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Counterparty risk valuation on credit-linked notes under a Markov Chain framework

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Abstract. A credit-linked note (CLN) is a note paying an enhanced coupon to investors for bearing the credit risk of a reference entity. In this paper, we study the counterparty risk on CLNs under a Markov chain framework, and introduce a Markov copula model to describe joint defaults between the reference entity underlying the CLN and CLN issuer. Assuming that the respective default intensities are directly and inversely proportional to the interest rate, which follows a CIR process, we obtain the explicit formulae for CLN values through a PDE approach. Finally, credit valuation adjustment (CVA) formula is derived to price counterparty credit risk.

§1 Introduction

Counterparty risk has been a crucial issue related to valuation and risk management of credit derivatives since the 2008 financial crisis. In general terms, counterparty risk is the risk that a party to an OTC derivatives contract may fail to perform on its contractual obligations, causing losses to the other party (see [5]). How to value counterparty risk in the form of credit valuation adjustment (CVA) is a hot topic recently.

A CLN is a security issued by a commercial bank or an investment bank, which has credit risk related to another reference entity. Actual CLN structures can appear quite complicating, but they are basically standard notes with an embedded credit default swap (CDS). In a typical CLN structure, the issuer of the note is the protection buyer and the investor is the protection seller. A CLN will have a coupon rate, maturity date and par value just like a standard bond. However, in contrast to a standard bond, the maturity value will depend on the performance of the reference entity. In particular, if a credit event occurs with respect to the reference entity, the note is paid off with the maturity value adjusted down (see [8]). The constructing process of a standard CLN can be described as Figure 1. In order to consider the pricing problem of

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CLN and the counterparty credit risk, we need to model the default correlation between the reference entity and CLN issuer.

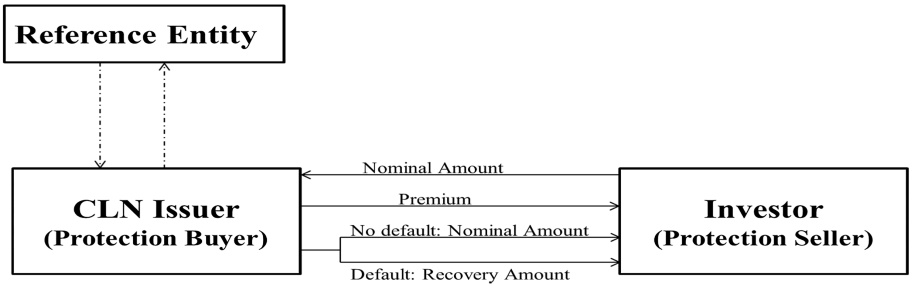


Figure 1. The cash flow of a standard CLN.

In the exiting literature, different approaches in modeling credit risk have been developed. Generally, it can be classified into two main categories, that is, structural models and reduced-form models. The structural model, first introduced by Merton [18], retains the rational hypothesis implicitly assumed in the option pricing approach. By investigating the effects of bond indenture in valuing corporate securities, Black et al. [4] developed the structural model further. Jiang et al. [14] used the structure model to consider an optimal repayment strategy for debit. Liang et al. [15] applied optimal stochastic control theory to the study of structural models as well. Hui et al. [11] developed a new approach to study the credit risk premiums of credit-linked notes using the structural model.

The reduced-form model is not based on a firm's financial position, and default events are usually modeled as a function of a set of state variables (see [21]). The origin of the reduced-form model dated back to Pye [19]. Litterman et al. [16] presented a reduced-form model to recognize the term structure of credit spreads and value callable corporate bonds. Jarrow et al. [13] used the reduced-form model to price derivative securities involving credit risk by giving a stochastic term structure of default-free interest rates and a stochastic maturity specific credit-risk spread. Madan et al. [17] made great contribution to the development of reduce-form models as well by dealing with the complicated securities. Using the reduces-form model, Wang et al. derived the analytic formulas for the CLN prices and proved that the values of CLNs issued by an SPV were higher than that issued by the protection buyer (see [6] and [20]). Ge et al. [10] obtained the explicit formula for the CLN values under the reduced-form model through a partial differential equation (PDE) approach.

In this paper, we try to provide a credit risk model incorporating the effects of macroeconomics factors, such as interest rates, on the defaults in the framework of Markov chain, which is also a kind of reduced-form model. We will use markov copula to model the default correlation between the reference entity and CLN issuer where the two credit entities may default simultaneously. With the PDE techniques we can obtain the explicit formulas for the values of CLN and its CVA.

The arrangement of the rest is as follows. In section 2, we give the value processes of

CLNs with and without counterparty risk by analyzing their cash flows. The CVA process is investigated here as well. In section 3, we introduce the Markov chain approach in detail, including a Markov copula model, and the application of Markov chains on CLNs. In section 4, we apply a PDE approach to our pricing models and derive the explicit formulas for CLN values under the given assumptions. Numerical results are presented in section 5. Finally, we conclude this paper.

§2 Cash Flows and Pricing in a General Setup

In this section, we will briefly introduce the basics of CLNs with and without counterparty risk under Markov chain framework where simultaneous defaults could occur. Firstly, We label by 1 the reference entity, and by 2 the counterparty, i.e. the CLN issuer as well. Each of them may default before the maturity T. We denote by R_1 and R_2 the constant recovery rates of the reference entity and the CLN issuer respectively. Given a complete probability space $(\Omega,\mathcal{G},\mathbb{Q})$, where $\mathcal{G}$ stands for the whole market information and $\mathbb{Q}$ is a risk-neutral measure. Let the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ represents all the market information except default events. We denote the default times of the reference entity and the CLN issuer by two strictly positive random variables τ_1 and τ_2 on $(\Omega,\mathcal{G},\mathbb{Q})$ respectively. We introduce the right-continuous filtration $\mathcal{H}_t^i = \sigma\left\{\tau_i \le s : s \le t\right\}$, and define $\mathbb{H}^i = \left\{\mathcal{H}_t^i\right\}_{t\in[0,T]}, i = 1,2$. Let $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$, i.e. $\mathcal{G}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t^1, \mathcal{H}_t^2)$ for every $t \in [0,T]$. We also define the filtration $\mathcal{G}_t^1 = \sigma(\mathcal{F}_t, \mathcal{H}_t^1)$ for every $t \in [0,T]$ to explain the whole market information except the counterparty's default. We assume the spot interest rate r_t is a process adapted to the filtration $\mathbb{F}$.

Without loss of generality, we always assume the face value of CLN is 1. The cash flows are considered from the perspective of the investor.

In the case of a CLN with counterparty risk, the investor pays the face value to the counterparty to get the note at the issue date. We define $\tau = \tau_1 \wedge \tau_2$.

- If $\tau > T$, none of the reference entity and the counterparty default during the life of the CLN. The counterparty pays a constant premium rate k to the investor from the issue date to T and returns the face value 1 to the investor at the maturity.

- If $\tau \le T$, either the reference entity or the counterparty (or both) may default within the lifetime of the CLN.

  — $\tau = \tau_1 < \tau_2$: the reference entity defaults earlier than the counterparty. The counterparty continuously pays constant premium rate k until time τ_1 and returns a constant recovery payoff R_1 to the investor.

  — $\tau = \tau_2 < \tau_1$: the counterparty defaults earlier than the reference entity. The counterparty continuously pays the premium rate k until time τ_2 and returns a recovery payoff to the investor, which is R_2 times the CLN's market value.

— $\tau = \tau_1 = \tau_2$: the reference entity and the counterparty default simultaneously. The counterparty continuously pays the premium rate k until time τ and returns a constant recovery payoff $R_1 \cdot R_2$ to the investor.

We can obtain the cash flows for a CLN without counterparty risk by taking $\tau_2 = \infty$ from the above analysis.

Let v_t and u_t separately refer to the values of a CLN without counterparty risk and with counterparty risk at time t. From the cash flows summarized in the former, we can give the following pricing model.

