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Approximation properties for the genuine modified Bernstein-Durrmeyer-Stancu operators

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Abstract. In this paper, we study on the genuine modified Bernstein-Durrmeyer-Stancu operators $\overline{G}_n(f,x)$ and investigate some approximation properties of them. Furthermore, we present a Voronovskaja type theorem for these operators. We also give some graphs and numerical examples to illustrate the convergence properties of these operators for certain functions.

§1 Introduction

One of the most important sequences of positive linear operators is Bernstein operators. These operators were first given by Bernstein [1] and concentratedly studied by some other authors. Readers might check [2-4] for more intense discussions on this topic. The Bernstein operators are given by

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x),$$

where

$$P_{n,k}(x)=\binom{n}{k} x^k(1-x)^{n-k} \text{ for } k=0,...,n \tag{1}$$

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which are the Bernstein fundamental polynomials, $x \in [0,1]$, $f \in C[0,1]$. In [5], Durrmeyer gave the integral type modification of the well-known Bernstein operators as

$$M_n(f,x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x)\int_0^1 P_{n,k}(t)f(t)dt\,,\ x \in [0,1]\,,$$

where $P_{n,k}(x)$ is as in (1). In [6], Derriennic firstly studied these operators in deeper aspects and obtained some results in ordinary and simultaneous approximations. Later, Gupta and Agrawal in [2] disccussed some direct estimates in simultaneous approximations for linear combinations. Furthermore, Rahman et al. in [7] introduced the Kantorovich variant of Lupaş operators based on Pólya distribution with shifted knots. They gave a Voronovskaja type theorem and some basic results for convergence of these operators. In [8], Mursaleen et al. constructed a (p,q)-analogue of Bernstein-Schurer-Stancu type generalized Boolean sum operators for approximating B-continuous and B-differentiable functions. Also, they established a uniform convergence theorem and gave the degree of approximation of B-continuous and B-differentiable functions. In [9], Mursaleen et al. introduced Stancu type modification of Jakimoski-Leviatan-Durrmeyer operators. They studied the problem of simultaneous approximation by these operators and found on upper bound for the approximation to rth derivative of a function by these operators. Also, the authors of [10] and [11] gave the q-analogue of the Bernstein-Durrmeyer operators and obtained several direct results.

The genuine Bernstein-Durrmeyer operators appeared firstly in [12] and [13] as follows:

$$\begin{aligned}(U_n)f(x) &= f(0)\,P_{n,0}(x) + f(1)\,P_{n,n}(x)\\ &\quad +(n-1)\sum_{k=1}^{n-1} P_{n,k}(x)\int_0^1 P_{n-2,k-1}(t)f(t)\,dt, \end{aligned} \tag{2}$$

where $P_{n,k}(x)$ is as in (1), $x \in [0,1]$, $f \in C[0,1]$. In [13], these operators were discussed by the authors as a particular case of a modified Schoenberg spline operator. Then, they investigated the operator U_n on the simplex and derived many properties of the operator U_n in [14]. In [15], Gonska et al. presented some direct and inverse results for the approximation by the genuine Bernstein-Durrmeyer operators U_n. There are many articles related to these operators U_n [15-24].

The aim of this paper is to introduce the genuine modified Bernstein-Durrmeyer-Stancu operators in order to obtain a better convergence of genuine Bernstein-Durrmeyer-Stancu operators. After investigating some approximation properties of these operators, we give a Voronovskaja type theorem for them. We also support the theoretical results with numerical examples and graphical illustrations.

Now, we consider the following operators as the situation in real space of the complex genuine Bernstein-Durrmeyer-Stancu (the genuine BDS) operators in [19]

$$G_n(f,x) \quad = \quad f\left(\frac{\alpha}{n+\beta}\right)P_{n,0}(x) + f\left(\frac{n+\alpha}{n+\beta}\right)P_{n,n}(x)$$

$$+(n-1)\sum_{k=1}^{n-1}P_{n,k}(x)\int_0^1 P_{n-2,k-1}(t)f\left(\frac{tn+\alpha}{n+\beta}\right)dt, \tag{3}$$

where $P_{n,k}(x)$ is given in (1), $x\in[0,1]$, $f\in C[0,1]$, and $0\le\alpha\le\beta$.

