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Protection zone in a diffusive predator-prey model with Ivlev-type functional response

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Abstract. The effect of a protection zone on a diffusion predator-prey model with Ivlev-type functional response is considered. We discuss the existence and non-existence of positive steady state solutions by using the bifurcation theory. It is shown that the protection zone for prey has beneficial effects on the coexistence of the two species when the growth rate of predator is positive. Moreover, we examine the dependence of the coexistence region on the efficiency of the predator capture of the prey and the protection zone.

§1 Introduction

Predator-prey interaction is one of the basic interspecies relations for ecological and social models. When the spatial distribution of the populations is also considered, a prototypical predator-prey system is of the form

$$\begin{cases} u_t = d_1 \triangle u + u(\lambda - u) - p(u)v, & x \in \Omega, \quad t > 0, \\ v_t = d_2 \triangle v + v[\mu - v + cp(u)], & x \in \Omega, \quad t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1}$$

where $u(x,t)$ and $v(x,t)$ are the population densities of the prey and the predator, respectively; d_1 and d_2 are the diffusion rates; Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, n is the outward unit normal vector on $\partial\Omega$; λ, μ denote the growth rates of the respective species; c is the conversion rate from the prey loss to the predator gain. Here both the predator and prey have a logistic growth rate. The function $p(u)$ represents the functional response of the predator. The classical Lotka-Volterra model assumes that $p(u) = u$. There is an important prey-dependent functional response: $p(u) = 1 - e^{-ru}$, due originally to Ivlev (1961) [7], where r means the efficiency of the predator capture of the prey. The parameters λ, r and c are all positive constants, and μ is a constant.

As pointed out in [2], in most predator-prey interactions, the prey population would extinguish if the growth rate of the predator is too large, or the predation rate is too high. Thus,

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human interference is often needed to save the endangered species. A natural idea is to set up a protection zone for the prey, where the prey species can enter and leave freely while the predator is kept out. Recently, many mathematicians are interested in studying the effects of protection zone on various predator-prey models, refer to the works for Holling II type predator-prey system (Du and Shi 2006) [2], Leslie type predator-prey system (Du et al. 2009) [3], Beddington-DeAngelis type predator-prey system (He and Zheng 2017) [6], as well as the ratio-dependent predator-prey model (Zeng et al. 2018) [16]. Moreover, there are other types of prey refuge created in [10,11].

When the protection zone is absent, many mathematicians have studied the Ivlev type predator-prey model and obtained some interesting results [5,13,14]. These results indicated that the Ivlev-type predator-prey model has wide applicabilities in ecology. But to our knowledge, there are few works on such Ivlev-type predator-prey model with a protection zone for prey. Following this line of thinking, in this paper we modify the Ivlev-type predator-prey model (1) to include a protection zone for the prey, that is, we consider the following system

$$\begin{cases} u_t = d_1\triangle u + u\,(\lambda - u) - b(x)v(1-e^{-ru}), \quad x\in\Omega, \quad t>0,\\ v_t = d_2\triangle v + v[\mu - v + c(1-e^{-ru})], \quad x\in\Omega\setminus\bar{\Omega}_0, \quad t>0,\\ \partial_n u = 0, \quad x\in\partial\Omega, \quad t>0, \quad \partial_n v = 0, \quad x\in\partial\Omega\cup\partial\Omega_0, \quad t>0,\\ u(x,0) = u_0(x)\ge 0, \quad x\in\Omega, \quad v(x,0) = v_0(x)\ge 0, \quad x\in\Omega\setminus\bar{\Omega}_0, \end{cases} \tag{2}$$

where Ω_0 is a subdomain of Ω satisfying $\overline{\Omega_0}\subset\Omega$ and the boundary $\partial\Omega_0$ is also smooth. The function $b(x)=1$ in $\Omega\setminus\Omega_0$ and $b(x)=0$ when $x\in\Omega_0$. The larger region Ω is the habitat of the prey with Ω_0 its protection zone; thus the predator species can only exist in $\Omega\setminus\Omega_0$. The boundary conditions mean that the predator and prey live in a closed ecosystem, and the prey could cross the boundary freely of the protection zone but the predator is prohibited from entering Ω_0.

In the present paper, to simplify the notations and make our analysis more transparent, we will assume that $d_1=d_2=1$. We mainly study the associate stationary problem with (2):

$$\begin{cases} \triangle u + u\,(\lambda - u) - b(x)v(1-e^{-ru}) = 0, & x\in\Omega,\\ \triangle v + v[\mu - v + c(1-e^{-ru})] = 0, & x\in\Omega_1,\\ \partial_n u = 0, & x\in\partial\Omega,\\ \partial_n v = 0, & x\in\partial\Omega_1, \end{cases} \tag{3}$$

where $\Omega_1=\Omega\setminus\bar{\Omega}_0$. Among other things, we are interesting in the positive solutions of (3). From an ecological point of view, a positive solution implies a coexistence steady state. We aim to obtain a sufficient condition of coefficients for the existence of positive solutions of (3). Our approach to the proof is based on the local and global bifurcation arguments.

The rest of this paper is structured in the following way. In Section 2, we state the main results of this paper. In Section 3, we present some basic results on the set of steady-state solutions and get the nonexistence of coexistence states. In Section 4, we obtain positive solutions from the viewpoint of the bifurcation theory. In Section 5, we examine the effects of the predator capture of the prey r and the protection zone Ω_0 on the coexistence region. Finally, we give a conclusion in the last section.

The eigenvalues of the $-\Delta$ operator with various boundary conditions, domains and potential functions will play an important role in our analysis. In this paper, we denote by $\lambda_1^D(\phi, O)$

and $\lambda_1^N(\phi, O)$ the principle eigenvalue of $-\Delta + \phi$ over the bounded domain O, with Dirichlet and Neumann boundary conditions respectively. If the potential function $\phi = 0$, we simply denote them by $\lambda_1^D(O)$ and $\lambda_1^N(O)$.

§2 Main results

In this section, we give our main results of this paper. In order to better explain the meaning of our main results, we introduce the following three sets in the (λ, μ) plane:

$$\begin{aligned} S_1 &:= \left\{(\lambda,\mu) : \mu = -c(1-e^{-r\lambda})\right\}, \\ S_2 &:= \left\{(\lambda,\mu) : \lambda_1^N\left(-\lambda + b(x)r\mu, \Omega\right) = 0\right\}, \\ S_3 &:= \left\{(\lambda,\mu) : \lambda_1^N\left(-\lambda + b(x)r\mu e^{-r\lambda}, \Omega\right) = 0\right\}. \end{aligned} \tag{4}$$

Let $\mu_*(\lambda) = -c(1-e^{-r\lambda})$. It is easy to see that $\mu = \mu_*(\lambda)$ is monotone decreasing with respect to $\lambda \in (0, \infty)$, $\mu_*(0) = 0$, and $\lim_{\lambda\to\infty} \mu_*(\lambda) = -c$.

