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Martingale method for optimal investment and proportional reinsurance

LIU Shuang-sui GUO Wen-jing* TONG Xin-le

Abstract. Numerous researchers have applied the martingale approach for models driven by Lévy processes to study optimal investment problems. This paper considers an insurer who wants to maximize the expected utility of terminal wealth by selecting optimal investment and proportional reinsurance strategies. The insurer's risk process is modeled by a Lévy process and the capital can be invested in a security market described by the standard Black-Scholes model. By the martingale approach, the closed-form solutions to the problems of expected utility maximization are derived. Numerical examples are presented to show the impact of model parameters on the optimal strategies.

§1 Introduction

Recently, there has been much attention to the problem of optimal reinsurance and investment in the financial and actuarial literature for an insurer. See for example, Browne (1995), Hipp and Plum (2000), Schmidli (2002), Liu and Yang (2004), Yang and Zhang (2005), Wang et al. (2007), Gu et al. (2010), Zou and Cadenillas (2014) and references therein. This is due to the fact that insurers are permitted to invest and reinsure in financial markets in practice, and in the meantime, this is a very interesting portfolio selection and risk control problem for institutions in the financial theory.

Optimal insurance investment problems with various objectives are always popular topics, such as, maximizing the expected utility of terminal wealth and minimizing the probability of ruin. Browne (1995) considers a model where the risk process is modeled by a Brownian motion with drift and obtains the optimal investment strategy to maximize the expected exponential utility of terminal wealth. Hipp and Plum (2000) use the classical Cramér-Lundberg model to describe the risk process and assume that the insurer can invest in a risky asset to minimize

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*Corresponding author.

the ruin probability. Explicit solutions are obtained in the case of exponential distributed claim-size. However, they do not incorporate a risk-free asset in their model. Liu and Yang (2004) reconsider the model in Hipp and Plum (2000) by incorporating risk-free interest rates. In this case, closed-form solutions cannot be obtained, but they provide numerical results for optimal strategies that maximize survival probability under different claim-size distribution assumptions. Yang and Zhang (2005) use a jump-diffusion process to model the risk process and consider the portfolio selection problems to optimize multiple objective functions. In particular, they obtain a closed-from solution to maximize the expected constant absolute risk aversion (CARA) utility.

In order to avoid the risk, insurance companies will transfer part of the underwriting risk to other reinsurance companies. For this purpose, a lot of literature consider the optimal reinsurance strategy while considering the optimal investment strategy. Schmidli (2002) considers a classical risk model and allows investment into a risky asset described by a Black-Scholes model as well as proportional reinsurance to minimize the ruin probability, and is able to obtain some analytical results. Bäuerle (2005) investigates a dynamic Cramér-Lundberg model with proportional reinsurance and finds the optimal reinsurance strategy which minimizes the expected quadratic distance of the risk reserve to a given benchmark. The result can then be used to solve the corresponding mean-variance problem. However, the investment problem is not considered. Under the controls of multi-asset investment and proportional reinsurance, Bai and Guo (2008) use a Brownian motion with drift model to investigate two optimization problems of maximizing the expected exponential utility and minimizing the probability of ruin, and obtain the optimal value functions and optimal strategies for two problems respectively. Liang et al. (2011) study the similar problem to Gu et al. (2010), but allow for jump-diffusion risk model by maximizing the expected exponential utility of terminal wealth and derive explicit expressions for the optimal strategy and value function. More results from different points of view can refer to Zeng and Li (2011), Chen and Yam (2013), and Zou and Cadenillas (2014), etc. Because the risk process of insurance company is modeled by the jump process, it is difficult to obtain the explicit solutions of the optimal strategies by using the dynamic programming principle under the general optimization criterion. So far, the explicit solutions of the optimal strategies are obtained by maximizing expected utility function of exponential function or mean-variance criterion. Compared with the dynamic programming principle, the martingale method is easier to get the explicit solutions of the optimal strategies. To our knowledge, there is little literature that uses the martingale method to solve the optimal investment and reinsurance problem.

In this paper, we study the optimal investment and proportional reinsurance problem via the martingale approach. When the risk process is modeled by a Lévy process and security market is described by the standard Black-Scholes model, the optimal strategies are worked out explicitly for the exponential utility and the quadratic utility, respectively. The organization of this paper is as follows. Section 2 describes the model and formulates the optimal investment and proportional reinsurance problem. Then, we discuss the problem under the exponential utility in Section 3 and the quadratic utility in Section 4 respectively. In Section 5, some numerical examples are presented to illustrate the impact of some model parameters on the optimal strategies. Finally, Section 6 concludes the study.

§2 Formulation of the problem

Under the classical risk model perturbed by diffusion, the surplus process is given by

$$dU(t) = c_0 dt - dR(t) + \sigma_0 dW_t^{(2)}, \tag{1}$$

where c_0 is the premium rate, $\sigma_0 \geq 0$, and $W_t^{(2)}$ is a 1-dimensional standard Brownian motion on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. $R(t) = \sum_{i=1}^{K(t)} Y_i$ is a compound Poisson process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. $K(t)$ is a homogeneous Poisson process with intensity λ, and represents the number of claims occurring in time interval [0,t]. Y_i is the size of the i-th claim. Thus the compound Poisson process $R(t) = \sum_{i=1}^{K(t)} Y_i$ represents the cumulative amount of claims in time interval [0,t]. The claim's sizes $Y = \{Y_i, i \geq 1\}$ are assumed to be an i.i.d. sequence with a common cumulative distribution function F. We assume that $\mu_0 = EY = \int_0^\infty y dF(y) < \infty$ and $E(Y^2) = \int_0^\infty y^2 dF(y) < \infty$. The diffusion term $\sigma_0 dW_t^{(2)}$ represents the additional small claims, which are the uncertainty associated with the insurance market or the economic environment. Let L_t denote the compensated compound Poisson process, i.e.

$$L_t := \sum_{i=1}^{K(t)} Y_i - \mu_0 \lambda t. \tag{2}$$

Then L_t is a 1-dimensional compensated pure Lévy process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and is a martingale (see, e.g., Shreve, 2004). Let N denote the Poisson random measure of L, and V denote the Lévy measure that satisfies $V(0) = 0$ and $\int_R (1 + |z|^2) V(dz) < \infty$. Intuitively speaking, the Lévy measure describes the expected number of jumps of a certain height in a time interval of length 1. For the surplus process (1), we have $V(dz) = \lambda F(dz)$. According to Øksendal and Sulem (2005), L_t has the following Lévy decomposition:

$$L_t = \int_0^t \int_R z \left[N(ds, dz) - V(dz) ds \right]. \tag{3}$$

Then the surplus process (1) can be written as

$$dU(t) = (c_0 - \lambda \mu_0) dt - dL_t + \sigma_0 dW_t^{(2)}. \tag{4}$$

As an effective way to reduce risk, we allow the insurance company to reinsure a fraction of its claim with the level $q \in [0, 1]$. That is, for claim Y_i, the reinsurer pays qY_i, and the insurer pays $(1 - q)Y_i$. Let $c_1(q)$ be the premium rate for the reinsurance. So, the premium rate remaining for insurer is $c_0 - c_1(q)$. Throughout the paper, we assume that the reinsurance premium is calculated according to the expected value principle: $c_1 = (1 + \eta) q \lambda \mu_0$, where η is the safety loading of the reinsurer. Let $c_0 = (1 + \theta) \lambda \mu_0$, where θ is the safety loading of the insurer. In general, we assume that $\eta > \theta > 0$.