Definition 2.1. *The value of a CLN without counterparty risk at t before maturity T is*

$$v_t = \mathbb{E}\big[V_t \mid \mathcal{G}_t^1\big], \quad 0 \le t \le T, \tag{2.1}$$

where

$$V_t = I_{\tau_1>t} \int_t^T k I_{\tau_1>s} e^{-\int_t^s r_\theta d\theta} ds + I_{t<\tau_1\le T} R_1 e^{-\int_t^{\tau_1} r_\theta d\theta} + I_{\tau_1>T} e^{-\int_t^T r_\theta d\theta}.$$

Definition 2.2. *The value of a CLN with counterparty risk at time t before maturity T is*

$$u_t = \mathbb{E}\big[U_t \mid \mathcal{G}_t\big], \quad 0 \le t \le T, \tag{2.2}$$

where

$$\begin{aligned} U_t = I_{\tau>t} & \int_t^T k I_{\tau_1>s} I_{\tau_2>s} e^{-\int_t^s r_\theta d\theta} ds + I_{t<\tau_1\le T} I_{\tau_2>\tau_1} R_1 e^{-\int_t^{\tau_1} r_\theta d\theta} \\ & + I_{t<\tau_2\le T} I_{\tau_1>\tau_2} R_2 u_{\tau_2} e^{-\int_t^{\tau_2} r_\theta d\theta} + I_{t<\tau_1\le T} I_{\tau_1=\tau_2} R_1 R_2 e^{-\int_t^{\tau_2} r_\theta d\theta} \\ & + I_{\tau_1>T} I_{\tau_2>T} e^{-\int_t^T r_\theta d\theta}. \end{aligned}$$

According the values of CLN in (2.1) and (2.2), we can give the following Definition, which follows [7].

Definition 2.3. *(i) The Exposure at Default (ED) is $\mathcal{G}_{\tau_2}$-measurable random variable ζ_{τ_2} defined by,*

$$\zeta_{\tau_2} = \begin{cases} R_1 - R_1 R_2, & \tau_2 = \tau_1 \le T, \\ v_{\tau_2} - R_2 u_{\tau_2}, & \tau_2 < \tau_1,\ \tau_2 \le T, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.3}$$

(ii) The Credit Valuation Adjustment (CVA) is the process killed at $\tau_1 \wedge \tau_2 \wedge T$ defined by, for $t \in [0, T]$,

$$cva_t = \mathbb{E}[I_{\tau_2>t} e^{-\int_t^{\tau_2} r_\theta d\theta} \zeta_{\tau_2} \mid \mathcal{G}_t]. \tag{2.4}$$

As is well known, Credit Valuation Adjustment (CVA) measures the counterparty credit risk of a credit derivative. In rough terms, the value of CVA is the difference between the risk-free value and the risky value. Before giving the exact expression of CVA, we first introduce the following Lemma.

***Lemma* 2.1.** *Let the stochastic process* $CVA_t = I_{\tau_2>t}V_t - U_t,\ \ 0 \le t \le T$. *Then we can obtain*

$$CVA_t = I_{t<\tau_2\le T} e^{-\int_t^{\tau_2} r_\theta d\theta}\Big[I_{\tau_1>\tau_2}\Big(V_{\tau_2} - R_2 u_{\tau_2}\Big) + I_{\tau_1=\tau_2}\Big[R_1\big(1-R_2\big)\Big]\Big], \tag{2.5}$$

where V_t, U_t *and* u_t *are defined by Definition 2.1 and Definition 2.2.*

Proof:

$$\begin{aligned}
CVA_t &= I_{\tau_2>t}V_t - U_t \\
&= I_{t<\tau_2\le T} I_{\tau_1>\tau_2} e^{-\int_t^{\tau_2} r_\theta d\theta}\Big(\int_{\tau_2}^{T} I_{\tau_1>s} k e^{-\int_{\tau_2}^{s} r_\theta d\theta} + I_{\tau_2<\tau_1\le T} R_1 e^{-\int_{\tau_2}^{\tau_1} r_\theta d\theta} \\
&\quad + I_{\tau_1>T} e^{-\int_{\tau_2}^{T} r_\theta d\theta} - R_2 u_{\tau_2}\Big) + I_{t<\tau_2\le T} I_{\tau_1=\tau_2} e^{-\int_t^{\tau_2} r_\theta d\theta}\Big(R_1 - R_1 R_2\Big) \\
&= I_{t<\tau_2\le T} e^{-\int_t^{\tau_2} r_\theta d\theta}\Big[I_{\tau_1>\tau_2}\Big(V_{\tau_2} - R_2 u_{\tau_2}\Big) + I_{\tau_1=\tau_2}\Big[R_1\big(1-R_2\big)\Big]\Big].
\end{aligned}$$

Let us denote by $\bar{v}_t = \mathbb{E}[V_t \mid \mathcal{G}_t]$. We introduce the following assumption which is standard in the literature..

Assumption 2.1.

$$\bar{v}_\tau = v_\tau \tag{2.6}$$

Assumption 2.1 has an intuitive financial interpretation that the default information of the counterparty is invalid for a derivative without counterparty risk.

Then, We can get the expression of cva_t.

***Corollary* 2.1.**

$$cva_t = I_{\tau_2>t} v_t - u_t. \tag{2.7}$$

Proof: According to Definition 2.1 and 2.2, we have

$$\begin{aligned}
I_{\tau_2>t}(v_t - u_t) &= I_{\tau_2>t}\mathbb{E}[V_t \mid \mathcal{G}_t^1] - \mathbb{E}[U_t \mid \mathcal{G}_t] \\
&\overset{(2.6)}{=} I_{\tau_2>t}\mathbb{E}[V_t \mid \mathcal{G}_t] - \mathbb{E}[U_t \mid \mathcal{G}_t] \\
&= \mathbb{E}[I_{\tau_2>t}V_t - U_t \mid \mathcal{G}_t] \\
&= \mathbb{E}[CVA_t \mid \mathcal{G}_t] \\
&\overset{(2.5)}{=} \mathbb{E}\Big[I_{t<\tau_2\le T} e^{-\int_t^{\tau_2} r_\theta d\theta}\Big[I_{\tau_1>\tau_2}\Big(V_{\tau_2} - R_2 u_{\tau_2}\Big) + I_{\tau_1=\tau_2}\Big[R_1\big(1-R_2\big)\Big]\Big]\Big|\mathcal{G}_t\Big] \\
&= \mathbb{E}\Big[I_{t<\tau_2\le T} e^{-\int_t^{\tau_2} r_\theta d\theta}\Big[I_{\tau_1>\tau_2}\Big(v_{\tau_2} - R_2 u_{\tau_2}\Big) + I_{\tau_1=\tau_2}\Big[R_1\big(1-R_2\big)\Big]\Big]\Big|\mathcal{G}_t\Big] \\
&\overset{(2.3)}{=} \mathbb{E}[I_{\tau_2>t} e^{-\int_t^{\tau_2} r_\theta d\theta} \zeta_{\tau_2} \mid \mathcal{G}_t] \\
&\overset{(2.4)}{=} cva_t.
\end{aligned}$$

□

§3 Markov Chain Model

In this section, we shall consider a Markov chain model of credit risk, in which, the counterparty and the reference entity can default simultaneously. By setting-up in the form of a bivariate Markov chain, we can make it possible to value a CLN and its CVA in a dynamic manner. In order to properly define the relationship between the generator functions of the bivariate Markov chain and its components, we first introduce the concept of Markov copula.

3.1 Markov Copula

Consider two $\mathbb{F}$-conditionally $\mathbb{G}$-Markov chains X_1 and X_2 relative to their own filtrations F^{X_1} and F^{X_2}, with values in $\mathcal{O}^1$ and $\mathcal{O}^2$ respectively. Suppose the generator of Markov chain X_1 is $A^1(t) = [\alpha^i_j(t)]_{i,j\in\mathcal{O}^1}$, where $\alpha^i_j(t)$ is the time-t intensity of jump from state i to state j. Similarly, we define the generator of Markov chain X_2 is $A^2(t) = [\beta^h_k(t)]_{h,k\in\mathcal{O}^2}$, where $\beta^h_k(t)$ is the time-t intensity of jump from state h to state k. Here $A^1(t)$, $A^2(t)$ are $\mathbb{F}$-progressively measurable matrix-value processes. To construct a bivariate Markov chain Z, whose components are X_1 and X_2, we introduce the following proposition (see [2]).