The profiles of curves S_2 and S_3 will be useful in later discussions, we will discuss them in Subsection 3.1. Precisely, the set S_2 is an unbounded curve and can be expressed by $S_2 = \{(\lambda,\mu) \in \mathbf{R}_+^2 : \mu = \mu^*(\lambda)\}$, where $\mu^*(\lambda)$ is a positive continuous and monotone increasing function for $\lambda \in (0, \lambda_1^D(\Omega_0))$, which satisfies $\lim_{\lambda\to 0^+} \mu^*(\lambda) = 0$ and $\lim_{\lambda\to[\lambda_1^D(\Omega_0)]^-} \mu^*(\lambda) = \infty$ (see Lemma 3.2); S_3 can be expressed by $S_3 = \{(\lambda,\bar\mu) \in \mathbf{R}_+^2 : \bar\mu = \bar\mu(\lambda)\}$, where S_3 has similar properties with S_2 and satisfies $\bar\mu(\lambda) > \mu^*(\lambda)$ (see Lemma 3.3).

Our first main result is to show that (3) has no positive solution if $\mu \le \mu_*$ or $\mu \ge \bar\mu$ holds (see Theorem 3.6). This result means that the lower or higher growth rate of the predator wipes out any positive solution even if there exists a protection zone for the prey.

Our second main result is to obtain the coexistence region of positive solutions for (3): (i) When $\lambda \ge \lambda_1^D(\Omega_0)$, (3) possesses at least one positive solution if $\mu > \mu_*$ (see Theorem 4.2); (ii) When $\lambda < \lambda_1^D(\Omega_0)$, (3) possess at least one positive solution for $\mu_* < \mu < \mu^*$ (see Theorem 4.4).

Combining Theorem 3.6, Theorem 4.2 and Theorem 4.4, we can draw the coexistence region and non-existence regions of (3) in the $\lambda - \mu$ plane (see Fig. 1). If (λ,μ) lies in the region surrounded by S_1 and S_2, (3) admits at least one positive solution.

Our third main result is to analyze the dependence of the coexistence region of (3) on r and Ω_0. We find that the coexistence region in the lower half plane spreads as r increases, while the coexistence region in the upper half plane narrows as r increases (see Proposition 1). The limiting coexistence regions as $r \to \infty$ is given in Fig. 2 (i). On the other hand, the coexistence region becomes larger as Ω_0 spreads to Ω (see Proposition 2), and the limiting coexistence regions as $\Omega_0 \to \Omega$ is given in Fig. 2 (ii).

§3 Preliminaries

3.1 Monotone behaviors of S_2 and S_3

Firstly, we discuss the profiles of curves S_2 and S_3. In order to do this, we need the following Lemma 3.1. Its proof is essentially the same as that of Theorem 2.1 in [8], so we omit it here.

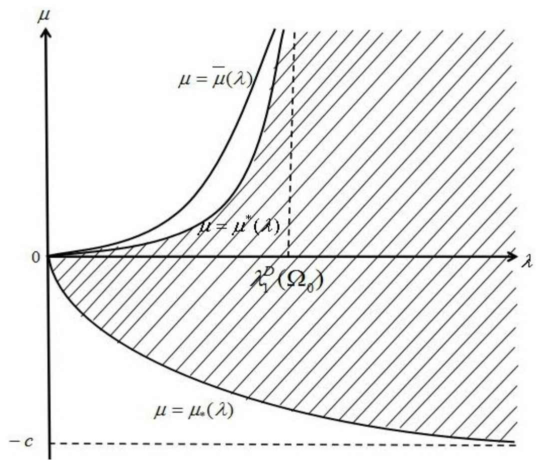

Fig. 1: Coexistence regions of (3) with fixed r and Ω_0.

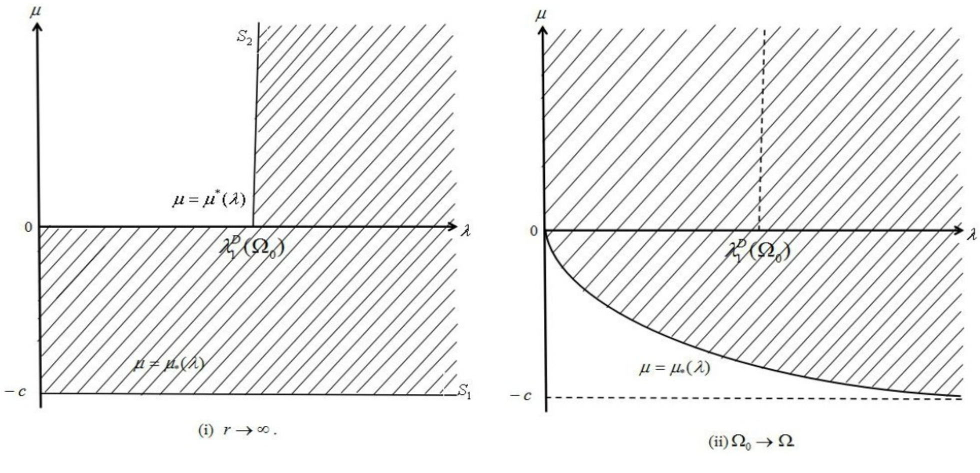

Fig. 2: Limiting coexistence regions of (3). (i)$r \to \infty$; (ii) $\Omega_0 \to \Omega$.

Lemma 3.1. *For any fixed Ω_0 and $r > 0$. Let $\lambda_1^N(b(x)r\mu, \Omega)$ be the principle eigenvalue of the following eigenvalue problem*

$$-\triangle u + b(x)r\mu u = \xi u, \quad x \in \Omega; \quad \partial_n u = 0, \quad x \in \partial\Omega,$$

then $\lambda_1^N(b(x)r\mu, \Omega) < \lambda_1^D(\Omega_0)$ and $\lim_{\mu\to\infty} \lambda_1^N(b(x)\mu r, \Omega) = \lambda_1^D(\Omega_0)$.

The following Lemmas 3.2 and 3.3 yield the profiles of curves S_2 and S_3, respectively.