In addition to reinsurance, the company is allowed to invest all its surplus in a financial market consisting of one risk-free asset and one risky asset. The price process of the risk-free asset $P_0(t)$ is given by

$$dP_0(t) = P_0(t) r dt, \ \ P_0(0) = 1, \tag{5}$$

and the price process of the risky asset $P_1(t)$ satisfies

$$dP_1(t) = P_1(t) \left[\mu dt + \sigma dW_t^{(1)} \right], \ \ P_1(0) > 0, \tag{6}$$

where r is the risk-free interest rate, μ is the appreciation rate, $\sigma > 0$ is the volatility, and $W_t^{(1)}$ is another standard Brownian motion. In general, we assume that $\mu > r \geq 0$. For simplicity, we assume that the two standard Brownian motion $W_t^{(1)}$ and $W_t^{(2)}$ are independent, and it is usually assumed that $W_t^{(1)}$, $W_t^{(2)}$, $K(t)$ and $\{Y_i, i \geq 1\}$ are mutually independent. Moreover $W_t^{(1)}$, $W_t^{(2)}$ and L_t are mutually independent.

Let $\pi(t)$ denote the total amount of money invested in the risky asset at time t, and $q(t)$ denote the reinsurance strategy. Let $X(t)_{t\geq 0}$ denote the associated surplus process, that is, $X(t)$ is the wealth of the company at time t if investment strategy $\pi(t)$ and reinsurance strategy $q(t)$ are adopted. Since any amount not invested in risky asset is held in the risk-free asset, this process then evolves as

$$\begin{aligned} dX_t &= \pi_t \frac{dP_1(t)}{P_1(t)} + (X_t - \pi_t)\frac{dP_0(t)}{P_0(t)} + [c_0 - c_1(q)]dt - (1-q_t)dR_t + (1-q_t)\sigma_0 dW_t^{(2)} \\ &= [X_t r + \pi_t(\mu - r) + (\theta - \eta q_t)\lambda\mu_0]\, dt + \pi_t \sigma dW_t^{(1)} + (1-q_t)\sigma_0 dW_t^{(2)} - (1-q_t)dL_t. \end{aligned} \tag{7}$$

In general, we assume $X_0 = x$. Then we have

$$\begin{aligned} X_t = xe^{rt} &+ \int_0^t e^{r(t-s)}\left[\pi_s(\mu - r) + (\theta - \eta q_s)\lambda\mu_0\right] ds + \int_0^t e^{r(t-s)}\pi_s \sigma dW_s^{(1)} \\ &+ \int_0^t e^{r(t-s)}(1-q_s)\sigma_0 dW_s^{(2)} - \int_0^t e^{r(t-s)}(1-q_s)dL_s. \end{aligned} \tag{8}$$

In this paper, we assume that continuous trading is allowed, all assets are infinitely divisible and no transaction cost or tax is involved in trading.

Definition 2.1. A control policy $h(t) = [\pi(t), q(t)]$ is said to be admissible if $\pi(t)$ and $q(t)$ are predictable with respect to $\mathcal{F}$ and for each $t \geq 0$,
(1) $0 \leq q(t) \leq 1$, and
(2) $E\left[\int_0^T (\pi(t))^2 dt\right] < \infty$, for all $T < \infty$.

The set of all admissible controls $h(t)$ is denoted by $\prod$.

Assume the aim of the insurer is to maximize the expected utility of terminal wealth, say at time T. Our problem can be formulated as follows:

$$\max_{h(t)\in\prod} E\left[U(X(T))\right]. \tag{9}$$

The utility function U is assumed to be a strictly concave function and continuously differentiable on $(-\infty, +\infty)$. Since the utility function U is strictly concave, there exists at most a unique optimal terminal wealth for the company.

The following Lemma 2.1 gives the condition that optimal control must satisfy, which is well-known in the martingale approach to the optimal investment problem, cf. Karatzas et al. (1991), among others. Lemma 2.2 is a generalized version of martingale representation theorem, which is equally important in this article.

Lemma 2.1. *(see Proposition 2.1 in Wang et al.(2007)). If there exists a control* $h^*(t) = [\pi^*(t),\ q^*(t)] \in \prod$*, such that* $E\left[U'(X_T^{h^*})X_T^h\right]$ *is constant over all admissible controls, then* $h^*(t)$ *is the optimal control of problem (9).*

Lemma 2.2. *(see Proposition 9.4 in Cont and Tankov(2003)). For any local (resp. square-integrable) martingale* Z_t*, there exist predictable processes* $\widetilde{\theta} = (\theta_1, \theta_2, \theta_3) \in \Theta$ *(resp.* $\widetilde{\theta} =$

$(\theta_1, \theta_2, \theta_3) \in \Theta^2$), *such that*

$$Z_t = Z_0 + \int_0^t \theta_1(s) dW_s^{(1)} + \int_0^t \theta_2(s) dW_s^{(2)} + \int_0^t \int_R \theta_3(s,z)\left[N(ds,dz) - V(dz)ds\right]$$

for all $t \in [0,T]$.

In the following sections, we will apply the previous lemmas to work out the optimal strategies explicitly for the commonly used exponential utility and quadratic utility. To conclude this section, we introduce some notations that will be used.

Some notations:

(i) P: the predictable σ-algebra on $\Omega \times [0,T]$, which is generated by all left-continuous and $(\mathcal{F}_t)$-adapted processes.

(ii) $\widetilde{\mathrm{P}} := \mathrm{P} \otimes B(R)$ where B is the Borel σ-algebra on $\Omega \times [0,T]$.

(iii) $L(\mathrm{P})$: the set of all $(\mathcal{F}_t)$-predictable, R-value processes, θ_1 such that $\int_0^T |\theta_1(t)|^2 dt < \infty$ a.s.

(iv) $L^2(\mathrm{P})$: the set of all $(\mathcal{F}_t)$-predictable, R-value processes, θ_1 such that $E\left[\int_0^T |\theta_1(t)|^2 dt\right] < \infty$.