***Proposition* 3.1.** *Consider the system of equations in unknowns $\lambda^{ih}_{jk}(t)$, where $i, j \in \mathcal{O}^1$, $h, k \in \mathcal{O}^2$ and $(i, h) \neq (j, k)$:*

$$\begin{aligned}\sum_{k\in\mathcal{O}^2} \lambda^{ih}_{jk}(t) = \alpha^i_j(t), \forall h \in \mathcal{O}^2, \forall i, j \in \mathcal{O}^1, i \neq j,\\ \sum_{j\in\mathcal{O}^1} \lambda^{ih}_{jk}(t) = \beta^h_k(t), \forall i \in \mathcal{O}^1, \forall h, k \in \mathcal{O}^2, h \neq k.\end{aligned} \tag{3.1}$$

Suppose that the above system admits solution such that the matrix function

$$\Lambda(t) = [\lambda^{ih}_{jk}(t)]_{i,j\in\mathcal{O}^1,h,k\in\mathcal{O}^2} \tag{3.2}$$

with

$$\lambda^{ih}_{ih}(t) = -\sum_{\substack{(j,k)\in\mathcal{O}^1\times\mathcal{O}^2\\(j,k)\neq(i,h)}} \lambda^{ih}_{jk}(t), \tag{3.3}$$

properly defines an infinitesimal generator function of a Markov chain with values in $\mathcal{O}^1 \times \mathcal{O}^2$. Consider, a bivariate Markov chain $Z = (Z_1, Z_2)$ on $\mathcal{O}^1 \times \mathcal{O}^2$ with generators $\Lambda(t)$. Then, the components Z_1 and Z_2 are Markov chains w.r.t their own filtrations, and their generators are $A^1(t)$ and $A^2(t)$.

The system (3.1) and (3.2) serve as a "copula" between the Markovian margins X_1, X_2 and the bivariate Markov chain Z. This observation leads to the following definition, which can be found in [2].

Definition 3.1. *A Markov Copula between the Markov chains X_1 and X_2 is any solution to system (3.1) such that the matrix function $\Lambda(t) = [\lambda^{ih}_{jk}(t)]_{i,j\in\mathcal{O}^1,h,k\in\mathcal{O}^2}$ with $\lambda^{ih}_{ih}(t)$ given in*

(3.2), properly defines an infinitesimal generator function of a Markov chain with values in $\mathcal{O}^1 \times \mathcal{O}^2$.

3.2 Credit-Linked Notes: Markov Chain Model

Suppose X_1 and X_2 are the indicator processes of the reference entity and the counterparty in a CLN contract. We define the pair $Z = (X_1, X_2)$ as a bivariate Markov chain relative to its own filtration on probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ with the state space $E = \{(0,0),(1,0),(0,1),(1,1)\}$, where

- $Z_t = (0,0)$ represents none of the two credit entities defaults at time t;
- $Z_t = (1,0)$ represents the reference entity defaults but the counterparty doesn't at time t;
- $Z_t = (0,1)$ represents the counterparty defaults but reference entity doesn't at time t;
- $Z_t = (1,1)$ represents both of the reference entity and the counterparty default at time t.

Since the defaults are absorbing states, we can assume the time-t matrix-generator of Z is given by the following 4×4 matrix $\Lambda(t)$:

$$\begin{array}{c} \\ (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (1,0) & (0,1) & (1,1) \end{array} \\ \begin{pmatrix} -(\lambda_1(t)+\lambda_2(t)+\lambda_3(t)) & \lambda_1(t) & \lambda_2(t) & \lambda_3(t) \\ 0 & -\alpha_1^0(t) & 0 & \alpha_1^0(t) \\ 0 & 0 & -\beta_1^0(t) & \beta_1^0(t) \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{3.4}$$

Suppose that Markov copula conditions (3.1) are satisfied. Then we can get the following equations:

$$\begin{aligned} \lambda_2(t)+\lambda_3(t) &= \alpha_1^0(t), \quad \forall t \in [0,T], \\ \lambda_1(t)+\lambda_3(t) &= \beta_1^0(t), \quad \forall t \in [0,T]. \end{aligned} \tag{3.5}$$

The financial interpretation for the first equation of (3.4) is that whether the reference entity defaults or not, there would be no effect on the counterparty's default. The second equation of (3.4) means the counterparty's default would not affect the reference entity's default.

We can construct the Markov chain Z by the canonical method discussed in [3]. Then we have

$$\begin{aligned} \mathbb{Q}(\tau_1 > t \mid \mathcal{F}_t) &= e^{-\int_0^t (\lambda_1(s)+\lambda_3(s))ds}, \\ \mathbb{Q}(\tau_2 > t \mid \mathcal{F}_t) &= e^{-\int_0^t (\lambda_2(s)+\lambda_3(s))ds}, \\ \mathbb{Q}(\tau_1 > t, \tau_2 > t \mid \mathcal{F}_t) &= e^{-\int_0^t (\lambda_1(s)+\lambda_2(s)+\lambda_3(s))ds}. \end{aligned} \tag{3.6}$$

Let $H_t^1 = I_{\{Z_t=(0,0)\}}$, $H_t^2 = I_{\{Z_t=(1,0)\}}$, $H_t^3 = I_{\{Z_t=(0,1)\}}$, $H_t^4 = I_{\{Z_t=(1,1)\}}$ and $H_t^{ij} = \sum_{0<s\le t} H_{s-}^i H_s^j$ with $i,j = 1,2,3,4$ and $i \neq j$, where I_A is the indicator function, which means

$I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \text{otherwise} \end{cases}$. Here, we assume $H_0^1 = 1$.

From the symbols above, We can conclude that

- $I_{t<\tau_1 \leq T} I_{\tau_1 < \tau_2} = \sum_{t<s\leq T} H_{s-}^1 H_s^2$;
- $I_{t<\tau_2 \leq T} I_{\tau_2 < \tau_1} = \sum_{t<s\leq T} H_{s-}^1 H_s^3$;
- $I_{t<\tau_1 \leq T} I_{\tau_1 = \tau_2} = \sum_{t<s\leq T} H_{s-}^1 H_s^4$;
- $I_{\tau_1 > t} I_{\tau_2 > t} = H_t^1$.

Next, we introduce three lemmas from [3].

***Lemma* 3.1.** *For any $\mathcal{G}$-measurable and integrable random variable X, we have, for any $t \in \mathbb{R}_+$,*

$$\mathbb{E}[I_{\tau>t} X \mid \mathcal{G}_t] = I_{\tau>t} \frac{\mathbb{E}[I_{\tau>t} X \mid \mathcal{F}_t]}{\mathbb{E}[I_{\tau>t} \mid \mathcal{F}_t]}. \tag{3.7}$$

***Lemma* 3.2.** *Assume that Y is a bounded, $\mathbb{F}$-predictable process such that the random variable Y_t is integrable, then we have, for every $t < s \leq T$,*

$$\mathbb{E}[I_{t<\tau\leq s} Y_\tau \mid \mathcal{G}_t] = I_{\tau>t} \frac{\mathbb{E}[\int_t^s Y_u dF_u \mid \mathcal{F}_t]}{\mathbb{E}[I_{\tau>t} | \mathcal{F}_t]}, \tag{3.8}$$

where $F_t = 1 - \mathbb{Q}(\tau > t \mid \mathcal{F}_t) = 1 - \mathbb{E}[I_{\tau>t} \mid \mathcal{F}_t]$.

***Lemma* 3.3.** *For any $\mathcal{G}$-measurable and integrable random variable X, we have, for any $t \in \mathbb{R}_+$,*

$$\mathbb{E}[I_{\tau_1>} I_{\tau_2>t} X \mid \mathcal{G}_t] = I_{\tau_1>t} I_{\tau_2>t} \frac{\mathbb{E}[I_{\tau_1>t} I_{\tau_2>t} X \mid \mathcal{F}_t]}{\mathbb{E}[I_{\tau_1>t} I_{\tau_2>t} \mid \mathcal{F}_t]}. \tag{3.9}$$

Using Lemma 3.1–3.3 and (3.5), we can get the following result.