Lemma 3.2. *The set S_2 is an unbounded curve and can be expressed by $S_2 = \{(\lambda, \mu) \in \mathbf{R}_+^2 : \mu = \mu^*(\lambda)\}$, where $\mu^*(\lambda)$ is a positive continuous and monotone increasing function for $\lambda \in (0, \lambda_1^D(\Omega_0))$, which satisfies $\lim_{\lambda\to 0^+} \mu^*(\lambda) = 0$ and $\lim_{\lambda\to[\lambda_1^D(\Omega_0)]^-} \mu^*(\lambda) = \infty$.*

Proof. Firstly, the set S_2 can be expressed by the curve

$$\lambda^*(\mu) = \lambda_1^N \left(b(x)r\mu, \Omega\right).$$

By the continuity and monotone increasing property of $\lambda_1^N(q)$ with respect to $q \in L^\infty$, we see that $\lambda^*(\mu)$ is a continuous and strictly increasing function for $\mu \in (0, \infty)$, and $\lambda^*(0) = \lambda_1^N(\Omega) = 0$. It follows from Lemma 3.1 that

$$\lambda^*(\mu) < \lambda_1^D(\Omega_0), \qquad \lim_{\mu\to+\infty} \lambda^*(\mu) = \lambda_1^D(\Omega_0).$$

Due to the monotonicity of $\lambda^*(\mu)$, for $\lambda \in (0, \lambda_1^D(\Omega_0))$, there exists a unique function $\mu^*(\lambda)$ such that $\lambda_1^N(-\lambda + b(x)r\mu, \Omega) = 0$, and the continuity and monotonicity of $\mu^*(\lambda)$ also follow. Moreover, we easily see that $\mu^*(0) \to 0$ as λ decreases to 0, and $\mu^*(\lambda) \to \infty$ as λ increases to $\lambda_1^D(\Omega_0)$. □

Lemma 3.3. *The set S_3 is an unbounded curve and can be expressed by $S_3 = \{(\lambda, \bar{\mu}) \in \mathbf{R}_+^2 : \bar{\mu} = \bar{\mu}(\lambda)\}$, where $\bar{\mu}(\lambda)$ is a positive continuous and monotone increasing function for $\lambda \in (0, \lambda_1^D(\Omega_0))$, which satisfies $\bar{\mu}(\lambda) > \mu^*(\lambda)$ for $\lambda \in (0, \lambda_1^D(\Omega_0))$, $\lim_{\lambda\to 0^+} \bar{\mu} = 0$, and $\lim_{\lambda\to[\lambda_1^D(\Omega_0)]^-} \bar{\mu}(\lambda) = \infty$.*

Proof. In view of (4), we put

$$\overline{S}(\lambda, \mu) = \lambda_1^N(-\lambda + b(x)r\mu e^{-r\lambda}, \Omega).$$

Obviously, $\overline{S}(0, \mu) = \lambda_1^N(b(x)r\mu, \Omega) > 0$. On the other hand, we notice that

$$\overline{S}(\lambda, \mu) = -\lambda + \lambda_1^N(b(x)r\mu e^{-r\lambda}, \Omega) < -\lambda + \lambda_1^N(b(x)r\mu, \Omega).$$

Thus, $\overline{S}(\lambda, \mu) < 0$ for any $\lambda \in (0, \lambda_1^N(b(x)r\mu, \Omega))$. Notice that $\overline{S}(\lambda, \mu)$ is continuous and strictly decreasing with λ. Therefore, the intermediate value theorem ensures that, for any fixed positive μ, there exists a unique $\overline{\lambda} \in (0, \lambda_1^N(b(x)r\mu, \Omega))$ such that $\overline{S}(\overline{\lambda}, \mu) = 0$. So, we obtain a function $\overline{\lambda} = \overline{\lambda}(\mu) : (0, \infty) \mapsto (0, \lambda_1^N(b(x)r\mu, \Omega))$. By implicit differentiation method, we easily get

$$\frac{d\overline{\lambda}}{d\mu} = -\frac{\bar{S}_\mu}{\bar{S}_{\overline{\lambda}}} > 0$$

since $\overline{S}$ is continuous and strictly increasing with respect μ and decreasing with λ.

Obviously, $\overline{\lambda}(0) = 0$. We next prove that $\lim_{\mu\to\infty} \overline{\lambda}(\mu) = \lambda_1^D(\Omega_0)$. Choose a sequence μ_n satisfying $\mu_n \to \infty$ as $n \to \infty$ and denote $\overline{\lambda}_n = \overline{\lambda}(\mu_n)$. By the monotone bounded theorem, we can assume $\overline{\lambda}_n \to \lambda_\infty \in (0, \lambda_1^D(\Omega_0)]$ *as* $n \to \infty$. Let $\phi_n > 0$ in Ω satisfy

$$-\triangle\phi_n + \left(-\bar{\lambda}_n + \mu_n b(x) r e^{-r\overline{\lambda}_n}\right)\phi_n = 0, \text{ in } \Omega, \quad \partial_n\phi_n = 0 \text{ on } \partial\Omega, \tag{5}$$

where $\|\phi_n\|_{\infty,\Omega} = 1$. Then, by (5), we deduce that $-\triangle\phi_n \le \lambda_1^D(\Omega_0)\phi_n$, and moreover,

$$\int_\Omega |\nabla\phi_n|^2 dx + \int_\Omega \phi_n^2 dx \le (\lambda_1^D(\Omega_0)+1)\int_\Omega \phi_n^2 dx \le (\lambda_1^D(\Omega_0)+1)|\Omega|. \tag{6}$$

This implies that ϕ_n is uniformly bounded in $H^1(\Omega)$ independent of n, and hence we may assume that $\phi_n \to \phi_\infty$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Due to $\|\phi_n\|_{\infty,\Omega} = 1$, we further obtain $\phi_n \to \phi_\infty$ in $L^p(\Omega)$ for each $p > 1$. By Lemma 2.2 in [4], we see that $\|\phi_\infty\|_{\infty,\Omega} = 1$. Multiplying (5) by ϕ_n and integrating the obtained result over Ω_1, we find that

$$\int_{\Omega_1} b(x)\phi_n^2 dx = -\frac{e^{r\bar{\lambda}_n}}{r\mu_n}\left[\int_\Omega |\nabla\phi_n|^2 dx - \bar{\lambda}_n\int_\Omega \phi_n^2 dx\right]. \tag{7}$$