(v) $L(\widetilde{\mathrm{P}})$: the set of all $\widetilde{\mathrm{P}}$-measurable, R-value functions θ_3 defined on $\Omega \times [0,T] \times R$ such that $\sqrt{\sum_{0<s\le t} |\theta_3(s,\Delta L_s)|^2 I_{\{\Delta L_s \ne 0\}}}$ is a locally integrable increasing process and for all $t \in [0,T]$, $\int_R |\theta_3(t,z)|\, V(dz) < \infty$ a.s., where $I_{\{\dots\}}$ is the indicator function.

(vi) $L^2(\widetilde{\mathrm{P}})$: the set of all $\widetilde{\mathrm{P}}$-measurable, R-value functions θ_3 defined on $\Omega \times [0,T] \times R$ such that $E\left[\int_0^T \int_R |\theta_3(t,z)|^2 V(dz)dt\right] < \infty$.

(vii) $\Theta := \left\{\widetilde{\theta} = (\theta_1, \theta_2, \theta_3) : (\theta_1, \theta_2, \theta_3) \in L(\mathrm{P}) \times L(\mathrm{P}) \times L(\widetilde{\mathrm{P}})\right\}$.

(viii) $\Theta^2 := \left\{\widetilde{\theta} = (\theta_1, \theta_2, \theta_3) : (\theta_1, \theta_2, \theta_3) \in L^2(\mathrm{P}) \times L^2(\mathrm{P}) \times L^2(\widetilde{\mathrm{P}})\right\}$.

(ix) L_F^2: the set of all $(\mathcal{F}_t)$-adapted processes (X_t) with cadlag paths such that $E\left[\sup_{0\le t\le T} |X_t|^2\right] < \infty$.

It is well-known that every square-integrable martingale belongs to $L^2_{\mathcal{F}}$, and it is easy to see that if $h(t) \in \prod$, then $X_t^h \in L^2_{\mathcal{F}}$.

§3 The analysis for exponential utility

In this section, we consider problem (9) for the exponential utility. For the utility function $U(x) = -\frac{1}{m}e^{-mx}$, where $m > 0$, Lemma 2.1 can be written as: $E\left[e^{-mX_T^{h^*}} X_T^h\right]$ is constant over $h(t) \in \prod$.

By (8), it is equivalent to that

$$E\Big[e^{-mX_T^{h^*}} \int_0^{\mathrm{T}} e^{r(T-t)}(\pi_t(\mu - r) - \eta q_t \lambda \mu_0)dt + e^{r(T-t)}\pi_t \sigma dW_t^{(1)} - e^{r(T-t)} q_t \sigma_0 dW_t^{(2)} + e^{r(T-t)} q_t dL_t\Big] \tag{10}$$

is constant over $h(t) \in \prod$.

Here, we show that the optimal strategies can be obtained by the martingale method.

Step 1.

In this step, we conjecture the form of $h^*(t)$ that satisfies condition (10).

Put

$$Z_T := \frac{e^{-mX_T^{h^*}}}{E\left[e^{-mX_T^{h^*}}\right]} \tag{11}$$

and $Z_t := E\left[Z_T \,|\mathcal{F}_t\right]$ for all $t \in [0,T]$. Then $Z_\tau = E\left[Z_T \,|\mathcal{F}_\tau\right]$ a.s. for any stopping time $\tau \le T$ a.s. Let Q be a probability measure on $(\Omega, \mathcal{F})$ such that $\frac{dQ}{dP} = Z_T$.

For any stopping time $\tau \le T$, we choose $\pi(t) = 1_{t\le\tau}$ and $q(t) = 0$, which is apparently an admissible control. Substituting this control into (10), we have

$$E\left[Z_T \int_0^\tau e^{r(\tau-t)}(\mu-r)dt + e^{r(\tau-t)}\sigma dW_t^{(1)}\right] = E_Q\left[\int_0^\tau e^{r(\tau-t)}(\mu-r)dt + e^{r(\tau-t)}\sigma dW_t^{(1)}\right]$$

is constant over all $\tau \le T$, which implies that

$$\int_0^t e^{r(t-s)}(\mu-r)ds + e^{r(t-s)}\sigma dW_s^{(1)} \tag{12}$$

is a martingale under Q.
Then we choose $\pi(t) = 0$ and $q(t) = 1_{t\le\tau}$. Following a similar argument as above, we have

$$\int_0^t e^{r(t-s)}\eta\lambda\mu_0 ds + e^{r(t-s)}\sigma_0 dW_s^{(2)} - e^{r(t-s)}dL_s \tag{13}$$

is a martingale under Q.

Since Z_t is a martingale, then $K_t := \int_0^t \frac{1}{Z_{s-}}dZ_s$, $t \in [0,T]$, is a local martingale. According to Lemma 2.2, there exist predictable processes $\widetilde{\theta} = (\theta_1, \theta_2, \theta_3) \in \Theta$, such that

$$dK_t = \theta_1(t)dW_t^{(1)} + \theta_2(t)dW_t^{(2)} + d\int_0^t\int_R \theta_3(s,z)\left[N(ds,dz) - V(dz)ds\right],$$

i.e.,

$$dZ_t = Z_{t-}\left\{\theta_1(t)dW_t^{(1)} + \theta_2(t)dW_t^{(2)} + d\int_0^t\int_R \theta_3(s,z)\left[N(ds,dz) - V(dz)ds\right]\right\}.$$

Furthermore, by the Doléans-Dade exponential formula, we have

$$\begin{aligned} Z_t = \exp\Bigg\{&\int_0^t \left(\theta_1(s)dW_s^{(1)} + \theta_2(s)dW_s^{(2)}\right) - \frac{1}{2}\int_0^t \left[(\theta_1(s))^2 + (\theta_2(s))^2\right]ds \\ &+ \int_0^t\int_R \theta_3(s,z)\left[N(ds,dz) - V(dz)ds\right] \\ &+ \int_0^t\int_R \left[\ln\left(1+\theta_3(s,z)\right) - \theta_3(s,z)\right]N(ds,dz)\Bigg\}. \end{aligned} \tag{14}$$

According to Girsanov's Theorem, we know that $W_t^{(1)} - \int_0^t \theta_1(s)ds$, and $W_t^{(2)} - \int_0^t \theta_2(s)ds$ are respectively Brownian motions under Q, and $\int_0^t\int_R \left[N(ds,dz) - \left(1+\theta_3(s,z)\right)V(dz)ds\right]$ is a local martingale under Q. Therefore, the following equations must hold,

$$\theta_1(t) = -\frac{\mu-r}{\sigma}\ ,\ \ t \in [0,T]\,, \tag{15}$$

$$\eta\lambda\mu_0 + \sigma_0\theta_2(t) - \int_R z\theta_3(t,z)V(dz) = 0\ ,\ \ t \in [0,T]\,. \tag{16}$$