§2 Genuine Modified BDS Operators

We define the genuine modified BDS operators denoted by $\overline{G}_n(f,x)$ as

$$\overline{G}_n(f,x)=G_n(f,\varrho(x)),\ x\in I:=\left[\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}\right], \tag{4}$$

where $\varrho(x)=\frac{n+\beta}{n}x-\frac{\alpha}{n}$ and $f\in C[0,1]$.

Now, we give the following lemma for the Korovkin test functions of $G_n(f(t),x)$ with the help of Korovkin test functions of the complex genuine Bernstein-Durrmeyer-Stancu (the genuine BDS) operators in [19].

Lemma 2.1. *([19]) Let $e_j(t)=t^j$, $j=0,1,2,3,4$. The following equations related to the genuine modified BDS operators are obtained*

$$\begin{aligned}
G_n(1,x) &= 1,\\
G_n(t,x) &= \frac{n}{n+\beta}x+\frac{\alpha}{n+\beta},\\
G_n(t^2,x) &= \frac{n^2(n-1)}{(n+\beta)^2(n+1)}x^2+\frac{2\alpha n(n+1)+2n^2}{(n+\beta)^2(n+1)}x+\frac{\alpha^2}{(n+\beta)^2},\\
G_n(t^3,x) &= \frac{n^5-3n^4+2n^3}{(n+\beta)^3(n+2)(n+1)}x^3+\frac{6n^3(n-1)+3\alpha n^2(n-1)(n+2)}{(n+\beta)^3(n+2)(n+1)}x^2\\
&\quad+\frac{6n^3+6\alpha n^2(n+2)+3\alpha^2n(n+2)(n+1)}{(n+\beta)^3(n+2)(n+1)}x+\frac{\alpha^3}{(n+\beta)^3},\\
G_n(t^4,x) &= \frac{n^7-6n^6+11n^5-6n^4}{(n+\beta)^4(n+3)(n+2)(n+1)}x^4\\
&\quad+\frac{12n^6-36n^5+24n^4+(4\alpha n^5-12\alpha n^4+8\alpha n^3)(n+3)}{(n+\beta)^4(n+3)(n+2)(n+1)}x^3\\
&\quad+\frac{36n^5-36n^4+\left(24\alpha n^4-24\alpha n^3\right)(n+3)+\left(6\alpha^2n^3-6\alpha^2n^2\right)(n+2)(n+3)}{(n+\beta)^4(n+3)(n+2)(n+1)}x^2\\
&\quad+\frac{24n^4+24\alpha n^3(n+3)+12\alpha^2n^2(n+2)(n+3)+4\alpha^3n(n+3)(n+2)(n+1)}{(n+\beta)^4(n+3)(n+2)(n+1)}x\\
&\quad+\frac{\alpha^4}{(n+\beta)^4}.
\end{aligned}$$

Hence we give the following lemma for the Korovkin test functions of the genuine modified BDS operators without proof.

Lemma 2.2. *For $x\in I$ and $e_j(t)=t^j$, $j=0,1,2,3,4$, the following properties for the operators $\overline{G}_n(e_j(t),x)$ are hold*

