $(0, \infty)$, which, together with Lemma 2.1 (b), indicates that $\overline{x}$ is a global attractor of all positive solutions of Eq.(3.1), i.e, Eq.(1.8).

Case(II) $q > pr$.

In this case, the function F in (2.2) has the form

$$F(x) = \frac{B}{(1+rx)^k}, x \in (0,\infty), \quad \text{where} \quad B = (p+q\overline{x})(1+r\overline{x})^{k-1}. \tag{3.2}$$

It suffices to show that the function F has no periodic points of prime period 2. Let $L = F(M)$ and $M > 0$ is the fixed point of $F^2(x)$, that is to say, $F^2(M) = M$. Then $\frac{B}{(1+rM)^k} = L$ and $\frac{B}{(1+rL)^k} = M$, which imply

$$\frac{(1+rM)^k}{M} = \frac{(1+rL)^k}{L}. \tag{3.3}$$

If $k = 1$, then it follows from (3.3) that $L = M$. So, $M = F(M)$. Namely, M is the fixed point of $F(x)$. But from the obvious facts that F is nonincreasing, $F(x) > 0$ for $x \in (0,\infty)$ and $F(+\infty) = 0$, it follows that F has a unique fixed point. Therefore, $M = \overline{x}$. So $\overline{x}$ is the unique fixed point of $F^2(x)$ in $x \in (0,\infty)$. Namely, the function F has no periodic points of prime period 2 for $k = 1$.

Now, suppose $k > 1$. Since $F(x)$ in (3.2) is decreasing, setting $I = [0,\ B]$, for every $x \in I$, we have $0 < F(x) \leq F(0) =: \lim_{x\to 0^+} F(x) = B$, i.e., $F(I) \subset I$. In order to apply Lemma 2.3, it requires to show that the Schwarzian derivative of F is negative. By calculation one gets

$$F'(x) = \frac{-krB}{(1+rx)^{k+1}}, F''(x) = \frac{k(k+1)r^2B}{(1+rx)^{k+2}}, \frac{F''(x)}{F'(x)} = \frac{-(k+1)r}{1+rx} \quad \textit{and} \quad [\frac{F''(x)}{F'(x)}]' = \frac{(k+1)r^2}{(1+rx)^2}.$$

Thus,

$$SF(x) = [\frac{F''(x)}{F'(x)}]' - \frac{1}{2}(\frac{F''(x)}{F'(x)})^2 = \frac{(1-k^2)r^2}{2(1+rx)^2} < 0.$$

By Lemma 2.3, $\overline{x}$ is a global attractor of all positive solutions of Eq.(2.4). Thereout, in view of Lemma 2.2, $\overline{x}$ is the only fixed point of F^2 in $(0,\infty)$. So, Lemma 2.1 (b) tells us that the unique positive equilibrium $\bar{x}$ of Eq.(1.8) is a global attractor of all of its positive solutions.

Combining the cases (I) and (II) completes the proof of Theorem XY.

4. Applications

In this section, we present some applications for our results.

Example 4.1 [3, Eq.4.1.1 in P_{75} or Eq.4.1.6 in P_{76}]. Consider discrete logistic model

$$x_{n+1} = \frac{\alpha x_n}{1+\beta x_{n-k}}, \ n = 0, 1, \cdots \tag{4.1}$$

where

$$\alpha \in (1,\ \infty), \beta \in (0,\ \infty), k \in \{0, 1, \cdots\} \tag{4.2}$$

and the initial conditions $x_{-k}, \cdots, x_{-1}, x_0 \in (0,\infty)$, which was proposed by Pielou in her books [12, P_{22}] and [13, P_{79}] as a discrete analogue of the delay logistic equation

$$x'(t) = rx(t)[1 - \frac{x(t-\tau)}{P}], \ t \geq 0.$$

The following results are obtained.

Theorem 4.1 [3, Theorem 4.1.1 (b) P_{76}] The positive equilibrium $\bar{x} = (\alpha-1)/\beta$ of Eq.(4.1) is globally asymptotically stable if $(\alpha-1)(k-1) \leq 1$. In particular, the positive equilibrium of Eq.(4.1) is globally asymptotically stable if $k = 0$ or $k = 1$.

On the basis of computer observations the authors in [3] believe that the following conjecture is true.

Conjecture [3, P_{76}] Assume that (4.2) holds. Show that the positive equilibrium $\bar{x} = (\alpha-1)/\beta$ of Eq.(4.1) is globally asymptotically stable if and only if it is locally asymptotically stable.

Indeed, we show that the above conjecture holds. The results are as follows.

Theorem XY-1 Assume that (4.2) holds. Then the positive equilibrium $\bar{x} = (\alpha-1)/\beta$ of Eq.(4.1) is globally asymptotically stable if and only if it is locally asymptotically stable.

Proof By Theorem 4.1, the positive equilibrium $\bar{x} = (\alpha-1)/\beta$ of Eq.(4.1) is globally attractive for $k = 0$. We now consider the case $k \geq 1$. Evidently, Eq.(4.1) is a special of Eq.(1.8) with $p = 0, q = \alpha > 1$ and $r = \beta > 0$. It follows from Theorem XY that the positive equilibrium $\bar{x} = (\alpha-1)/\beta$ of Eq.(4.1) is globally attractive. So, the positive equilibrium of Eq.(4.1) is globally asymptotically stable if and only if it is locally asymptotically stable.

Remark 4.1 Without loss of generality, we may suppose $r = 1$ in Eq.(1.8). To coincide with Example 4.1, we still write it as r.

Example 4.2 [4, P_{129}] Consider difference equation

$$x_{n+1} = \frac{p + qx_n}{1 + x_{n-k}},\ n = 0, 1, \cdots \tag{4.5}$$

where

$$p, q \in [0,\ \infty), k \in \{1, \cdots\} \tag{4.6}$$

and the initial values $x_{-k}, \cdots, x_{-1}, x_0 \in (0, \infty)$. Kulenovic and Ladas gave the following questions.

Open Problem 6.10.17 [4, P_{129}] Assume that $p, q \in [0,\ \infty), k \in \{2, 3, \cdots\}$. Investigate the global behavior of all positive solutions of Eq.(4.5).

Invoking Lemma 2.1 and Theorem 4.1, Mehdi Dehghan and Reza Mazrooei-Sebdani studied in [15] the global asymptotic stability of Eq.(4.5) and derived some results which requires $p > 0$.

Anyway, according to Theorem XY, one can easily derive the following results.

Theorem XY-2 Assume that (4.6) holds. If $p = 0$ and $0 < q \leq 1$, then the zero equilibrium of Eq.(4.5) is a global attractor of all positive solutions of Eq.(4.3). Assume that $p > 0$ or $q > 1$. Then the unique positive equilibrium of Eq.(4.5) is globally attractive.

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