***Corollary* 3.1.** *The value of a CLN with counterparty risk at time t before maturity T can be presented as*

$$\begin{aligned} u_t = I_{\tau_1>t} I_{\tau_2>t} \mathbb{E}\Big[\int_t^T e^{-\int_t^s \left(r_\theta + \lambda_1(\theta) + \lambda_2(\theta) + \lambda_3(\theta)\right) d\theta} \big[k + R_1 \lambda_1(s) \\ + R_2 \lambda_2(s) u_s + R_1 R_2 \lambda_3(s)\big] ds + e^{-\int_t^T \left(r_\theta + \lambda_1(\theta) + \lambda_2(\theta) + \lambda_3(\theta)\right) d\theta} \big| \mathcal{F}_t \Big]. \end{aligned} \tag{3.10}$$

Proof: It's easy to derive the following equations from (3.6)–(3.8).

$$\mathbb{E}\Big[\int_t^T kI_{\tau_1>s}I_{\tau_2>s}e^{-\int_t^s r_\theta d\theta}ds\big|\mathcal{G}_t\Big]$$
$$=\frac{I_{\tau_1>t}I_{\tau_2>t}}{\mathbb{E}[I_{\tau_1>t}I_{\tau_2>t}\big|\mathcal{F}_t]}\mathbb{E}\Big[\int_t^T \mathbb{E}\big[I_{\tau_1>s}I_{\tau_2>s}\big|\mathcal{F}_s\big]ke^{-\int_t^s r_\theta d\theta}ds\big|\mathcal{F}_t\Big]$$
$$=I_{\tau_1>t}I_{\tau_2>t}\mathbb{E}\Big[\int_t^T ke^{-\int_t^s \big(r_\theta+\lambda_1(\theta)+\lambda_2(\theta)+\lambda_3(\theta)\big)d\theta}ds\big|\mathcal{F}_t\Big],$$

and

$$\mathbb{E}\Big[I_{t<\tau_1\le T}I_{\tau_2>\tau_1}R_1e^{-\int_t^{\tau_1} r_\theta d\theta}\big|\mathcal{G}_t\Big]$$
$$=\frac{I_{\tau_1>t}I_{\tau_2>t}}{\mathbb{E}[I_{\tau_1>t}I_{\tau_2>t}\big|\mathcal{F}_t]}\mathbb{E}\Big[\sum_{t<s\le T} H^1_{s-}H^2_sR_1e^{-\int_t^{\tau_1} r_\theta d\theta}\big|\mathcal{F}_t\Big]$$
$$=\frac{I_{\tau_1>t}I_{\tau_2>t}}{\mathbb{E}[I_{\tau_1>t}I_{\tau_2>t}\big|\mathcal{F}_t]}\mathbb{E}\Big[\int_t^T R_1e^{-\int_t^s r_\theta d\theta}\lambda_1(s)H^1_sds\big|\mathcal{F}_t\Big]$$
$$=I_{\tau_1>t}I_{\tau_2>t}\mathbb{E}\Big[\int_t^T R_1\lambda_1(s)e^{-\int_t^s \big(r_\theta+\lambda_1(\theta)+\lambda_2(\theta)+\lambda_3(\theta)\big)d\theta}ds\big|\mathcal{F}_t\Big],$$

where the fourth equality is because $H^{12}_t-\int_0^t\lambda_1(s)H^1_sds$ is a $\mathbb{G}$-martingale (see [3]).

Similarly, we have

$$\mathbb{E}\Big[I_{t<\tau_2\le T}I_{\tau_1>\tau_2}R_2u_{\tau_2}e^{-\int_t^{\tau_2} r_\theta d\theta}\big|\mathcal{G}_t\Big]$$
$$=I_{\tau_1>t}I_{\tau_2>t}\mathbb{E}\Big[\int_t^T e^{-\int_t^s \big(r_\theta+\lambda_1(\theta)+\lambda_2(\theta)+\lambda_3(\theta)\big)d\theta}R_2\lambda_2(s)u_sds\big|\mathcal{F}_t\Big],$$
$$\mathbb{E}\Big[I_{t<\tau_1\le T}I_{\tau_1=\tau_2}R_1R_2e^{-\int_t^{\tau_2} r_\theta d\theta}\big|\mathcal{G}_t\Big]$$
$$=I_{\tau_1>t}I_{\tau_2>t}\mathbb{E}\Big[\int_t^T e^{-\int_t^s \big(r_\theta+\lambda_1(\theta)+\lambda_2(\theta)+\lambda_3(\theta)\big)d\theta}R_1R_2\lambda_3(s)ds\big|\mathcal{F}_t\Big],$$
$$\mathbb{E}\Big[I_{\tau_1>T}I_{\tau_2>T}e^{-\int_t^T r_\theta d\theta}\big|\mathcal{G}_t\Big]=I_{\tau_1>t}I_{\tau_2>t}\mathbb{E}\big[e^{-\int_t^T \big(r_\theta+\lambda_1(\theta)+\lambda_2(\theta)+\lambda_3(\theta)\big)d\theta}\big|\mathcal{F}_t\big].$$

Then we can obtain (3.9). □

We can also get the following results from equation (2.1) and equation (2.4).

***Corollary* 3.2.** *The value of a CLN without counterparty risk at time t before maturity T can be presented as*

$$v_t=I_{\tau_1>t}\mathbb{E}\Big[\int_t^T e^{-\int_t^s(r_\theta+\lambda_1(\theta)+\lambda_3(\theta))d\theta}[k+R_1\big(\lambda_1(s)+\lambda_3(s)\big)]ds+e^{-\int_t^T(r_\theta+\lambda_1(\theta)+\lambda_3(\theta))d\theta}\mid\mathcal{F}_t\Big]. \tag{3.11}$$

***Corollary* 3.3.** *The CVA of a CLN at time t before maturity T can be presented as*

$$cva_t=I_{\tau_1>t}I_{\tau_2>t}\mathbb{E}\Big[\int_t^T e^{-\int_t^s \big(r_\theta+\lambda_1(\theta)+\lambda_2(\theta)+\lambda_3(\theta)\big)d\theta}\big[\lambda_2(s)\big(v_s-R_2u_s\big)+\lambda_3(s)R_1\big(1-R_2\big)\big]ds\big]\big|\mathcal{F}_t\Big]. \tag{3.12}$$

§4 PDE Method

In this section, we first introduce how to model the default intensities. Then we derive the pricing PDEs for CLNs and give the explicit formulas.

We assume $\mathcal{F}_t = \sigma(r_s, 0 \le s \le t)$, and the spot interest rate r_t satisfies CIR model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t, \tag{4.1}$$

where W_t is a Brownian motion under risk-neutral probability measure $\mathbb{Q}$, and κ, θ, σ are positive constant parameters satisfying the Feller condition (see [9])

$$2\kappa\theta > \sigma^2. \tag{4.2}$$

Suppose that the default intensities $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ are smooth functions of r_t. By the strong markov property of r_t, on $\{\tau_1 > t\}$ and $\{\tau_1 > t, \tau_2 > t\}$, the equations (3.10) and (3.9) can be written separately as:

$$v_t = \mathbb{E}\Big[\int_t^T e^{-\int_t^s \left(r_\theta+\lambda_1(\theta)+\lambda_3(\theta)\right)d\theta}[k + R_1\big(\lambda_1(s)+\lambda_3(s)\big)]ds + e^{-\int_t^T \left(r_\theta+\lambda_1(\theta)+\lambda_3(\theta)\right)d\theta}\big|r_t\Big], \tag{4.3}$$

$$\begin{aligned} u_t = \mathbb{E}\Big[\int_t^T & e^{-\int_t^s \left(r_\theta+\lambda_1(\theta)+\lambda_2(\theta)+\lambda_3(\theta)\right)d\theta}[k + R_1\lambda_1(s) + R_2\lambda_2(s)u_s + R_1R_2\lambda_3(s)]ds \\ &+ e^{-\int_t^T \left(r_\theta+\lambda_1(\theta)+\lambda_2(\theta)+\lambda_3(\theta)\right)d\theta}\big|r_t\Big]. \end{aligned} \tag{4.4}$$