$$\overline{G}_n(1,x) = 1,$$

$$
\begin{aligned}
\overline{G}_n(t,x) &= x,\\
\overline{G}_n(t^2,x) &= \frac{n-1}{n+1}x^2+\frac{2n+4\alpha}{(n+1)(n+\beta)}x-\frac{2n\alpha+2\alpha^2}{(n+1)(n+\beta)^2},\\
\overline{G}_n(t^3,x) &= \frac{n^2-3n+2}{(n+1)\,(n+2)}x^3+\frac{12\alpha n-12\alpha-6n+6n^2}{(n+1)\,(n+2)\,(n+\beta)}x^2\\
&\quad+\frac{24\alpha n-6\alpha n^2-6\alpha^2 n+24\alpha^2+6n^2}{(n+1)(n+2)(n+\beta)^2}x-\frac{6\alpha n^2+18\alpha^2 n+12\alpha^3}{(n+1)(n+2)(n+\beta)^3},\\
\overline{G}_n(t^4,x) &= \frac{n^3-6n^2+11n-6}{(n+1)\,(n+2)\,(n+3)}x^4+\frac{12n^3-36n^2+24n+24\alpha n^2-72\alpha n+48\alpha}{(n+1)\,(n+2)\,(n+3)\,(n+\beta)}x^3\\
&\quad+\frac{36n^3-12\alpha n^3+156\alpha n^2-144\alpha n-12\alpha^2n^2+156\alpha^2 n-144\alpha^2-36n^2}{(n+1)\,(n+2)\,(n+3)\,(n+\beta)^2}x^2\\
&\quad+\frac{24n^3-48\alpha n^3+144\alpha n^2-144\alpha^2n^2-96\alpha^3 n+192\alpha^3+288\alpha^2 n}{(n+1)\,(n+2)\,(n+3)\,(n+\beta)^3}x\\
&\quad+\frac{12\alpha^4 n-84\alpha^4+24\alpha^3n^2-168\alpha^3 n+12\alpha^2n^3-108\alpha^2n^2-24\alpha n^3}{(n+1)\,(n+2)\,(n+3)\,(n+\beta)^4}.
\end{aligned}
$$

We need to compute the second moment before giving our main results.

Lemma 2.3. *For the genuine modified BDS operators, the following equalities for $x \in I$ are obtained*

$$
\begin{aligned}
\overline{G}_n((t-x)^2,x) &= \frac{-2}{n+1}x^2+\frac{2n+4\alpha}{(n+1)(n+\beta)}x-\frac{2\alpha n+2\alpha^2}{(n+1)(n+\beta)^2}, \qquad (5)\\
\overline{G}_n((t-x)^4,x) &= \frac{10n-84}{(n+1)(n+2)(n+3)}x^4+\frac{168n-24n^2+336\alpha-48\alpha n}{(n+1)(n+2)(n+3)(n+\beta)}x^3\\
&\quad+\frac{72\alpha n^2+72\alpha^2 n+12n^3-504\alpha n+504\alpha^2-108n^2}{(n+1)(n+2)(n+3)(n+\beta)^2}x^2\\
&\quad+\frac{24n^3-24\alpha n^3-72\alpha^2n^2+216\alpha n^2+504\alpha^2 n-48n\alpha^3+336\alpha^3}{(n+1)(n+2)(n+3)(n+\beta)^3}x\\
&\quad+\frac{12\alpha^2n^3-24\alpha n^3+24\alpha^3n^2-108\alpha^2n^2+12\alpha^4 n-168\alpha^3 n-84\alpha^4}{(n+1)(n+2)(n+3)(n+\beta)^4}.
\end{aligned}
$$

§3 Some Approximation Properties for the Genuine Modified BDS Operators

All continuous functions in $[0,1]$ is denoted by $C[0,1]$ with the norm

$$\|f\| = \max\{|f(x)| : x \in [0,1]\}.$$

The classical modulus of continuity for $f \in C[0,1]$ and $x \in I$ is denoted as follows:

$$\omega(f,\delta) = \sup_{|t-x|\le\delta,\ t\in[0,1]} |f(t)-f(x)|. \qquad (6)$$

The Peetre's K-functional is defined with the help of the following representation

$$K(f,\delta) = \inf\left\{\|f-g\| + \delta\,\|g''\| : g \in C^2[0,1]\right\},\ \delta > 0, \qquad (7)$$

where $C^2[0,1]=\{g\in C[0,1]: g', g''\in C[0,1]\}$. There is a positive constant M at Theorem 2.4 on p. 177 in [25], such that

$$K(f,\delta)\le M\omega_2(f,\sqrt{\delta}), \tag{8}$$

where the second order modulus of continuity is

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}}\ \sup_{x,x+2h\in[0,1]}|f(x+2h)-2f(x+h)+f(x)|. \tag{9}$$

Thus, we are ready to give direct results.