Recall that $b(x) = 1$ in Ω_1. By virtue of (6) and $\mu_n \to \infty$ as $n \to \infty$, we let $n \to \infty$ in (7) and obtain

$$\int_{\Omega_1} \phi_\infty^2 dx = 0.$$

Then $\phi_\infty(x) = 0$ almost everywhere in Ω_1, which implies that $\phi_\infty|_{\Omega_0} \in H_0^1(\Omega_0)$ due to the smoothness of $\partial\Omega_0$. Multiplying (5) by ϕ_n, integrating the obtained result over Ω_0 and letting $n \to \infty$, we derive

$$\int_{\Omega_0} |\nabla\phi_\infty|^2 dx - \lambda_\infty\int_{\Omega_0} \phi_\infty^2 dx = 0.$$

Here we use the assumption that $b(x) \equiv 0$ in Ω_0. Therefore, $\phi_\infty|_{\Omega_0}$ satisfies the following equation (weakly)

$$-\triangle\phi = \lambda_\infty\phi \ \text{ in } \ \Omega_0, \quad \phi = 0 \ \text{ on } \ \partial\Omega_0.$$

By the strong maximum principle, we have $\phi_\infty(x) \equiv 0$ or $\phi_\infty(x) > 0$ in Ω_0. If the former occurs, then it contradicts with $\|\phi_\infty(x)\|_{\infty,\Omega} = 1$ due to $\phi_\infty(x) \equiv 0$ in Ω_0. Hence, $\phi_\infty(x) > 0$ holds, which implies that $\lambda_\infty = \lambda_1^D(\Omega_0)$. Consequently, we prove $\lim_{\mu\to\infty}\overline{\lambda}(\mu) = \lambda_1^D(\Omega_0)$.

We denote the inverse function of $\overline{\lambda} = \overline{\lambda}(\mu)$ by $\overline{\mu} = \overline{\mu}(\lambda)$, the continuity and monotonicity of $\overline{\mu}$ are as same as $\overline{\lambda}$ and we easily see that $\overline{\mu}(\lambda) \to 0$ as μ decreases to 0, and $\overline{\mu}(\lambda) \to \infty$ as λ increases to $\lambda_1^D(\Omega_0)$. ☐

3.2 Stability of trivial and semi-trivial steady-state solutions

Obviously, the steady state problem (3) has the following three non-negative solutions: The trivial solution $(0,0)$, two semi-trivial solutions $(\lambda,0)$ and $(0,\mu)$. The local stability of these trivial and semi-trivial solutions can be determined through linear stability analysis as follows.

***Theorem* 3.4.** (a) *The trivial solution* $(0,0)$ *is unstable for all* μ.

(b) *The semi-trivial solution* $(\lambda,0)$ *is locally asymptotically stable when* $\mu < \mu_*$ *and it is unstable if* $\mu > \mu_*$.

(c) *The semi-trivial solution* $(0,\mu)$ *is locally asymptotically stable if* $\mu > \mu^*$ *and it is unstable if* $\mu < \mu^*$.

Proof. The linearization operator of (3) at a constant solution $e^* = (u,v)$ can be expressed by

$$L = \begin{pmatrix} \triangle + A(u,v) & B(u,v) \\ C(u,v) & \triangle + D(u,v) \end{pmatrix}$$

on domain $X=\{(\phi,\psi)\in H^2(\Omega)\times H^2(\Omega_1):\partial_n\phi=0,\ x\in\partial\Omega,\ \partial_n\psi=0,\ x\in\partial\Omega_1\}$, where

$$A(u,v)=\lambda-2u-b(x)rve^{-ru},\quad B(u,v)=-b(x)(1-e^{-ru}),$$
$$C(u,v)=crve^{-ru},\qquad D(u,v)=\mu-2v+c(1-e^{-ru}).$$

It is known that if all the eigenvalues of the operator L have negative real parts, then e^* is locally asymptotically stable. If there is an eigenvalue with positive real part, then e^* is unstable.

(a) If $e^*=(0,0)$, then

$$\begin{pmatrix} A(u,v) & B(u,v)\\ C(u,v) & D(u,v)\end{pmatrix}\Bigg|_{(u,v)=(0,0)}=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}.$$

Hence, λ is an eigenvalue of the linearization operator L at $(0,0)$. So, $(0,0)$ is unstable.

(b) The eigenvalue problem for the corresponding linearized system to (3) at $(\lambda,0)$ is

$$\begin{cases}\triangle h-\lambda h-b(x)(1-e^{-r\lambda})k-\zeta h=0, & x\in\Omega,\\ \triangle k+\mu k+c(1-e^{-r\lambda})k-\zeta k=0, & x\in\Omega_1,\\ \partial_n h=0,\quad x\in\partial\Omega,\quad \partial_n k=0,\quad x\in\partial\Omega_1,\end{cases}$$

which has a sequence of real eigenvalues $\zeta_1>\zeta_2>\zeta_3\ldots>\zeta_n>\ldots\to-\infty$. Since ζ_1 is determined by the equation of k only, the solution $(\lambda,0)$ is stable when $\zeta_1<0$, that is

$$-\mu-c(1-e^{-r\lambda})=\lambda_1^N\left(-\mu-c(1-e^{-r\lambda}),\Omega_1\right)>0.$$

Then the semi-trivial solution $(\lambda,0)$ is locally asymptotically stable if $\mu<\mu_*$, and it is unstable if $\mu>\mu_*$.

(c) The eigenvalue problem for the corresponding linearized system to (3) at $(0,\mu)$ is

$$\begin{cases}\triangle h+(\lambda-b(x)r\mu)h-\eta h=0, & x\in\Omega,\\ \triangle k-\mu k+cr\mu h-\eta k=0, & x\in\Omega_1,\\ \partial_n h=0,\quad x\in\partial\Omega,\quad \partial_n k=0,\quad x\in\partial\Omega_1,\end{cases}$$

which has a sequence of real eigenvalues $\eta_1>\eta_2>\eta_3\ldots>\eta_n>\ldots\to-\infty$. Obviously, η_1 is determined by the equation of h only, and the solution $(0,\mu)$ is stable if $\eta_1<0$, that is

$$\lambda_1^N\left(-\lambda+b(x)r\mu,\Omega\right)>0.$$

By the continuity and monotone increasing property of $\lambda_1^N(q)$ with respect to $q\in L^\infty$, we can see that

$$\lambda_1^N\left(-\lambda+b(x)r\mu,\Omega\right)>0\Leftrightarrow\mu>\mu^*.$$

Thus, the semi-trivial solution $(0,\mu)$ is locally asymptotically stable if $\mu>\mu^*$, and it is unstable if $\mu<\mu^*$. □

3.3 A prior estimates and nonexistence regions

In this subsection, we first derive a prior estimates of positive solutions of (3).