4.1 Default Intensities Modelling

The CLN issuer usually holds some risky assets of reference entity. When the reference entity defaults, the CLN issuer would suffer some losses. So the default intensities of the CLN issuer and the reference entity usually change in the same direction. Here we assume that the interest rate r_t is the only market factor that affects the default intensities and may have two types of effects, that is, positive correlation or negative correlation between default intensities and the interest rate. For the sake of simplicity, we only consider the following two cases:

- Positive Correlation:

$$\begin{cases} \lambda_1(t) = h_1 r_t + k_1, \\ \lambda_2(t) = h_2 r_t + k_2, \\ \lambda_3(t) = h_3 r_t + k_3. \end{cases} \tag{4.5}$$

- Negative Correlation:

$$\begin{cases} \lambda_1(t) = \dfrac{\xi_1}{r_t} + \eta_1, \\ \lambda_2(t) = \dfrac{\xi_2}{r_t} + \eta_2, \\ \lambda_3(t) = \dfrac{\xi_3}{r_t} + \eta_3, \end{cases} \tag{4.6}$$

where, h_i, k_i, ξ_i and η_i $(i = 1, 2, 3)$ are all positive constant parameters. In (4.5), the default densities satisfy CIR process. If they follow (4.6), the default densities satisfy inverse CIR (ICIR) processes, which can be found in [1] and [12].

4.2 Pricing PDE of CLNs and Explicit Formulas

We denote by $v_t = v(t, r_t)$ and $u_t = u(t, r_t)$ as the pre-default values of CLNs. It's easy to obtain the pricing PDEs for CLN pre-default values $v(t, r)$ and $u(t, r)$ from (4.3) and (4.4) separately by Feynman-Kac formula.

4.2.1 Positive Correlation

Under the positive correlation assumption (4.5), we can get the following pricing PDEs.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\sigma^2 r \dfrac{\partial^2 u}{\partial r^2} + \kappa(\theta - r)\dfrac{\partial u}{\partial r} - \big[[1 + h_1 + (1 - R_2)h_2 + h_3]r + k_1 + (1 - R_2)k_2 + k_3\big]u \\ \quad + [k + R_1(h_1 r + k_1) + R_1 R_2(h_3 r + k_3)] = 0, \qquad (0 < r < \infty, 0 \le t < T), \\ u(T, r) = 1, \qquad (0 < r < \infty). \end{cases} \tag{4.7}$$

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 r \dfrac{\partial^2 v}{\partial r^2} + \kappa(\theta - r)\dfrac{\partial v}{\partial r} - [(1 + h_1 + h_3)r + k_1 + k_3]v \\ \quad + \big[k + R_1[(h_1 + h_3)r + k_1 + k_3]\big] = 0, \qquad (0 < r < \infty, 0 \le t < T), \\ v(T, r) = 1, \qquad (0 < r < \infty). \end{cases} \tag{4.8}$$

To solve (4.7), we first give the following transformation

$$P(t, r) = -e^{-a_1 r - a_2 t}\frac{\partial u}{\partial t}, \tag{4.9}$$

which means

$$u(t, r) = 1 + \int_t^T e^{(a_1 r + a_2 s)} P(s, r) ds. \tag{4.10}$$

An easy calculation will show that $P(t, r)$ satisfies the following PDE

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 r \dfrac{\partial^2 P}{\partial r^2} + \kappa'(\theta' - r)\dfrac{\partial P}{\partial r} = 0, \qquad (0 < r < \infty, 0 \le t < T), \\ P(T, r) = -e^{-a_1 r - a_2 T}\big[[1 + (1 - R_1)h_1 + (1 - R_2)h_2 + (1 - R_1 R_2)h_3]r + (1 - R_1)k_1 \\ \qquad + (1 - R_2)k_2 + (1 - R_1 R_2)k_3 - k\big], \qquad (0 < r < \infty). \end{cases} \tag{4.11}$$

where

$$\begin{aligned} a_1 &= \frac{\kappa - \sqrt{\kappa^2 + 2\sigma^2(h_1 + h_2 + h_3 + 1 - R_2 h_2)}}{\sigma^2}, \\ a_2 &= -\kappa\theta a_1 - k_2 R_2 + (k_1 + k_2 + k_3), \\ \kappa' &= \kappa - a_1\sigma^2, \theta' = \frac{\kappa\theta}{\kappa - a_1\sigma^2}. \end{aligned} \tag{4.12}$$

In order to solve (4.11), we introduce the following lemma (see [9]).

***Lemma* 4.1.** *The fundamental solution of the following equations*

$$\begin{cases} \dfrac{\partial P}{\partial t}+\dfrac{1}{2}\sigma^2 r\dfrac{\partial^2 P}{\partial r^2}+\kappa'(\theta'-r)\dfrac{\partial P}{\partial r}=0, & (0<r<\infty, 0\le t<T), \\ P(T,r;y)=\delta(r-y), & (0<r<\infty). \end{cases} \tag{4.13}$$

is $G(t,r;y)$, where $\delta(\cdot)$ is Dirac delta function and the function G is denoted by

$$G(t,r;y)=ce^{-p-q}\Big(\frac{q}{p}\Big)^{\frac{z}{2}}I_z\big(2(pq)^{\frac{1}{2}}\big) \tag{4.14}$$

with

$$\begin{aligned} &c=\frac{2\kappa'}{\sigma^2(1-e^{-\kappa'(T-t)})}, z=\frac{2\kappa'\theta'}{\sigma^2}-1, p=cre^{-\kappa'(T-t)}, q=cy, \\ &I_z\Big(2(pq)^{\frac{1}{2}}\Big)=\big(pq\big)^{\frac{z}{2}}\sum_{k=0}^{\infty}\frac{\big(pq\big)^k}{k!}. \end{aligned} \tag{4.15}$$

Then we can get the solution of (4.11) as follows.

$$\begin{aligned} P(t,r)=&\int_0^\infty P(T,y)\cdot G(t,r;y)dy \\ =&\Big(\frac{c}{a_1+c}\Big)^{z+1}\cdot e^{-\frac{a_1 p}{a_1+c}-a_2T} \\ &\cdot\Bigg(\frac{\Big((R_1-1)h_1+(R_2-1)h_2+(R_1R_2-1)h_3-1\Big)\Big(cp+(z+1)(a_1+c)\Big)}{\big(a_1+c\big)^2} \\ &+(R_1-1)k_1+(R_2-1)k_2+(R_1R_2-1)k_3+k\Bigg). \end{aligned} \tag{4.16}$$

We can conclude by the following theorem.

***Theorem* 4.1.** *The pre-default value of a CLN with counterparty risk and default intensities satisfying (4.5) is*

$$u(t,r)=1+\int_t^T e^{a_1r+a_2s}P(s,r)ds, \tag{4.17}$$

where a_1, a_2 and $P(s,t)$ are defined by (4.12) and (4.16).

A similar calculation leads to the following result.

***Theorem* 4.2.** *The pre-default of a CLN without counterparty risk and default intensities satisfying (4.5) is*

$$v(t,r)=1+\int_t^T e^{b_1r+b_2s}Q(s,r)ds, \tag{4.18}$$

where

$$Q(t,r) = \Big(\frac{\big((R_1-1)(h_1+h_3)-1\big)\big(cp+(z+1)(b_1+c)\big)}{(b_1+c)^2} + (R_1-1)(k_1+k_3) + k\Big)\cdot\Big(\frac{c}{b_1+c}\Big)^{z+1}\cdot e^{-\frac{b_1p}{a_1+c}-b_2T}, \tag{4.19}$$

$$b_1 = \frac{\kappa-\sqrt{\kappa^2+2\sigma^2(h_1+h_3+1)}}{\sigma^2}, \qquad b_2 = -\kappa\theta b_1 + k_1 + k_3.$$