Theorem 3.1. *Let f is continuous function on $[0,1]$. Then we have*

$$\lim_{n\to\infty}\max_{x\in I}\left|\overline{G}_n(f,x)-f(x)\right|=0.$$

Proof. From Lemma 2.2, we get

$$\lim_{n\to\infty}\max_{x\in I}\left|\overline{G}_n(t^\gamma,x)-x^\gamma\right|=0,\ \gamma=0,1,2. \tag{10}$$

Considering the following operators

$$\widetilde{G}_n(f,x)=\begin{cases}\overline{G}_n(f,x) & ;\ \text{if } \frac{\alpha}{n+\beta}\le x\le\frac{n+\alpha}{n+\beta}\\ f(x) & ;\ \text{if } x\in\left[0,\frac{\alpha}{n+\beta}\right)\cup\left(\frac{n+\alpha}{n+\beta},1\right]\end{cases},$$

we have

$$\left\|\widetilde{G}_n(f,x)-f(x)\right\|=\max_{x\in I}\left|\overline{G}_n(f,x)-f(x)\right|. \tag{11}$$

Thus by (10), we get

$$\lim_{n\to\infty}\left\|\widetilde{G}_n(t^\gamma,x)-x^\gamma\right\|=0,\ \gamma=0,1,2.$$

Applying the Korovkin theorem [26] for the operators $\widetilde{G}_n$, we have

$$\lim_{n\to\infty}\left\|\widetilde{G}_n(f,x)-f\right\|=0$$

for any continuous function f in $[0,1]$. Hence using (11), we obtain

$$\lim_{n\to\infty}\max_{x\in I}\left|\overline{G}_n(f,x)-f(x)\right|=0.$$

Theorem 3.1 is proved. □

Theorem 3.2. *For the genuine modified BDS operators, the inequality for every $x\in I$ and $f\in C^2[0,1]$ is obtained*

$$\left|\overline{G}_n(f(t),x)-f(x)\right|\le 2\zeta_n(x)\left\|f''\right\|, \tag{12}$$

where

$$\zeta_n(x):=\frac{1}{n+1}x^2+\frac{n+2\alpha}{(n+1)(n+\beta)}x+\frac{\alpha n+\alpha^2}{(n+1)(n+\beta)^2}.$$

Proof. Using the following Taylor's expansion

$$f(t)=f(x)+(t-x)f'(x)+\int_x^t(t-u)f''(u)du$$

and Lemma 2.2, we obtain the following equation

$$\overline{G}_n(f(t),x)-f(x)=\overline{G}_n\left(\int_x^t(t-u)f''(u)du,x\right).$$

On the other hand, combining the inequality

$$\left|\int_x^t (t-u)f''(u)du\right| \le \|f''\|(t-x)^2,$$

and Lemma 2.3, we get the following inequalities

$$\begin{aligned}\left|\overline{G}_n(f(t),x)-f(x)\right| &= \left|\overline{G}_n\left(\int_x^t (t-u)f''(u)du, x\right)\right| \\ &\le \|f''\|\overline{G}_n((t-x)^2,x) \\ &\le 2\|f''\|\left\{\frac{1}{n+1}x^2+\frac{n+2\alpha}{(n+1)(n+\beta)}x+\frac{\alpha(n+\alpha)}{(n+1)(n+\beta)^2}\right\}.\end{aligned}$$

Thus, the proof of the Theorem 3.2 is completed. □

Theorem 3.3. *For the genuine modified BDS operators and any $f \in C[0,1]$, the following inequality is obtained*

$$\left|\overline{G}_n(f(t),x)-f(x)\right| \le 2Mw_2(f,\sqrt{\zeta_n(x)}),$$

where M is a constant,

$$\zeta_n(x) := \frac{1}{n+1}x^2+\frac{n+2\alpha}{(n+1)(n+\beta)}x+\frac{\alpha n+\alpha^2}{(n+1)(n+\beta)^2}$$

and $x \in I$.