Lemma 3.5. *Suppose that (u,v) is any positive solution of (3). Then*

$$0<u(x)\le\lambda\ \text{ in }\overline{\Omega},\quad 0<v(x)\le\mu+c(1-e^{-r\lambda})\ \text{ in }\overline{\Omega}_1.$$

Moreover, if $\mu\ge0$, then $v(x)>\mu$ in $\overline{\Omega}_1$.

Proof. Denote $u(x_0)=\max_{x\in\overline{\Omega}}u(x)$. By the maximum principle, we have

$$0\le-\triangle u(x_0)=u(x_0)\left(\lambda-u(x_0)\right)-b(x_0)v(x_0)(1-e^{-ru(x_0)})$$

and so $u(x_0) \le \lambda$. This implies that $0 < u(x_0) \le \lambda$ for any $x \in \overline{\Omega}$. Suppose that $v(x_1) = \max_{x\in\overline{\Omega}_1} v(x)$. It follows from the second equation of (3) that

$$0 \le -\triangle v(x_1) = v(x_1)\left[\mu - v(x_1) + c(1-e^{-ru(x_1)})\right].$$

Thus, we see that $v(x_1) \le \mu + c(1-e^{-ru(x_1)})$. Therefore, for any $x \in \overline{\Omega}_1$, we obtain $v(x) \le \mu + c(1-e^{-r\lambda})$.

On the other hand, we notice that

$$-\triangle v = v[\mu - v + c(1-e^{-ru})] > v(\mu - v) \text{ in } \Omega_1, \quad \partial_n v = 0 \text{ on } \partial\Omega_1.$$

By the well-known comparison result, we derive $v(x) > \mu$ in $\overline{\Omega}_1$ if $\mu \ge 0$. Thus, the proof is complete. □

Next, we proof the nonexistence of coexistence states of (3).

***Theorem* 3.6.** (a) *If $\mu \le \mu_*$, then* (3) *has no positive solution.*

(b) *If $\mu \ge \bar{\mu}$, then* (3) *has no positive solution.*

Proof. (a) Suppose for contradiction that (u, v) is a positive solution of (3) with $\mu \le \mu_*$. Then v is a positive solution of

$$-\triangle v + \left[v - \mu - c(1-e^{-ru})\right] v = 0 \text{ in } \Omega_1, \quad \partial_n v = 0 \text{ on } \partial\Omega_1.$$

By Lemma 3.5, we know $u \le \lambda$ in $\overline{\Omega}$. Hence, we obtain

$$0 = \lambda_1^N\left(v - \mu - c(1-e^{-ru}), \Omega_1\right) > \lambda_1^N\left(-\mu - c(1-e^{-r\lambda}), \Omega_1\right) = -\mu - c(1-e^{-r\lambda})$$

and thus $\mu > \mu_*$, which contracts the assumption $\mu \le \mu_*$. This completes the proof of (a).

(b) Let (u, v) be a positive solution of (3) with $\mu \ge \bar{\mu}$. Then u is a positive solution of

$$-\triangle u + [-u(\lambda - u) + b(x)v(1-e^{-ru})] = 0 \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \partial\Omega.$$

We notice that

$$1 - e^{-ru} > re^{-r\lambda}u, \quad \text{for } 0 < u \le \lambda. \tag{8}$$

In fact, let $Q(u) = 1 - e^{-ru} - re^{-r\lambda}u$. Then $Q'(u) = re^{-ru} - re^{-r\lambda} > 0$ for all $u \in (0, \lambda)$. Hence $Q(u)$ is increasing with u and $Q(u) > Q(0) = 0$ for all $u \in (0, \lambda]$, that is, $1 - e^{-ru} > re^{-r\lambda}u$. By Lemma 3.5, we know $v > \mu$ if $\mu \ge 0$. Hence, we find that

$$0 = \lambda_1^N\left(-\lambda + u + \frac{1}{u}b(x)v(1-e^{-ru}), \Omega\right) > \lambda_1^N\left(-\lambda + b(x)r\mu e^{-r\lambda}, \Omega\right).$$

From Lemma 3.3, the above inequality yields $\mu < \bar{\mu}$. This contracts the assumption $\mu \ge \bar{\mu}$. Thus, (3) has no positive solution if $\mu \ge \bar{\mu}$. □

§4 Existence of coexistence states

Since the curve S_2 exists only when $\lambda < \lambda_1^D(\Omega_0)$, we will divide our discussions below into two cases: (a) $\lambda \ge \lambda_1^D(\Omega_0)$ and (b) $0 < \lambda < \lambda_1^D(\Omega_0)$. The value of $\lambda_1^D(\Omega_0)$ will plays a crucial role in determining the bifurcation structure of (3) in our analysis to come. A well-known property of $\lambda_1^D(O)$ is that $\lambda_1^D(O_1) \ge \lambda_1^D(O_2)$ if $O_1 \subset O_2$. So, from an ecological point of view, the value of $\lambda_1^D(\Omega_0)$ implies that, for any fixed prey growth rate λ, there exists a critical path size of the protection zone determined by $\lambda = \lambda_1^D(\Omega_0)$. Case (a) corresponds to the large protection zone case and Case (b) corresponds to the small protection zone case.

4.1 The large protection zone case: $\lambda \geq \lambda_1^D(\Omega_0)$

In this subsection, we consider the set of positive steady state solutions when the protection zone Ω_0 is large so that $\lambda \geq \lambda_1^D(\Omega_0)$. We start our analysis by a standard local bifurcation argument by regarding μ as a bifurcation parameter. Obviously, (3) has two semi-trivial solution curves in the space of (μ, u, v):

$$\Gamma_u = \{(\mu, u, v) : -\infty < \mu < \infty, (u, v) = (\lambda, 0)\},$$
$$\Gamma_v = \{(\mu, u, v) : 0 < \mu < \infty, (u, v) = (0, \mu)\}.$$

For $p > N$, we define Banach spaces X_1 and X_2 as

$$X_1 = W_n^{2,p}(\Omega) \times W_n^{2,p}(\Omega_1) \text{ and } X_2 = L_n^p(\Omega) \times L_n^p(\Omega_1)$$

where $W_n^{2,p}(O) = \{w \in W_n^{2,p}(O) : \partial_n w = 0, x \in \partial O\}$. It follows from Sobolev embedding theorem that

$$X_1 \subset E = C_n^1(\overline{\Omega}) \times C_n^1(\overline{\Omega}_1).$$

We introduce a positive functions ϕ^* by solving the problem

$$-\triangle\phi^* = \lambda\phi^* - b(x)r\mu^*\phi^* \text{ in } \Omega, \ \partial_n\phi^* = 0 \text{ on } \partial\Omega. \tag{9}$$

In addition, we define

$$\psi^* = (-\triangle + \mu)_{\Omega_1}^{-1}(\mu^* cr\phi^*), \tag{10}$$
$$\phi_* = -(-\triangle + \lambda)_{\Omega}^{-1}\left(b(x)(1 - e^{-r\lambda})\right) < 0. \tag{11}$$

The following properties hold true.