Remark 4.1. *If we don't consider the possibility of simultaneous defaults, that is, the default density* λ_3 *equals to* 0 *in the generator matrix, the pricing formula (4.17) for the CLN can be simplified to*

$$\bar{u}(t,r) = 1 + \int_t^T e^{\bar{a}_1 r + \bar{a}_2 s}\bar{P}(s,r)ds, \tag{4.20}$$

where $\bar{P}(s,t)$ *is defined by*

$$\bar{P}(s,t) = \Bigg(\frac{\Big((R_1-1)h_1+(R_2-1)h_2-1\Big)\Big(cp+(z+1)(\bar{a}_1+c)\Big)}{\Big(\bar{a}_1+c\Big)^2} + (R_1-1)k_1 + (R_2-1)k_2 + k\Bigg)\cdot\Big(\frac{c}{\bar{a}_1+c}\Big)^{z+1}\cdot e^{-\frac{\bar{a}_1p}{\bar{a}_1+c}-\bar{a}_2T} \tag{4.21}$$

with

$$\bar{a}_1 = \frac{\kappa-\sqrt{\kappa^2+2\sigma^2(h_1+h_2+1-R_2h_2)}}{\sigma^2}, \quad \bar{a}_2 = -\kappa\theta a_1 - k_2R_2 + (k_1+k_2),$$

and c, p, z *defined in (4.15). This is the result obtained by Ge et. al [10] under conditional independence assumption.*

Remark 4.2. *Similarly, by (3.11), we can denote the pre-default values of CVA by* $cva_t = cva(r_t,t)$ *and derive the following PDE.*

$$\begin{cases} \dfrac{\partial cva}{\partial t} + \dfrac{1}{2}\sigma^2 r\dfrac{\partial^2 cva}{\partial r^2} + \kappa(\theta-r)\dfrac{\partial cva}{\partial r} - [(1+h_1+h_2+h_3)r + k_1+k_2+k_3]cva \\ \quad + (h_2r+k_2)(v-R_2u) + R_1(1-R_2)(h_3r+k_3) = 0, \qquad (0<r<\infty, 0\le t<T), \\ cva(T,r) = 0, \qquad (0<r<\infty). \end{cases} \tag{4.22}$$

Noticing that equation (4.22) can be obtained by taking the difference between equations (4.7) and (4.8), we can get the explicit formula for pre-default CVA by calculating

$$\begin{aligned} cva(t,r) &= v(t,r) - u(t,r) \\ &= \int_t^T \big[e^{b_1r+b_2s}Q(s,r) - e^{a_1r+a_2t}P(s,r)\big]ds, \end{aligned} \tag{4.23}$$

where, $\{a_i\}_{i=1,2}$*,* $\{b_i\}_{i=1,2}$*,* $P(s,t)$ *and* $Q(s,t)$ *are defined by (4.12), (4.16) and (4.19).*

4.2.2 Negative Correlation

When default densities satisfy (4.6), we can obtain by using the Feynman-Kac formula that $u(r,t)$ and $v(r,t)$ satisfy the following PDEs.

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{1}{2}\sigma^2 r\dfrac{\partial^2 u}{\partial r^2}+\kappa(\theta-r)\dfrac{\partial u}{\partial r}-\big[r+\dfrac{\xi_1+(1-R_2)\xi_2+\xi_3}{r}+\eta_1+(1-R_2)\eta_2+\eta_3]u \\ \quad +[k+R_1(\dfrac{\xi_1}{r}+\eta_1)+R_1R_2(\dfrac{\xi_3}{r}+\eta_3)]=0, \qquad (0<r<\infty, 0\le t<T), \\ u(T,r)=1, \qquad (0<r<\infty). \end{cases} \tag{4.24}$$

$$\begin{cases} \dfrac{\partial v}{\partial t}+\dfrac{1}{2}\sigma^2 r\dfrac{\partial^2 v}{\partial r^2}+\kappa(\theta-r)\dfrac{\partial v}{\partial r}-(r+\dfrac{\xi_1+\xi_3}{r}+\eta_1+\eta_3)v \\ \quad +\big[k+R_1(\dfrac{\xi_1+\xi_3}{r}+\eta_1+\eta_3)\big]=0, \qquad (0<r<\infty, 0\le t<T), \\ v(T,r)=1, \qquad (0<r<\infty). \end{cases} \tag{4.25}$$

To solve (4.24), we first define

$$P_1(t,r)=-r^{-a_3}e^{-a_1 r-a_2 t}\frac{\partial u}{\partial t}. \tag{4.26}$$

Then we can get the following PDE.

$$\begin{cases} \dfrac{\partial P_1}{\partial t}+\dfrac{1}{2}\sigma^2 r\dfrac{\partial^2 P_1}{\partial r^2}+\kappa'(\theta'-r)\dfrac{\partial P_1}{\partial r}=0, \qquad (0<r<\infty, 0\le t<T), \\ P_1(T,r)=-r^{-a_3}e^{-a_1 r-a_2 T}\Big[r+(1-R_1)\Big(\dfrac{\xi_1}{r}+\eta_1\Big)+(1-R_2)\Big(\dfrac{\xi_2}{r}+\eta_2\Big) \\ \qquad +(1-R_1R_2)\Big(\dfrac{\xi_3}{r}+\eta_3\Big)-k\Big], \qquad (0<r<\infty), \end{cases} \tag{4.27}$$

where

$$\begin{aligned} a_1&=\frac{\kappa-\sqrt{\kappa^2+2\sigma^2}}{\sigma^2}, a_2=\kappa a_3-\kappa\theta a_1-a_1a_3\sigma^2+\eta_1+(1-R_2)\eta_2+\eta_3, \\ a_3&=\frac{\frac{1}{2}\sigma^2-\kappa\theta+\sqrt{(\kappa\theta-\frac{1}{2}\sigma^2)^2+2\sigma^2(\xi_1+(1-R_2)\xi_2+\xi_3)}}{\sigma^2}, \\ \kappa'&=\kappa-a_1\sigma^2, \theta'=\frac{\sigma^2 a_3+\kappa\theta}{\kappa-a_1\sigma^2}. \end{aligned} \tag{4.28}$$

Using Lemma 4.1, we can get

$$\begin{aligned} P_1(t,r)&=\int_0^\infty P_1(T,y)G(t,r;y)dy \\ &=[(R_1-1)\xi_1+(R_2-1)\xi_2+(R_1R_2-1)\xi_3]c^{a_3+1}e^{-a_2T-p}(\frac{c}{a_1+c})^{z-a_3}M(z-a_3, \\ &\quad z+1,\frac{cp}{a_1+c})\frac{\Gamma(z-a_3)}{\Gamma(z+1)}+[k+(R_1-1)\eta_1+(R_2-1)\eta_2+(R_1R_2-1)\eta_3] \\ &\quad \cdot c^{a_3}e^{-a_2T-p}(\frac{c}{a_1+c})^{z-a_3+1}M(z-a_3+1,z+1,\frac{cp}{a_1+c})\frac{\Gamma(z-a_3+1)}{\Gamma(z+1)} \\ &\quad -c^{a_3-1}e^{-a_2T-p}(\frac{c}{a_1+c})^{z-a_3+2}M(z-a_3+2,z+1,\frac{cp}{a_1+c})\frac{\Gamma(z-a_3+2)}{\Gamma(z+1)}, \end{aligned} \tag{4.29}$$

where

$$M(\mu,\nu,z)=\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!(\nu)_k}z^k \tag{4.30}$$

is the confluent hypergeometric function with $(\mu)_k$ defined by $(\mu)_k=\mu(\mu+1)\cdots(\mu+k-1)$ for $k>0,(\mu)_0=1$.

We conclude it as the following theorem.

***Theorem* 4.3.** *The pre-default value of a CLN with counterparty risk and default intensities satisfying (4.6) is*

$$u(t,r)=1+\int_t^T r^{a_3}e^{(a_1r+a_2s)}P_1(s,r)ds, \tag{4.31}$$

where a_1, a_2 and $P_1(s,t)$ are defined by (4.28) and (4.29).

A similar calculation leads to the following result.