Proof. For any $g \in C^2[0,1]$, we obtain the inequality

$$\begin{aligned}&\left|\overline{G}_n(f(t),x)-f(x)\right| \\ \le\ &\left|\overline{G}_n((f-g)(t),x)-(f-g)(x)\right| + \left|\overline{G}_n(g(t),x)-g(x)\right| \\ \le\ &2\|f-g\| + 2\zeta_n(x)\|g''\|.\end{aligned}$$

Taking infimum over $g \in C^2[0,1]$ on the right side of the above inequality and using the inequalities (8) and (9), we reach the desired result. □

§4 A Voronovskaja type theorem for the genuine modified BDS operators

In this section, we prove a Voronovskaja type theorem for the $\overline{G}_n(f(t),x)$ operators given by (4). We first need the following lemma.

Lemma 4.1. *For the genuine modified BDS operators, the following properties are satisfied*

$$(i)\ \lim_{n\longrightarrow\infty} n\overline{G}_n((t-x)^2,x) = -2x^2+2x,$$

$$(ii)\ \lim_{n\longrightarrow\infty} n^2\overline{G}_n((t-x)^4,x) = 10x^4-24x^3+12x^2,$$

where $x \in I$.

Proof. (i) Using the linearity property of the $\overline{G}_n$ operators for $n \in \mathbb{N}$ and $x \in I$, and the results of Lemma 2.3, we obtain

$$\lim_{n\longrightarrow\infty} n\overline{G}_n((t-x)^2,x)$$

$$\begin{aligned} &= \lim_{n\longrightarrow\infty}\left\{\frac{-2n}{n+1}x^2+\frac{2n^2+4\alpha n}{(n+1)(n+\beta)}x-\frac{2\alpha n^2+2\alpha^2 n}{(n+1)(n+\beta)^2}\right\} \\ &= -2x^2+2x. \end{aligned}$$

(ii) Using the same process for the proof of the equation in (i) and using Lemma 2.3, we find the following equation

$$\lim_{n\longrightarrow\infty} n^2\overline{G}_n((t-x)^4,x)=10x^4-24x^3+12x^2.$$

□

Now, we are ready to give a Voronovskaja type theorem for the operators $\overline{G}_n(f(t),x)$.

Theorem 4.2. *For the genuine modified BDS operators, the following property is satisfied*

$$\lim_{n\longrightarrow\infty} n\left(\overline{G}_n(f(t),x)-f(x)\right)=(x-x^2)f''(x)$$

for any $f\in C^2[0,1]$ and $x\in I$.

Proof. By using the following Taylor's expansion of f, we get

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\varsigma(t,x)(t-x)^2,$$

where $\varsigma(t,x)\to 0$ as $t\to x$. From the linearity of the operators $\overline{G}_n(f(t),x)$, we have

$$\overline{G}_n(f(t),x)-f(x)=\frac{1}{2}f''(x)\overline{G}_n(\left(t-x\right)^2,x)+\overline{G}_n(\varsigma(t,x)\left(t-x\right)^2,x).$$

From Lemma 2.2, we obtain

$$\begin{aligned} \overline{G}_n(f(t),x)-f(x) &= \frac{1}{2}f''(x)\left\{\frac{-2}{n+1}x^2+\frac{2n+4\alpha}{(n+1)(n+\beta)}x-\frac{2\alpha n+2\alpha^2}{(n+1)(n+\beta)^2}\right\} \\ &\quad +\overline{G}_n(\varsigma(t,x)\left(t-x\right)^2,x). \end{aligned} \tag{13}$$

If we use Cauchy-Schwartz inequality for $\overline{G}_n(\varsigma(t,x)\left(t-x\right)^2,x)$ in the last equation and then take the limit for $n\to\infty$, we get

$$\lim_{n\longrightarrow\infty} n\overline{G}_n(\varsigma(t,x)\left(t-x\right)^2,x)\le\sqrt{\lim_{n\longrightarrow\infty}\overline{G}_n(\varsigma^2(t,x),x)}\sqrt{\lim_{n\longrightarrow\infty} n^2\overline{G}_n(\left(t-x\right)^4,x)}.$$