On the other hand, by (8), we have

$$\begin{aligned} e^{-mX_T^{h^*}} =\exp\Bigg\{ & -mxe^{rT} - m\int_0^T e^{r(T-t)}\left[\pi_t^*(\mu-r)+(\theta-\eta q_t^*)\lambda\mu_0\right]dt \\ & -m\int_0^T e^{r(T-t)}\pi_t^*\sigma dW_t^{(1)} - m\int_0^T e^{r(T-t)}(1-q_t^*)\sigma_0 dW_t^{(2)} \\ & +m\int_0^T e^{r(T-t)}(1-q_t^*)dL_t\Bigg\}. \end{aligned} \tag{17}$$

Comparing the $dW_t^{(1)}$-term, $dW_t^{(2)}$-term and dN-term respectively in (14) and (17), it is reasonable to conjecture that

$$\begin{cases} \theta_1(t) = -me^{r(T-t)}\pi_t^*\sigma \\ \theta_2(t) = -me^{r(T-t)}(1-q_t^*)\sigma_0 \\ \ln(1+\theta_3(t,z)) = me^{r(T-t)}(1-q_t^*)z. \end{cases} \tag{18}$$

Plugging (18) into (15) and (16), we obtain the following result:

$$\pi_t^* = \frac{\mu-r}{m\sigma^2}e^{-r(T-t)}, \ t\in[0,T], \tag{19}$$

and q_t^* satisfies the equation:

$$\eta\lambda\mu_0 - m(\sigma_0)^2 e^{r(T-t)}(1-q_t) - \int_R z\left[e^{mz(1-q_t)e^{r(T-t)}}-1\right]V(dz) = 0. \tag{20}$$

Define $n = m(1-q_t)e^{r(T-t)}$. Then (20) becomes

$$\eta\lambda\mu_0 - (\sigma_0)^2 n + \lambda\int_R zF(dz) = \lambda\int_R ze^{zn}F(dz). \tag{21}$$

Considering (21) as an equation of n, it is easy to prove that equation (21) has a unique positive root n_0 (Figure1). Define $q_0(t) = 1-\frac{n_0}{m}e^{-r(T-t)}$. Here, n_0 is a constant, and it only depends on the safety loading η, the claim's sizes distribution, and the parameter σ_0. It is clear that $q_0(t) < 1$, for $t\in[0,T]$. Since the reinsurance policy is assumed to satisfy $q_t\in[0,1]$, we discuss the optimal values in the following three cases.

Case 1: $\eta \le \frac{(\sigma_0)^2 m}{\lambda\mu_0}$ or $\eta > \frac{(\sigma_0)^2 m}{\lambda\mu_0}$ and $n_0 \le m$.

In this case, $\frac{n_0}{m}\le 1$, and thus $q_0(t)\ge 0$, for any $t\in[0,T]$. Then the optimal reinsurance strategy is given by

$$q^*(t) = 1-\frac{n_0}{m}e^{-r(T-t)}, \quad t\in[0,T].$$

Case 2: $\eta > \frac{(\sigma_0)^2 m}{\lambda\mu_0}$ and $m < n_0 < me^{rT}$.

Let $t_1 = T-\frac{1}{r}\ln\left[\frac{n_0}{m}\right]$. Then $\frac{n_0}{m} > 1$, and thus $q_0(t) > 0$, for $t\in[0,t_1)$; $q_0(t)\le 0$, for $t\in[t_1,T]$. Therefore, the optimal reinsurance strategy is given by

$$q^*(t) = \begin{cases} 1-\frac{n_0}{m}e^{-r(T-t)}, & t\in[0,t_1), \\ 0, & t\in[t_1,T]. \end{cases}$$

Case 3: $\eta > \frac{(\sigma_0)^2 m}{\lambda\mu_0}$ and $n_0 \ge me^{rT}$.

In this case, $\frac{n_0}{m}e^{-rT}\ge 1$, and thus $q_0(t)\le 0$, for any $t\in[0,T]$, and hence,

$$q^*(t) \equiv 0, \quad t\in[0,T].$$

Step 2.

In this step, we verify that Z_T satisfies its definition.

Rewrite (17) as $e^{-mX_T^{h^*}} = I_T H_T$ where

$$I_T = \exp\left\{-mxe^{rT} - m\int_0^T e^{r(T-t)}\left[\pi_t^*(\mu - r) + (\theta - \eta q_t^*)\lambda\mu_0\right]dt\right\},$$

$$H_T = \exp\left[-m\int_0^T e^{r(T-t)}\pi_t^*\sigma dW_t^{(1)} + e^{r(T-t)}(1-q_t^*)\sigma_0 dW_t^{(2)} - e^{r(T-t)}(1-q_t^*)dL_t\right].$$

Substituting (18) back into (14), we obtain $Z_T = J_T H_T$, where

$$J_T = \exp\left\{-\frac{1}{2}\int_0^T \left(me^{r(T-t)}\pi_t^*\sigma\right)^2 dt - \frac{1}{2}\int_0^T \left[me^{r(T-t)}(1-q_t^*)\sigma_0\right]^2 dt \right.$$
$$\left. + \int_0^T\int_R \left[me^{r(T-t)}(1-q_t^*)z - e^{mz(1-q_t^*)e^{r(T-t)}} + 1\right]V(dz)dt\right\}$$

is constant.

By definition, Z_t is a martingale and $E\left[Z_T\right] = 1$, and then $E\left[H_T\right] = \frac{1}{J_T}$.Therefore,

$$\frac{e^{-mX_T^{h^*}}}{E\left[e^{-mX_T^{h^*}}\right]} = \frac{I_T H_T}{I_T E\left[H_T\right]} = \frac{H_T}{{J_T}^{-1}} = J_T H_T = Z_T,$$

which shows that Z given by (14) with θ_i provided by (18) is the same as the definition: $Z_T = \frac{e^{-mX_T^{h^*}}}{E\left[e^{-mX_T^{h^*}}\right]}$.