***Theorem* 4.4.** *The pre-default value of the a CLN without counterparty risk and default intensities satisfying (4.6) is*

$$v(t,r)=1+\int_t^T r^{b_3}e^{(b_1r+b_2s)}Q_1(s,r)ds, \tag{4.32}$$

where $Q_1(t,r)$ is defined as

$$\begin{aligned}Q_1(t,r)=&(R_1-1)(\xi_1+\xi_3)c^{b_3+1}e^{-b_2T-p}(\frac{c}{b_1+c})^{z-b_3}M(z-b_3,z+1,\frac{cp}{b_1+c})\frac{\Gamma(z-b_3)}{\Gamma(z+1)}\\&+[k+(R_1-1)(\eta_1+\eta_3)]c^{b_3}e^{-b_2T-p}(\frac{c}{b_1+c})^{z-b_3+1}M(z-b_3+1,z+1,\frac{cp}{b_1+c})\\&\cdot\frac{\Gamma(z-b_3+1)}{\Gamma(z+1)}-c^{b_3-1}e^{-b_2T-p}(\frac{c}{b_1+c})^{z-b_3+2}M(z-b_3+2,z+1,\frac{cp}{b_1+c})\\&\cdot\frac{\Gamma(z-b_3+2)}{\Gamma(z+1)},\end{aligned} \tag{4.33}$$

with

$$\begin{aligned}b_1=&\frac{\kappa-\sqrt{\kappa^2+2\sigma^2}}{\sigma^2},b_2=\kappa b_3-\kappa\theta b_1-b_1b_3\sigma^2+\eta_1+\eta_3,\\b_3=&\frac{\frac{1}{2}\sigma^2-\kappa\theta+\sqrt{(\kappa\theta-\frac{1}{2}\sigma^2)^2+2(\xi_1+\xi_3)\sigma^2}}{\sigma^2},\end{aligned} \tag{4.34}$$

and κ', θ', c, p, z, $M(\mu,\nu,z)$ are defined by (4.28), (4.15) and (4.30).

Remark 4.3. *Similarly, the explicit formula for CVA with default intensities satisfying (4.6) is*

$$\begin{aligned}cva(t,r)&=v(t,r)-u(t,r)\\&=\int_t^T\left[r^{b_3}e^{b_1r+b_2s}Q_1(s,r)-r^{a_3}e^{a_1r+a_2t}P_1(s,r)\right]ds,\end{aligned} \tag{4.35}$$

where, $\{a_i\}_{i=1,2}$, $P_1(s,t)$, $\{b_i\}_{i=1,2,3}$, and $Q_1(s,t)$ are defined by (4.28), (4.29), (4.34) and (4.33) separately.

§5 Numerical Analysis

In this section, we will verify our theoretical results by numerical experiments. We will show the CLN values under different circumstance and analyze how parameter values affect the prices of CLNs and CVA.

In order to compare with Ge's results (see [10]), the numerical tests below will be done using the same parameter values of the CIR model in their paper, that is,

$$\kappa = 0.0904, \theta = 0.0594, \sigma = 0.0295. \tag{5.1}$$

The parameter values of the default intensities are assumed to be

- Positive Correlation:

$$h_1 = 2, k_1 = 0.16; h_2 = 1.5, k_2 = 0.1; h_3 = 1.0, k_3 = 0.06.$$

- Negative Correlation:

$$\xi_1 = 0.002, \eta_1 = 0.16; \xi_2 = 0.0015, \eta_2 = 0.1; \xi_3 = 0.001, \eta_3 = 0.06.$$

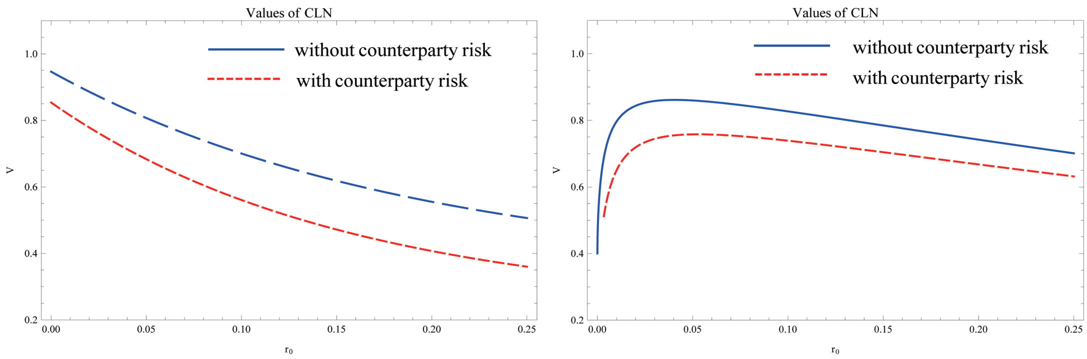
Other parameters are assumed to be $T = 1.5$, $k = 0.1$, $R_1 = 0.4$, and $R_2 = 0.4$.

In Figure 2, (a) shows the values of CLNs with and without counterparty risk while the interest rate r_0 changes from 0 to 0.25 under the positive correlation assumption. It verifies that the higher the interest rate, the smaller the discount factor, and the lower the CLN values. Meanwhile, since the investor of CLNs with counterparty risk has to face the default risk brought by the issuer, the values of these CLNs are always lower than those without couterparty risk. Also, (b) shows the values of CLNs when the interest rate changes from 0 to 0.25 under the negative correlation assumption. In the beginning, with the increase of r_0, the default density has more influence on the CLN values than the discount factor and the CLN values go up. However, when r_0 becomes large enough, the discount factor will affect more and the CLN values begin to go down.

The two charts in Figure 3 show the difference between two frameworks in pricing a CLN with counterparty risk. One is the reduced-form framework where defaults of the reference entity and the counterparty are conditionally independent (see [10]). The other is the Markov chain framework discussed in this paper where the two credit entities could default simultaneously and the possibility of the defaults is higher than that in [10]. Owing to this, the CLN values in this paper are lower than that in [10], which is showed in Figure 3. Under the positive correlation, the values' difference between two frameworks goes up at the very start, for the default intensities affect less than the discount factor in pricing. When r_0 is getting larger and larger, the difference tends to be stabilized gradually because the discount factor plays a more important role at this time. On the contrary, under the negative correlation, lower interest rate will bring stronger default densities and the default intensities affect more than the discount factor, that is, the difference between two pricing frameworks goes down with the increase of interest rate in the beginning. When r_0 is getting larger and larger, the difference tends to be stabilized as well.

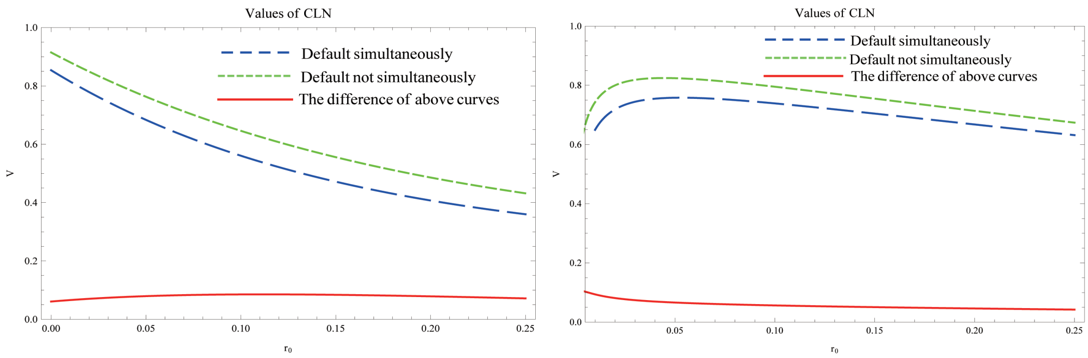
Figure 4 shows the CVA values of a CLN. In the case of positive correlation, the CVA value increases with the rise of the interest rate. Usually higher interest rate will bring stronger default intensity, and the investor will suffer from corresponding counterparty risk. On the contrary, under the negative correlation, the rise of the interest rate means that default probabilities will go down. Hence, the CVA value is negatively correlated with interest rate.

Figure 5 and Figure 6 show the sensitivity analysis of the CVA values to parameters of the default intensity $\lambda_3(t)$. The cases for other default intensities $\lambda_1(t)$ and $\lambda_2(t)$ are similar. We understand the rising of the default intensities will increase the default probabilities of the credit entities and decrease the CLN values with counterparty risk. Thus, the larger the default intensities are, the higher the CVA values will be under both positive correlation and negative correlation situations.