***Theorem* 4.1.** *If $\lambda \geq \lambda_1^D(\Omega_0)$, then*

(a) $(\mu_*, \lambda, 0)$ *is the only bifurcation point on the semi-trivial solution curve Γ_u, where $\mu_* = -c(1 - e^{-r\lambda})$.*

(b) *The set of positive solutions near $(\mu_*, \lambda, 0) \in \mathbb{R} \times X_1$ can be expressed as*

$$\Gamma_* = \{(\mu, u, v) = (\mu(s), \lambda + s(\phi_* + \hat{u}(s)), s(1 + \hat{v}(s))) : s \in (0, \delta)\},$$

where $\delta > 0$ is small. Here, $(\hat{u}(s), \hat{v}(s))$ is a smooth function with respect to s and satisfies $(\mu(0), \hat{u}(0), \hat{v}(0)) = (\mu_*, 0, 0)$ and $\int_{\Omega_1} \hat{v}(s)dx = 0$.

(c) *The bifurcation of Γ_* at $(\mu_*, \lambda, 0)$ is supercritical due to $\mu'(0) > 0$.*

(d) $(0, \mu)$ *is an unstable steady state of* (3) *for all μ, and there is no bifurcation of positive solutions occurring along Γ_v.*

Proof. We define a mapping $F : \mathbb{R} \times X_1 \to X_2$ by

$$F(\mu, u, v) = \begin{pmatrix} \triangle u + u(\lambda - u) - b(x)v(1 - e^{-ru}) \\ \triangle v + v\left[\mu - v + c(1 - e^{-ru}\right]) \end{pmatrix}. \tag{12}$$

Since $(u, v) = (\lambda, 0)$ is a semi-trivial solution of (3), $F(\mu, \lambda, 0) = 0$ for $\mu \in \mathbb{R}$. The Fréchet derivative of F at $(\mu, \lambda, 0)$ is given by

$$F_{(u,v)}(\mu, \lambda, 0)[\phi, \psi] = \begin{pmatrix} \triangle\phi - \lambda\phi - (1 - e^{-r\lambda})b(x)\psi \\ \triangle\psi + [\mu + c(1 - e^{-r\lambda})]\psi \end{pmatrix}.$$

By the Krein-Rutman theorem [15], we see that $F_{(u,v)}(\mu, \lambda, 0)[\phi, \psi] = (0, 0)$ has a solution with $\psi > 0$ only for $\mu = \mu_* = -c(1 - e^{-r\lambda})$. Hence, $(\mu_*, \lambda, 0)$ is the only bifurcation point and

$$\text{Ker}F_{(u,v)}(\mu_*, \lambda, 0) = \text{span}\{(\phi_*, 1)\}.$$

Here, ϕ_* is given by (11). If $(\tilde{\phi}, \tilde{\psi}) \in \text{Range}F_{(u,v)}(\mu_*, \lambda, 0)$, then

$$\begin{cases} \triangle\phi - \lambda\phi - (1 - e^{-r\lambda})b(x)\psi = \tilde{\phi} & \text{in } \Omega, \\ \triangle\psi = \tilde{\psi} & \text{in } \Omega_1, \\ \partial_n\phi = 0, \text{ on } \partial\Omega, \quad \partial_n\psi = 0, \text{ on } \partial\Omega_1 \end{cases}$$

for some $(\phi, \psi) \in X_1$. From the Fredholm alternative theorem [15], the second equation has a solution ψ if and only if $\int_{\Omega_1} \tilde{\psi} dx = 0$. For such a solution ψ, the first equation has a unique solution ϕ because $-\triangle + \lambda$ is invertible. Then it follows that codim-Range$F_{(u,v)}(\mu_*, \lambda, 0) = 1$. Moreover, some elementary calculations enable us to obtain

$$F_{\mu(u,v)}(\mu_*, \lambda, 0) \begin{pmatrix} \phi_* \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \not\subseteq \text{Range}F_{(u,v)}(\mu_*, \lambda, 0).$$

Consequently, the assertion (a) and (b) can be obtained by applying the local bifurcation theorem [1]. Furthermore, we use the result of Shi [12] to obtain

$$\mu'(0) = -\frac{\langle F_{(u,v)(u,v)}(\mu_*, \lambda, 0)[\phi_*, 1]^2, l\rangle}{2\langle F_{\mu,(u,v)}(\mu_*, \lambda, 0)[\phi_*, 1], l\rangle} = 1 - \frac{cre^{-r\lambda}}{|\Omega|} \int_{\Omega_1} \phi_* dx > 0,$$

where l is a linear functional on X_2 defined as $\langle[\phi, \psi], l\rangle = \int_{\Omega_1} \psi dx$. Therefore the bifurcation at $(\mu_*, \lambda, 0)$ is always supercritical.

In view of Lemma 3.1, for any μ, $\lambda_1^N(b(x)r\mu, \Omega) < \lambda_1^D(\Omega_0)$. Therefore, when $\lambda \geq \lambda_1^D(\Omega_0)$, $\lambda_1^N(-\lambda + b(x)r\mu, \Omega) = -\lambda + \lambda_1^N(b(x)r\mu, \Omega) < 0$. Thanks to Theorem 3.4 (c), we know that $(0, \mu)$ is an unstable steady state of (3) for all μ, and there is no bifurcation of positive solutions occurring along Γ_v. □

Next, we extend the local bifurcation branch Γ_* to global solution branch.