Theorem 3.1. *Assume that n_0 is the unique positive root to equation (21). Let $t_1 = T - \frac{1}{r}\ln\left[\frac{n_0}{m}\right]$. Then, for the exponential utility function $U(x) = -\frac{1}{m}e^{-mx}$, the optimal investment strategy for our investment and proportional reinsurance problem (9) is given by*

$$\pi^*(t) = \frac{\mu - r}{m\sigma^2}e^{-r(T-t)}, \quad t \in [0, T]. \tag{22}$$

Furthermore, we have the following cases for the optimal reinsurance strategy:

(a) If $\eta \le \frac{(\sigma_0)^2 m}{\lambda\mu_0}$ or $\eta > \frac{(\sigma_0)^2 m}{\lambda\mu_0}$ and $n_0 \le m$, then the optimal reinsurance strategy is given by

$$q^*(t) = 1 - \frac{n_0}{m}e^{-r(T-t)}, \quad t \in [0, T]. \tag{23}$$

(b) If $\eta > \frac{(\sigma_0)^2 m}{\lambda\mu_0}$ and $m < n_0 < me^{rT}$, then the optimal reinsurance strategy is given by

$$q^*(t) = \begin{cases} 1 - \frac{n_0}{m}e^{-r(T-t)}, & t \in [0, t_1), \\ 0, & t \in [t_1, T]. \end{cases} \tag{24}$$

(c) If $\eta > \frac{(\sigma_0)^2 m}{\lambda\mu_0}$ and $n_0 \ge me^{rT}$, then the optimal reinsurance strategy is given by

$$q^*(t) \equiv 0, \quad t \in [0, T]. \tag{25}$$

§4 The analysis for quadratic utility

In this section, we consider problem (9) for the quadratic utility function $U(x) = x - \frac{m_1}{2}x^2$, where $m_1 > 0$. Then, Lemma 2.1 can be written as:

$$E\left[(1 - m_1 X_T^{h^*})X_T^h\right] \text{ is constant over } h(t) \in \prod. \tag{26}$$

Put $Z_t := E\left[1 - m_1 X_T^{h^*} \,\middle|\, \mathcal{F}_t\right],\ t \in [0,T]$. Then $Z_T = 1 - m_1 X_T^{h^*}$ and $Z_t = E\left[Z_T \,|\mathcal{F}_t\right]$ a.s. for any stopping time $\tau \leq T$ a.s.
Since $h^*(t) \in \prod$, Z is a square-integrable martingale under P. According to Lemma 2.2, there exist progressively measurable processes $\widetilde{\theta} = (\theta_1, \theta_2, \theta_3) \in \Theta^2$, such that

$$dZ_t = \theta_1(t)dW_t^{(1)} + \theta_2(t)dW_t^{(2)} + d\int_0^t \int_R \theta_3(s,z)\left[N(ds,dz) - V(dz)ds\right].$$

Rewrite X_t as

$$X_t = e^{rt}\left[x + \int_0^t \lambda\theta\mu_0 e^{-rs}ds + \int_0^t e^{-rs}\sigma_0 dW_s^{(2)} - \int_0^t e^{-rs}dL_s + S_t\right],$$

where S_t is given by

$$S_t = \int_0^t e^{-rs}\left[\left(\pi_s(\mu - r) - \eta q_s\lambda\mu_0\right)ds + \pi_s\sigma dW_s^{(1)} - q_s\sigma_0 dW_s^{(2)} + q_s dL_s\right].$$

Then we obtain a sufficient condition for (26): $\{S_tZ_t\}_{t\in[0,T]}$ is a martingale under measure P. By Ito's formula for Ito-Lévy process, we have

$$\begin{aligned} dS_tZ_t &= S_{t-}dZ_t + Z_{t-}dS_t + d\left[S,Z\right](t) \\ &= S_{t-}dZ_t + Z_{t-}e^{-rt}\left[\left(\pi_t(\mu - r) - \eta q_t\lambda\mu_0\right)dt + \pi_t\sigma dW_t^{(1)} - q_t\sigma_0 dW_t^{(2)} + q_t dL_t\right] \\ &\quad + e^{-rt}\left[(\pi_t\sigma\theta_1 - q_t\sigma_0\theta_2)dt + \int_R q_t z\theta_3 N(dt,dz)\right]. \end{aligned} \tag{27}$$

Therefore a necessary condition for S_tZ_t to be a P-martingale is

$$Z_{t-}\left[\pi_t(\mu - r) - \eta q_t\lambda\mu_0\right] + \pi_t\sigma\theta_1 - q_t\sigma_0\theta_2 + q_t\int_R z\theta_3 V(dz) = 0.$$

Considering two admissible controls ($\pi_t = 1,\ q_t = 0$) and ($\pi_t = 0,\ q_t = 1$), we obtain

$$\theta_1 = -\frac{\mu - r}{\sigma}Z_{t-}, \tag{28}$$

$$Z_{t-}\eta\lambda\mu_0 + \sigma_0\theta_2 - \int_R z\theta_3 V(dz) = 0. \tag{29}$$

Define $B_t := \exp\left\{\int_0^t b_s ds\right\},\ t \in [0,T]$, where b_t is a deterministic function that will be determined later. Applying Ito's formula to B_tZ_t gives

$$\begin{aligned} B_TZ_T &= Z_0 + \int_0^T B_t dZ_t + \int_0^T Z_{t-}dB_t \\ &= Z_0 - \int_0^T \frac{\mu - r}{\sigma}B_tZ_{t-}dW_t^{(1)} + \int_0^T B_t\theta_2(t)dW_t^{(2)} \\ &\quad + \int_0^T \int_R B_t\theta_3(t,z)\left[N(dt,dz) - V(dz)dt\right] + \int_0^T B_tZ_{t-}b_t dt, \end{aligned} \tag{30}$$

which implies

$$\begin{aligned} \frac{1 - Z_T}{m_1} &= \frac{1}{m_1} - \frac{B_TZ_T}{m_1B_T} \\ &= \frac{1}{m_1} - \frac{1}{m_1B_T}Z_0 - \frac{1}{m_1B_T}\int_0^T Z_{t-}B_tb_t dt + \frac{1}{m_1B_T}\int_0^T \frac{\mu - r}{\sigma}B_tZ_{t-}dW_t^{(1)} \\ &\quad - \frac{1}{m_1B_T}\int_0^T B_t\theta_2(t)dW_t^{(2)} - \frac{1}{m_1B_T}\int_0^T \int_R B_t\theta_3(t,z)\left[N(dt,dz) - V(dz)dt\right]. \end{aligned} \tag{31}$$

On the other hand, by (8), we have

$$\begin{aligned} X_T^{h^*} &= xe^{rT} + \int_0^T e^{r(T-t)}\left[\pi_t^*(\mu - r) + (\theta - \eta q_t^*)\lambda\mu_0\right]dt + \int_0^T e^{r(T-t)}\pi_t^*\sigma dW_t^{(1)} \\ &\quad + \int_0^T e^{r(T-t)}(1-q_t^*)\sigma_0 dW_t^{(2)} - \int_0^T e^{r(T-t)}(1-q_t^*)dL_t\,. \end{aligned} \tag{32}$$