Because

$$\lim_{n\longrightarrow\infty}\overline{G}_n(\varsigma^2(t,x),x)=0,$$

and

$$\lim_{n\longrightarrow\infty} n^2\overline{G}_n(\left(t-x\right)^4,x)$$

is finite by using Lemma 2.3, we have

$$\lim_{n\longrightarrow\infty} n\overline{G}_n(\varsigma(t,x)\left(t-x\right)^2,x)=0.$$

Thus we obtain

$$\begin{aligned} &\lim_{n\longrightarrow\infty} n\left(\overline{G}_n(f(t),x)-f(x)\right) \\ &= \lim_{n\longrightarrow\infty} n\left\{\frac{1}{2}f''(x)\left(\frac{-2(n+\beta)^2x^2}{(n+1)(n+\beta)^2}+\frac{(2n+4\alpha)(n+\beta)x-2n\alpha-2\alpha^2}{(n+1)(n+\beta)^2}\right)\right\} \\ &= \left(x-x^2\right)f''(x) \end{aligned}$$

and the proof is completed. □

§5 Numerical results

In this section, we will illustrate the theoretical results presented in the previous sections by numerical examples, we write $||\cdot||$ as $||\cdot||_{C(I)}$ for convenience.

Example 5.1. *Let the trigonometric function $f_1(x) = 1 - \cos(4e^x)$, the graphs of $G_n(f_1, x)$, $\overline{G}_n(f_1, x)$ and $f_1(x)$ with $\alpha = 5$, $\beta = 6$ and $n = 50$ are shown in Figure 1. Table 1 is the absolute error bound of $G_n(f_1, x)$ and $\overline{G}_n(f_1, x)$ to $f_1(x)$ with different values of α, β and n.*

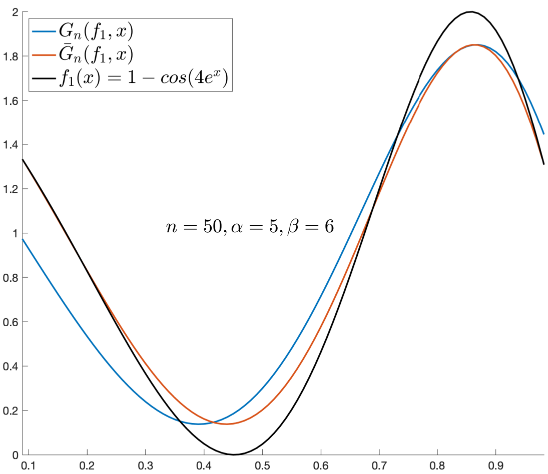

Figure 1: The figures of $G_n(f_1, x)$, $\overline{G}_n(f_1, x)$ and trigonometric function f_1.

Table 1: The absolute error bound of $G_n(f_1)$ and $\overline{G}_n(f_1)$ for the trigonometric function f_1.

n	$\alpha=5,\ \beta=6$		$\alpha=0.5,\ \beta=0.6$		$\alpha=10,\ \beta=12$	
	$\|\|f_1-G_n(f_1)\|\|$	$\|\|f_1-\overline{G}_n(f_1)\|\|$	$\|\|f_1-G_n(f_1)\|\|$	$\|\|f_1-\overline{G}_n(f_1)\|\|$	$\|\|f_1-G_n(f_1)\|\|$	$\|\|f_1-\overline{G}_n(f_1)\|\|$
10	0.749879885	0.293532128	0.6103936	0.58364897	1.01429322	0.17841767
20	0.546230698	0.252929227	0.39940364	0.38088268	0.70898824	0.17694818
30	0.521329118	0.211363313	0.29653123	0.28249708	0.5512643	0.16016665
50	0.359357702	0.15472218	0.19548752	0.18650351	0.58439882	0.12845532