***Theorem* 4.2.** *When $\lambda \geq \lambda_1^D(\Omega_0)$, (3) possesses at least one positive solution if $\mu > \mu_*$.*

Proof. We define a mapping $\overline{F} : \mathbb{R} \times E \to E$ by

$$\overline{F}(\mu, u, v) = \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} (-\triangle + I)_{\Omega}^{-1} [u + u(\lambda - u) - b(x)v(1 - e^{-ru})] \\ (-\triangle + I)_{\Omega_1}^{-1} [v + v[\mu - v + c(1 - e^{-ru}])] \end{pmatrix},$$

where the second term of $\overline{F}(\mu, u, v)$ is a compact operator. From the global bifurcation theorem, it follows that the local bifurcation branch Γ_* can be extended to the maximal connected set, denoted by Γ_M, and it satisfies

$$\Gamma_* \subset \Gamma_M \subset \{(\mu, u, v) \in (\mathbb{R} \times E) \setminus \{(\mu_*, \lambda, 0) : \overline{F}(\mu, u, v) = 0\}. \tag{13}$$

In order to complete the proof, we next prove

$$\Gamma_M \subset \mathbb{R} \times P_\Omega \times P_{\Omega_1}, \tag{14}$$

where $P_O = \{w \in C_n^1(\bar{O}) : w > 0 \text{ in } \overline{O}\}$. We argue it by contraction. Suppose that $\Gamma_M \not\subseteq \mathbb{R} \times P_\Omega \times P_{\Omega_1}$. Then there exists a sequence $\{(\mu_i, u_i, v_i)\}_{i=1}^{\infty} \subset \Gamma_M \cap (\mathbb{R} \times P_\Omega \times P_{\Omega_1})$ such that $\lim_{i\to\infty}(\mu_i, u_i, v_i) = (\mu_\infty, u_\infty, v_\infty)$ in $\mathbb{R} \times E$, where

$$(\mu_\infty, u_\infty, v_\infty) \in \Gamma_M \cap (\mathbb{R} \times \partial(P_\Omega \times P_{\Omega_1})). \tag{15}$$

We note that (u_∞, v_∞) is a non-negative solution of (3). By the strong maximum principle, we know that (u_∞, v_∞) must satisfy one of the following:

(i) $u_\infty \equiv 0$ in $\overline{\Omega}$, $v_\infty \equiv 0$ in $\overline{\Omega_1}$;

(ii) $u_\infty > 0$ in $\overline{\Omega}$, $v_\infty \equiv 0$ in $\overline{\Omega_1}$;

(iii) $u_\infty \equiv 0$ in $\overline{\Omega}$, $v_\infty > 0$ in $\overline{\Omega_1}$.

From the second equation of (3) with $(\mu, u, v) = (\mu_i, u_i, v_i)$, we derive

$$\int_{\Omega_1} v_i \left[\mu_i - v_i + c(1 - e^{-ru_i})\right] dx = 0 \ \text{ for any } \ i \in N. \tag{16}$$

By Theorem 3.6 (a), we know that $\mu_i > \mu_*$. Thus $\mu_\infty \geq \mu_*$.

If $\mu_\infty > 0$. Suppose that (i) or (ii) occurs, then

$$\mu_i - v_i + c(1 - e^{-ru_i}) \longrightarrow \mu_\infty + c(1 - e^{-ru_\infty}) > 0 \ \text{ as } \ i \to \infty.$$

Thus, for sufficiently large $i \in N$, the integration in (16) is positive, which contradicts with (16). If case (iii) occurs, then v_∞ satisfies

$$\triangle v_\infty + v_\infty(\mu_\infty - v_\infty) = 0 \ \text{ in } \Omega_1, \ \ \partial_n v_\infty = 0 \ \text{ on } \partial\Omega_1.$$

Thus, it is well know that $v_\infty \equiv \mu_\infty$ in $\overline{\Omega_1}$. Therefore, we must obtain $(\mu_\infty, u_\infty, v_\infty) = (\mu, 0, \mu)$, which contradicts with Theorem 4.1 (d).

If $\mu_\infty = 0$. Suppose that (ii) occurs, then

$$\mu_i - v_i + c(1 - e^{-ru_i}) \longrightarrow c(1 - e^{-ru_\infty}) > 0 \ \text{ as } \ i \to \infty.$$

If (iii) occurs, then

$$\mu_i - v_i + c(1 - e^{-ru_i}) \longrightarrow -v_\infty < 0 \ \text{ as } \ i \to \infty.$$

Similar to the above, we can get some contradictions if (ii) or (iii) occurs. If case (i) occurs, we can deduce $(\mu_\infty, u_\infty, v_\infty) = (\mu_\infty, 0, 0)$, which contradicts with Theorem 3.4 (a).

If $\mu_\infty \in [\mu_*, 0)$. Suppose that (i) or (iii) occurs, then

$$\mu_i - v_i + c(1 - e^{-ru_i}) \longrightarrow \mu_\infty - v_\infty < 0 \ \text{ as } \ i \to \infty.$$

Thus, for sufficiently large $i \in N$, the integration in (16) is negative, which contradicts with (16). On the other hand, if case (ii) occurs, then u_∞ satisfies

$$\triangle u_\infty + u_\infty(\lambda - u_\infty) = 0 \ \text{ in } \Omega, \ \ \partial_n u_\infty = 0 \ \text{ on } \partial\Omega.$$

Thus, it is well know that $u_\infty \equiv \lambda$ in $\overline{\Omega}$. Therefore, by Theorem 4.1 (a), we must obtain $(\mu_\infty, u_\infty, v_\infty) = (\mu_*, \lambda, 0)$, which contradicts with (13) and (15). Consequently, we prove (14).

Define

$$Y = \left\{(\phi, \psi) \in E : \int_{\Omega_1} \psi dx = 0\right\}. \tag{17}$$

Then Y is the supplement of span$\{(\phi_*, 1)\}$ in E. By Theorem 6.4.3 of [9], we see that one of the following non-excluding situations occurs:

(1) Γ_M is unbounded in $\mathbb{R} \times E$;

(2) Γ_M contains a point $(\widetilde{\mu}, \lambda, 0)$, where $\widetilde{\mu} \neq \mu_*$.

(3) Γ_M contains a point $(\widehat{\mu}, \widehat{\phi}, \widehat{\psi})$, where $(\widehat{\mu}, \widehat{\phi}, \widehat{\psi}) \in \mathbb{R} \times (Y \pm \{\lambda, 0\})$.