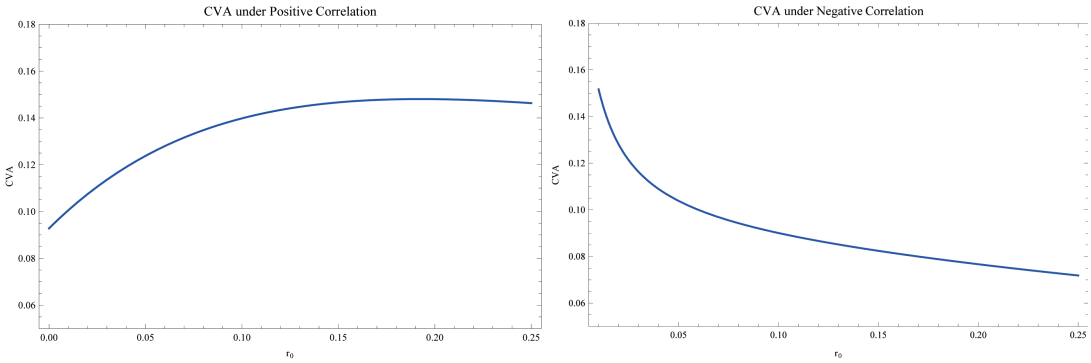
(a) Values of CLNs under positive correlation. (b) Values of CLNs under negative correlation.

Figure 2. Values of CLNs.



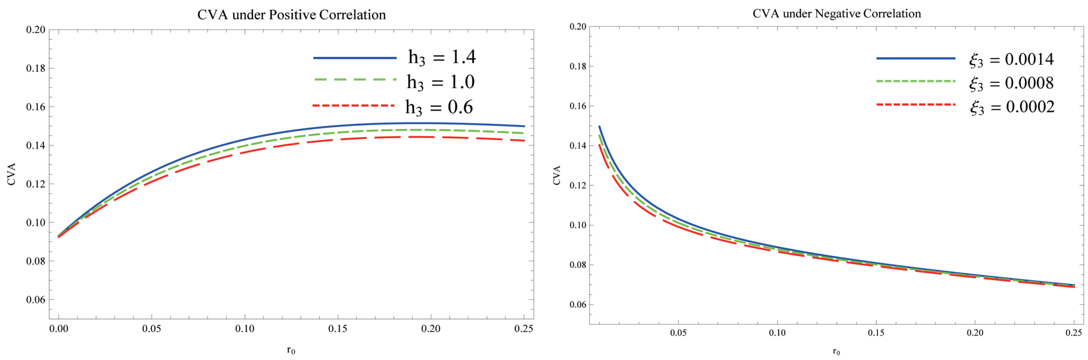
(a) Values of CLNs and the difference under positive correlation. (b) Values of CLNs and the difference under negative correlation.

Figure 3. Values of CLNs and the difference.



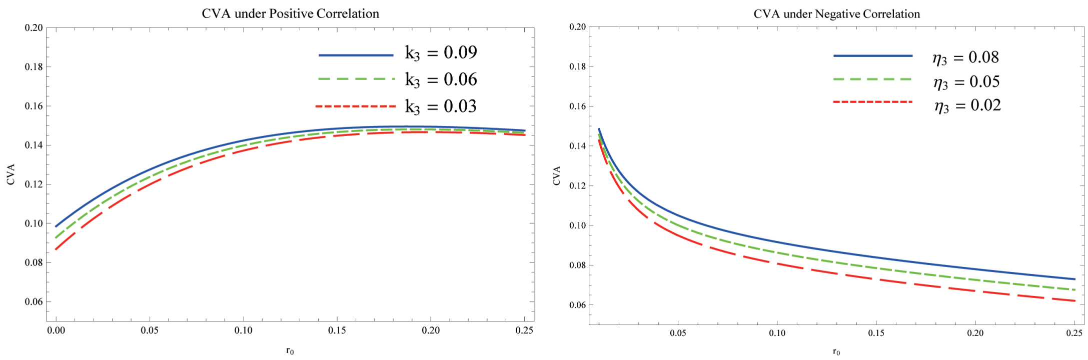
(a) Values of CVA under positive correlation. (b) Values of CVA under negative correlation.

Figure 4. Values of CVA.



(a) Values of CVA under positive correlation. (b) Values of CVA under negative correlation.

Figure 5. Values of CVA with different default densities.



(a) Values of CVA under positive correlation. (b) Values of CVA under negative correlation.

Figure 6. Values of CVA with different default densities.

§6 Conclusions

In this paper, we mainly investigate the valuation of CLN with counterparty risk and its CVA. Under a Markov chain framework, we use Markov copula to model the default correlation between the reference entity and CLN issuer. By using a PDE approach and assuming that the default densities are mainly affected by the interest rate, we get explicit formulas for the CLN values and CVAs. Based on these results, we present some numerical illustrations to show how the default correlation impact on CLN values and CVA.

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References

[1] D H Ahn, B Gao. *A parametric nonlinear model of term structure dynamics*, Rev Financ Stud, 1999, 12(4): 721-762.

[2] T R Bielecki, J Jakubowski, A Vidozzi, L Vidozzi. *Study of dependence for some stochastic processes*, Stoch Anal Appl, 2008, 26(4): 903-924.

[3] T R Bielecki, M Rutkowski. *Credit risk: modeling, valuation and hedging*, Berlin Heidelberg New York: Springer, 2001.

[4] F Black, J C Cox. *Valuing corporate securities: some effects of bond indenture*, J Finance, 1976, 31: 351-367.

[5] E Canabarro, D Duffie. *Asset/Liability Management of Financial institutions: Measuring and marking counterparty risk*, 2003.

[6] C C Chang, C W Wang. *The valuation of special purpose vehicles by issuing structured credit-linked notes*, Appl Financial Econ, 2009, 19(3): 227-256.

[7] S Crpey, M Jeanblanc, B Zargari. *Counterparty risk on a cds in a markov chain copula model with joint defaults*, Recent Advances in Financial Engineering, 2009, 2010, 91-126.

[8] F J Fabozzi, H A Davis, M Choudhry. *Credit-linked notes: A product primer*, J Struct Finance, 2007, 12(4): 67-77.

[9] W Feller. *Two singular diffusion problems*, Ann Math, 1951, 54: 173-182.

[10] L Ge, X S Qian, X Y Yue. *Explicit formulas for pricing credit-linked notes with counterparty risk under reduced-form framework*, IMA J Manag Math, 2015, 26(3): 325-344.

[11] C H Hui, C F Lo. *Effect of asset value correlation on credit-linked note values*, Int J Theor Appl Finance, 2002, 5(05): 455-478.

[12] T R Hurd, A Kuznetsov. *Explicit formulas for laplace transforms of stochastic integrals*, Markov Process Relat Fields, 2008, 14(2): 277-290.

[13] R Jarrow, S Turnbull. *Pricing derivatives on financial securities subject to credit risk*, J Finance, 1995, 50(1): 53-85.

[14] L S Jiang, B J Bian, F H Yi. *A parabolic variational inequality arising from the valuation fixed rate mortgages*, Eur J Appl Math, 2005, 16(3): 361-383.

[15] G C Liang, L S Jiang. *A modified structural model for credit risk*, IMA J Manag Math, 2012, 23(2): 147-170.

[16] R Litterman, T Iben. *Corporate bond valuation and the term structure of credit spreads*, J Portf Manage, 1991, 17(3): 52-64.

[17] D B Madan, H Unal. *Pricing the risks of default*, Rev Deriv Res, 1998, 2(2-3): 121-160.

[18] R C Merton. *On the pricing of corporate debt: the risk structure of interest rates*, J Finance, 1976, 29: 449-470.

[19] G Pye. *Gauging the default premium*, Financial Anal J, 1974, 30(1): 49-52.

[20] C W Wang, C C Chang. *Pricing credit-linked notes issued by the protection buyer and an spv*, In 3rd International Conference on Intelligent Information Hiding and Multimedia Signal Processing, 2007.

[21] S Wu, L S Jiang, J Liang. *Intensity-based models for pricing mortgage-backed securities with repayment risk under a cir process*, Int J Theor Appl Finance, 2012, 15(03): 1250021.

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