Example 5.2. *Let the polynomial function $f_2(x) = \left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{4}\right)$, the graphs of $G_n(f_2, x)$, $\overline{G}_n(f_2, x)$ and $f_2(x)$ with $\alpha = 5$, $\beta = 6$ and $n = 50$ are shown in Figure 2. Table 2 is the absolute error bound of $G_n(f_2, x)$ and $\overline{G}_n(f_2, x)$ to $f_2(x)$ with different values of α, β and n. We also compare the operators $U_n(f_2)$ (that is the case of $\alpha = \beta = 0$ of $G_n(f_2)$) and $\overline{G}_n(f_2)$ to f_2 from the point of absolute error bound, see Table 3.*

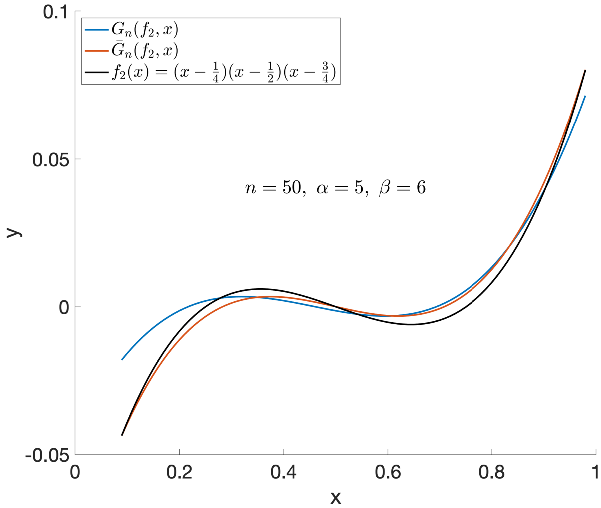

Figure 2: The figures of $G_n(f_2,x)$, $\overline{G}_n(f_2,x)$ and polynomial function f_2.

Table 2: The absolute error bound of $G_n(f_2)$ and $\overline{G}_n(f_2)$ to polynomial function f_2.

n	$\alpha=5,\ \beta=6$		$\alpha=0.5,\ \beta=0.6$		$\alpha=10,\ \beta=12$	
	$\|f_2-G_n(f_2)\|$	$\|f_2-\overline{G}_n(f_2)\|$	$\|f_2-G_n(f_2)\|$	$\|f_2-\overline{G}_n(f_2)\|$	$\|f_2-G_n(f_2)\|$	$\|f_2-\overline{G}_n(f_2)\|$
10	0.015094325	0.010291193	0.02044737	0.01990492	0.01391931	0.00613073
20	0.012176929	0.007982098	0.01213443	0.01187381	0.01419692	0.005625
30	0.021440603	0.006344659	0.00860268	0.00842992	0.0143708	0.00486388
50	0.025619689	0.004444794	0.00576814	0.00532672	0.0195961	0.00371261

Table 3: The absolute error bound of $U_n(f_2)$ $(G_n(f_2),\ \alpha=\beta=0)$, $\overline{G}_n(f_2)$ to polynomial function f_2.

n	$\alpha=0,\beta=0$	$\alpha=1,\beta=2$	$\alpha=5,\beta=6$	$\alpha=10,\beta=12$	$\alpha=20,\beta=21$	$\alpha=50,\beta=51$
	$\|f_2-U_n(f_2)\|$	$\|f_2-\overline{G}_n(f_2)\|$	$\|f_2-\overline{G}_n(f_2)\|$	$\|f_2-\overline{G}_n(f_2)\|$	$\|f_2-\overline{G}_n(f_2)\|$	$\|f_2-\overline{G}_n(f_2)\|$
20	0.01249636	0.00938808	0.0079821	0.005625	0.00461626	0.00201649
30	0.008729819	0.00719332	0.006344659	0.00486388	0.004171217	0.002152587
50	0.005442421	0.004838186	0.004444794	0.003712609	0.00332845	0.002062527

From the point of absolute error bound, one can see from Tables 1-3, the latter operators $\overline{G}_n(f,x)$ are better than the former ones $G_n(f,x)$ and $U_n(f)$, so it is important to consider the genuine modified operators $\overline{G}_n(f,x)$. When the following figures are examined carefully, it is seen that the figures of the functions taken with our operator $\overline{G}_n(f,x)$ move more synchronously according to other operators $G_n(f,x)$.

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