Recall the definition of Z_T, we obtain $X_T^{h^*} = \frac{1-Z_T}{m_1}$. Comparing the $dW_t^{(1)}$-term, $dW_t^{(2)}$-term and $d(N-V)$-term respectively in (31) and (32), it is reasonable to conjecture that

$$\begin{cases} \dfrac{\mu - r}{m_1 B_T \sigma} B_t Z_{t-} = e^{r(T-t)}\pi_t^*\sigma \\ \dfrac{1}{m_1 B_T} B_t\theta_2(t) = -e^{r(T-t)}(1-q_t^*)\sigma_0 \\ \dfrac{1}{m_1 B_T} B_t\theta_3(t,z) = e^{r(T-t)}(1-q_t^*)z. \end{cases} \tag{33}$$

Then (31) can be written as

$$\begin{aligned} \frac{1-Z_T}{m_1} &= \frac{1}{m_1} - \frac{1}{m_1 B_T}Z_0 + X_T^{h^*} - xe^{rT} - \frac{1}{m_1 B_T}\int_0^T Z_{t-}B_t b_t dt \\ &\quad - \int_0^T e^{r(T-t)}\left[\pi_t^*(\mu - r) + (\theta - \eta q_t^*)\lambda\mu_0\right]dt\,. \end{aligned} \tag{34}$$

From (28), (29) and (33), we obtain the following result:

$$\pi_t^* = \frac{B_t Z_{t-}}{m_1 B_T}\frac{\mu - r}{\sigma^2}e^{-r(T-t)}. \tag{35}$$

$$q_t^* = 1 - \frac{B_t Z_{t-}}{m_1 B_T}\frac{\eta\lambda\mu_0}{(\sigma_0)^2 + \int_R z^2 V(dz)}e^{-r(T-t)}. \tag{36}$$

Substituting (35) and (36) into (34), we obtain

$$\begin{aligned} \frac{1-Z_T}{m_1} &= \frac{1}{m_1} - \frac{1}{m_1 B_T}Z_0 + X_T^{h^*} - xe^{rT} - \int_0^T e^{r(T-t)}(\theta - \eta)\lambda\mu_0 dt \\ &\quad - \frac{1}{m_1 B_T}\int_0^T B_t Z_{t-}\left[b_t + \left(\frac{\mu - r}{\sigma}\right)^2 + \frac{(\eta\lambda\mu_0)^2}{(\sigma_0)^2 + \int_R z^2 V(dz)}\right]dt. \end{aligned} \tag{37}$$

To ensure equation (37) holds, we conjecture that

$$b_t = -\frac{(\eta\lambda\mu_0)^2}{(\sigma_0)^2 + \int_R z^2 V(dz)} - \left(\frac{\mu - r}{\sigma}\right)^2, \tag{38}$$

$$Z_0 = B_T - xm_1 B_T e^{rT} - m_1 B_T\int_0^T (\theta - \eta)\lambda\mu_0 e^{r(T-t)}dt\,. \tag{39}$$

From (33) and (36), we obtain

$$\theta_2(t) = -\sigma_0\phi Z_{t-}\,, \tag{40}$$

$$\theta_3(t,z) = z\phi Z_{t-}\,, \tag{41}$$

where ϕ is defined as

$$\phi = \frac{\eta\lambda\mu_0}{(\sigma_0)^2 + \int_R z^2 V(dz)}.$$

Therefore, we obtain the dynamics of Z as

$$dZ_t = Z_{t-}\left\{-\frac{\mu-r}{\sigma}dW_t^{(1)} - \sigma_0\phi dW_t^{(2)} + \phi d\int_0^t\int_R z\left[N(ds,dz)-V(dz)ds\right]\right\},$$

which admits a unique solution

$$\begin{aligned} Z_t =& Z_0\exp\left\{-\frac{\mu-r}{\sigma}W_t^{(1)} - \sigma_0\phi W_t^{(2)}\right. \\ & -\frac{1}{2}\left[\left(\frac{\mu-r}{\sigma}\right)^2 + (\sigma_0\phi)^2\right]t + \int_0^t\int_R\left(\ln(1+z\phi)-z\phi\right)V(dz)ds \\ & \left.+\int_0^t\int_R\left(\ln(1+z\phi)\right)\left[N(ds,dz)-V(dz)ds\right]\right\}. \end{aligned} \tag{42}$$

Then, we can rewrite the control in the following form:

$$\pi_t^* = \frac{\mu-r}{m_1\sigma^2}e^{-(r+b)(T-t)}Z_{t-}, \tag{43}$$

$$q_t^* = 1 - \frac{\phi e^{-(r+b)(T-t)}Z_{t-}}{m_1}, \tag{44}$$

where b_t and Z_t are given by (38) and (42) respectively.

Define

$$Y_t = \frac{\phi e^{-(r+b)(T-t)}Z_{t-}}{m_1}.$$

From the definition of Y_t, we know that Y_t is a stochastic process with jump process, and the continuous part Y_t^c of Y_t is continuous process with decreasing trend about t. Hence, the times for Y_t occurring in the interval $[0,T]$ satisfying $\{Y_t\,|Y_t=1 \text{ or } Y_{t^-}<1 \text{ and } Y_{t^+}>1\ ,\ 0\le t\le T\}$ could be represented by a random variable $N^{(1)}$. Define $\bar{t}_i$ as the time of the i-th appearance of $\{Y_t\,|Y_t=1 \text{ or } Y_{t^-}<1 \text{ and } Y_{t^+}>1\ ,\ 0\le t\le T\}$, where $i=1,2,3...N^{(1)}$.

Since the proportional reinsurance policy q_t is assumed to satisfy $q_t\in[0,1]$, we discuss the optimal values in the following cases.

Case 1: $Z_0\le 0$.

In this case, $1-Y_t\ge 1$, for any $t\in[0,T]$, and hence, the optimal reinsurance strategy is given by

$$q_t^*\equiv 1\ ,\quad t\in[0,T]\,. \tag{45}$$

Case 2: $Z_0>0$ and $Y_0=\frac{\phi e^{-(r+b)T}Z_0}{m_1}>1$.

In this case, it is clear that $1-Y_t<1$. First we consider $Y_T<1$. By analysis, we obtain $1-Y_t\le 0$, for $t\in\left[0,\bar{t}_1\right]$; $1-Y_t\ge 0$, for $t\in\left(\bar{t}_1,\bar{t}_2\right)$; $1-Y_t\le 0$, for $t\in\left[\bar{t}_2,\bar{t}_3\right]$;...$1-Y_t\ge 0$, for $t\in\left(\bar{t}_{N^{(1)}-2},\bar{t}_{N^{(1)}-1}\right)$; $1-Y_t\le 0$, for $t\in\left[\bar{t}_{N^{(1)}-1},\bar{t}_{N^{(1)}}\right]$; $1-Y_t\ge 0$, for $t\in\left(\bar{t}_{N^{(1)}},T\right]$. Therefore, when $Y_T<1$, the optimal reinsurance strategy is given by

$$q_t^* = \begin{cases} 0, & t\in\left[0,\bar{t}_1\right]\cup\left[\bar{t}_2,\bar{t}_3\right]\cup...\cup\left[\bar{t}_{N^{(1)}-3},\bar{t}_{N^{(1)}-2}\right]\cup\left[\bar{t}_{N^{(1)}-1},\bar{t}_{N^{(1)}}\right], \\ 1-\frac{\phi e^{-(r+b)(T-t)}Z_{t-}}{m_1}, & t\in\left(\bar{t}_1,\bar{t}_2\right)\cup\left(\bar{t}_3,\bar{t}_4\right)\cup...\cup\left(\bar{t}_{N^{(1)}-2},\bar{t}_{N^{(2)}-1}\right)\cup\left(\bar{t}_{N^{(1)}},T\right]. \end{cases} \tag{46}$$

Then, we consider $Y_T\ge 1$. By analysis, we obtain $1-Y_t\le 0$, for $t\in\left[0,\bar{t}_1\right]$; $1-Y_t\ge 0$, for $t\in\left(\bar{t}_1,\bar{t}_2\right)$; $1-Y_t\le 0$, for $t\in\left[\bar{t}_2,\bar{t}_3\right]$;...$1-Y_t\le 0$, for $t\in\left(\bar{t}_{N^{(1)}-2},\bar{t}_{N^{(1)}-1}\right)$; $1-Y_t\ge 0$, for $t \in \left[\bar{t}_{N^{(1)}-1}, \bar{t}_{N^{(1)}}\right]$; $1-Y_t \le 0$, for $t \in \left(\bar{t}_{N^{(1)}}, T\right]$. Therefore, when $Y_T \ge 1$, the optimal reinsurance strategy is given by

$$q_t^* = \begin{cases} 0, & t \in \left[0, \bar{t}_1\right] \cup \left[\bar{t}_2, \bar{t}_3\right] \cup ... \cup \left[\bar{t}_{N^{(1)}-2}, \bar{t}_{N^{(1)}-1}\right] \cup \left[\bar{t}_{N^{(1)}}, T\right], \\ 1 - \frac{\phi e^{-(r+b)(T-t)} Z_{t-}}{m_1}, & t \in \left(\bar{t}_1, \bar{t}_2\right) \cup \left(\bar{t}_3, \bar{t}_4\right) \cup ... \cup \left(\bar{t}_{N^{(1)}-3}, \bar{t}_{N^{(2)}-2}\right) \cup \left(\bar{t}_{N^{(1)}-1}, \bar{t}_{N^{(1)}}\right). \end{cases} \tag{47}$$

Case 3: $Z_0 > 0$ and $Y_0 = \frac{\phi e^{-(r+b)T} Z_0}{m_1} \le 1$.

In this case, it is clear that $1 - Y_t < 1$. First we consider $Y_T < 1$. By analysis, we obtain $1-Y_t \ge 0$, for $t \in \left[0, \bar{t}_1\right]$; $1-Y_t \le 0$, for $t \in \left(\bar{t}_1, \bar{t}_2\right)$; $1-Y_t \ge 0$, for $t \in \left[\bar{t}_2, \bar{t}_3\right]$;...$1-Y_t \ge 0$, for $t \in \left(\bar{t}_{N^{(1)}-2}, \bar{t}_{N^{(1)}-1}\right)$; $1-Y_t \le 0$, for $t \in \left[\bar{t}_{N^{(1)}-1}, \bar{t}_{N^{(1)}}\right]$; $1-Y_t \ge 0$, for $t \in \left(\bar{t}_{N^{(1)}}, T\right]$. Therefore, when $Y_T < 1$, the optimal reinsurance strategy is given by

$$q_t^* = \begin{cases} 1 - \frac{\phi e^{-(r+b)(T-t)} Z_{t-}}{m_1}, & t \in \left[0, \bar{t}_1\right] \cup \left[\bar{t}_2, \bar{t}_3\right] \cup ... \cup \left[\bar{t}_{N^{(1)}-2}, \bar{t}_{N^{(1)}-1}\right] \cup \left[\bar{t}_{N^{(1)}}, T\right], \\ 0, & t \in \left(\bar{t}_1, \bar{t}_2\right) \cup \left(\bar{t}_3, \bar{t}_4\right) \cup ... \cup \left(\bar{t}_{N^{(1)}-3}, \bar{t}_{N^{(1)}-2}\right) \cup \left(\bar{t}_{N^{(1)}-1}, \bar{t}_{N^{(1)}}\right). \end{cases} \tag{48}$$

Then, we consider $Y_T \ge 1$. By analysis, we obtain $1-Y_t \ge 0$, for $t \in \left[0, \bar{t}_1\right]$; $1-Y_t \le 0$, for $t \in \left(\bar{t}_1, \bar{t}_2\right)$; $1-Y_t \ge 0$, for $t \in \left[\bar{t}_2, \bar{t}_3\right]$;...$1-Y_t \le 0$, for $t \in \left(\bar{t}_{N^{(1)}-2}, \bar{t}_{N^{(1)}-1}\right)$; $1-Y_t \ge 0$, for $t \in \left[\bar{t}_{N^{(1)}-1}, \bar{t}_{N^{(1)}}\right]$; $1-Y_t \le 0$, for $t \in \left(\bar{t}_{N^{(1)}}, T\right]$. Therefore, when $Y_T \ge 1$, the optimal reinsurance strategy is given by

$$q_t^* = \begin{cases} 1 - \frac{\phi e^{-(r+b)(T-t)} Z_{t-}}{m_1}, & t \in \left[0, \bar{t}_1\right] \cup \left[\bar{t}_2, \bar{t}_3\right] \cup ... \cup \left[\bar{t}_{N^{(1)}-3}, \bar{t}_{N^{(1)}-2}\right] \cup \left[\bar{t}_{N^{(1)}-1}, \bar{t}_{N^{(1)}}\right], \\ 0, & t \in \left(\bar{t}_1, \bar{t}_2\right) \cup \left(\bar{t}_3, \bar{t}_4\right) \cup ... \cup \left(\bar{t}_{N^{(1)}-2}, \bar{t}_{N^{(1)}-1}\right) \cup \left(\bar{t}_{N^{(1)}}, T\right]. \end{cases} \tag{49}$$

Theorem 4.1. *Let $\pi^*(t)$ be defined as in (43) and $q^*(t)$ be defined as in the above three cases in Section 4 respectively, then $h^*(t) = [\pi^*(t), q^*(t)]$ is the optimal control to problem (9) for the quadratic utility function $U(x) = x - \frac{m_1}{2}x^2$, where $m_1 > 0$.*

§5 Numerical examples

This section analyzes the impact of market parameters on the optimal policies under the exponential utility. Two numerical examples are presented. The claim sizes $\{Y_i\}$ are assumed to be independent and exponentially distributed with parameter $\frac{1}{\mu_0}$.
Example 5.1

Let $\lambda = 3$, $\mu_0 = 1$, $T = 10$, $r = 0.3$, $m = 0.2$, and $\theta = 1$. The results are shown in Figures 2 and 3. From Figure 2, we see that the root of equation (21) increases with η, which means that a more expensive reinsurance premium leads to a lower optimal level of reinsurance (Figure 3). Moreover, the root decreases with the value of σ_0^2, which means that when the uncertainty of the claim increases, the insurer will transfer more risk to the reinsurer. In Figure 3, we set $\sigma_0^2 = 0.5$, and plot q_t^* against t for ($\eta = 1$, $t_1 = 8.8263$), ($\eta = 1.5$, $t_1 = 8.2577$), ($\eta = 2$, $t_1 = 7.5597$), and

($\eta = 3$, $t_1 = 6.9807$) with

$$q^*(t) = \begin{cases} 1 - \frac{n_0}{m} e^{-r(T-t)}, & t \in [0, t_1), \\ 0, & t \in [t_1, T]. \end{cases}$$

From Figure 3, we see that the optimal reinsurance strategy decreases with t. It can also be seen that a greater value of η yields a smaller value of q_t^*, which illustrates the intuitive observation that if the reinsurance premium increases, the insurer would rather purchase less reinsurance by retaining a greater share of each claim.

Example 5.2

We calculate the optimal investment strategy π_t^* by (22). Let $T = 10$, $r = 0.3$, the results are shown in Figures 4, 5 and 6. In Figure 4, we set $\sigma^2 = 2$, $m = 0.2$, and plot π_t^* against t for $\mu = 0.4, 0.6, 0.8, 1$. It can be found that π_t^* is an increasing function of μ, which describes the rate of the income of the risky asset will be, and hence the more the insurance company will wish to invest in the risky asset. We also see that larger investment is desired when we approaching time T. In Figure 5, we set $\mu = 0.4$, $m = 0.2$ and plot π_t^* against t for $\sigma^2 = 1, 2, 3.5, 5$. We find that π_t^* is decreasing in σ, which is the volatility of the risky asset. The larger σ is, the riskier the risky asset will be, and hence the less the insurance company will wish to invest in the risky asset. In Figure 6, we set $\mu = 0.4$, $\sigma^2 = 2$ and plot π_t^* against t for $m = 0.2, 0.4, 0.6, 0.8$. We find that π_t^* is decreasing in m. As m is the absolute risk aversion parameter, the larger m is, the less aggressive the insurance company will be, and hence the less the insurance company will wish to invest in the risky asset.

§6 Conclusions

This paper studies optimal investment and proportional reinsurance problems for an insurer. The insurer is allowed to invest in a financial market and purchase reinsurance. The surplus process of the insurer is assumed to follow a Lévy process and the financial market consists of one risk-free asset and one risky asset whose price process is described by the standard Black-Scholes model. And the closed-form solutions to the problems of expected utility maximization are derived by the martingale approach. Some numerical examples are presented to illustrate the impact of model parameters on the optimal strategies. Our work is preliminary. There is much work in this direction that deserves further exploration. For example, in recent years, behavioral economics has attracted a great deal of attention to the hypothesis of non-self-interest of economic individuals. How this will affect the insurance investment problems and how the martingale approach can be applied in this framework are both interesting questions.

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Figures

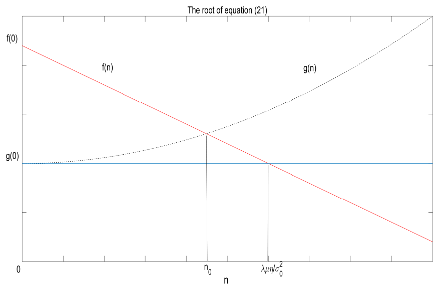

Figure 1: The root of equation (21).

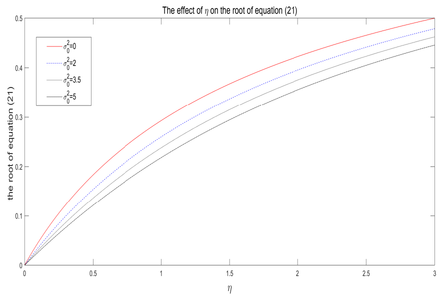

Figure 2: The effect of η on the root of equation (21).

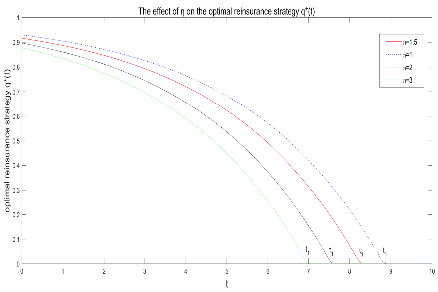

Figure 3: The effect of η on the optimal reinsurance strategy $q^*(t)$.

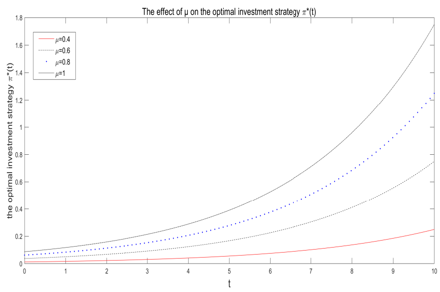

Figure 4: The effect of μ on the optimal investment strategy $\pi^*(t)$.

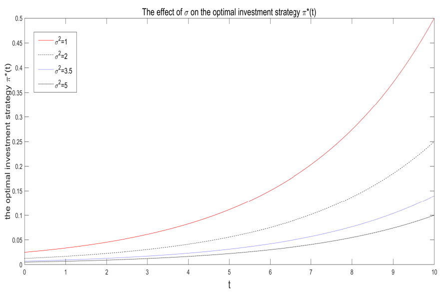

Figure 5: The effect of σ on the optimal investment strategy $\pi^*(t)$.

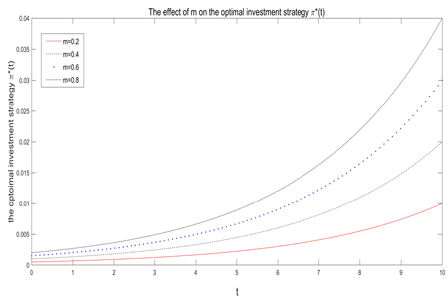

Figure 6: The effect of m on the optimal investment strategy $\pi^*(t)$.

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School of Finance, Nanjing University of Finance and Economics, Nanjing 210023, China.
Email: guowenj810919@sina.com