Case (2) is impossible because of (14). In view of (14) and (17), we find that case (3) cannot occur. Hence, case (1) must occur. By Theorem 4.1 (c), we know that the bifurcation of Γ_* at $(\mu_*, \lambda, 0)$ is supercritical. Hence, the local bifurcation curve Γ_* can be globally extended to infinity along μ, that is, $\text{Proj}_\mu \Gamma_* = (\mu_*, +\infty)$. Thus the proof of Theorem 4.2 is complete. □

4.2 The small protection zone case: $\lambda < \lambda_1^D(\Omega_0)$

In this subsection, we consider the set of positive steady state solutions when the protection zone Ω_0 satisfies $\lambda < \lambda_1^D(\Omega_0)$. We will see that the bifurcation structure of (3) is different from that in the above subsection.

***Theorem* 4.3.** *If $0 < \lambda < \lambda_1^D(\Omega_0)$, then*

(a) $(\mu_*, \lambda, 0)$ *is the only bifurcation point on the semi-trivial solution curve* Γ_u.

(b) *The set of positive solutions near $(\mu_*, \lambda, 0) \in \mathbb{R} \times X_1$ can be expressed as*

$$\Gamma_* = \{(\mu, u, v) = (\mu(s), \lambda + s(\phi_* + \hat{u}(s)), s(1 + \hat{v}(s))) : s \in (0, \delta)\},$$

where $\delta > 0$ is small. Here, $(\hat{u}(s), \hat{v}(s))$ is a smooth function with respect to s and satisfies $(\mu(0), \hat{u}(0), \hat{v}(0)) = (\mu_*, 0, 0)$ and $\int_{\Omega_1} \hat{v}(s)dx = 0$.

(c) $(\mu^*, 0, \mu^*)$ *is only bifurcation point of positive solution of* (3) *along* Γ_v.

(d) *The set of positive solutions near $(\mu^*, 0, \mu^*) \in \mathbb{R} \times X_1$ can be expressed as*

$$\Gamma^* = \{(\mu, u, v) = (\mu(s), s(\phi^* + \widetilde{u}(s)), \mu + s(\psi^* + \widetilde{v}(s))) : |s| < \tilde{\delta}\}$$

such that $(\mu(0), \widetilde{u}(0), \widetilde{v}(0)) = (\mu^*, 0, 0)$ and $\int_\Omega \widetilde{u}(s)\phi^* dx = 0$, where $\widetilde{\delta} > 0$ is small.

(e) *The bifurcation of Γ^* at $(\mu^*, 0, \mu^*)$ is supercritical if $I > 0$ and it is subcritical if $I < 0$, where I is defined*

$$I = \int_\Omega (\phi^*)^2 \left[(b(x)r^2\mu^* - 2)\phi^* + 2rb(x)\psi^*\right] dx. \tag{18}$$

Proof. The proofs of (a) and (b) are as same as that of Theorem 4.2. Now, We consider bifurcation at Γ_v. It is clear that $F(\mu, 0, \mu) = 0$ for any $\mu > 0$, where $F(\mu, u, v)$ is given by (12). By a simple calculation, we obtain

$$F_{(u,v)}(\mu, 0, \mu)[\phi, \psi] = \begin{pmatrix} \triangle\phi + \lambda\phi - b(x)r\mu\phi \\ \triangle\psi - \mu\psi + \mu cr\phi \end{pmatrix}.$$

Thus, we see that the equation $F_{(u,v)}(\mu, 0, \mu)[\phi, \psi] = (0, 0)$ has a solution with $\phi > 0$ if and only if $\lambda_1^N(-\lambda + b(x)r\mu, \Omega) = 0$. By some similar arguments as that in the proof of Theorem 4.1, we get

$$\mathrm{Ker} F_{(u,v)}(\mu^*, 0, \mu^*) = \mathrm{span}\{(\phi^*, \psi^*)\},$$

where, ϕ^* and ψ^* are given in (9) and (10). Moreover,

$$\mathrm{Range} F_{(u,v)}(\mu^*, 0, \mu^*) = \left\{(\phi, \psi) \in X_2 : \int_{\Omega_1} \phi\phi^* dx = 0\right\}.$$

The above equation yields

$$F_{\mu(u,v)}(\mu^*, 0, \mu)[\phi^*, \psi^*] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \not\subseteq \mathrm{Range} F_{(u,v)}(\mu^*, 0, \mu^*).$$

Thus, by the local bifurcation theorem, we obtain the corresponding results (c) and (d). Furthermore, we use the result of Shi [12] to obtain

$$\mu'(0) = -\frac{\langle F_{(u,v)(u,v)}(\mu^*, 0, \mu^*)[\phi^*, \psi^*]^2, l\rangle}{2\langle F_{\mu,(u,v)}(\mu^*, 0, \mu^*)[\phi^*, \psi^*], l\rangle} = \frac{\int_\Omega (\phi^*)^2 \left[(b(x)r^2\mu^* - 2)\phi^* + 2rb(x)\psi^*\right] dx}{2\int_\Omega b(x)r\phi^{*2} dx},$$

where l is a linear functional on X_2 defined as $\langle[\phi, \psi], l\rangle = \int_\Omega \phi\phi^* dx$. This completes the proof of Theorem 4.3. □

Next, we extend the local bifurcation branches Γ_* (obtained in Theorem 4.3 (b)) and Γ^* (obtained in Theorem 4.3 (d)) as global solution branches.

***Theorem* 4.4.** *When $\lambda < \lambda_1^D(\Omega_0)$,* (3) *possess at least one positive solution for $\mu_* < \mu < \mu^*$.*

Proof. We consider the positive solutions of (3) emanating from $(\mu^*, 0, \mu^*)$. From the global bifurcation theorem, it follows that the local bifurcation branch Γ^* can be extended to the maximal connected set, denoted by Γ^M, and it satisfies

$$\Gamma^* \subset \Gamma^M \subset \{(\mu, u, v) \in (\mathbb{R} \times E) \setminus \{(\mu^*, 0, \mu) : \overline{F}(\mu, u, v) = 0\}. \tag{19}$$

Similar to the discussion of Theorem 4.2, we can get

$$\Gamma^M \subset \mathbb{R} \times P_\Omega \times P_{\Omega_1}. \tag{20}$$

Define

$$Y = \left\{ (\phi, \psi) \in E : \int_{\Omega_1} \phi\phi^* dx = 0 \right\}. \tag{21}$$

We find that Γ^M satisfies one of (1)-(3) as follows:

(1) Γ^M is unbounded in $\mathbb{R} \times E$;

(2) Γ^M contains a point $(\widetilde{\mu}, 0, \widetilde{\mu})$, where $\widetilde{\mu} \neq \mu^*$.

(3) Γ^M contains a point $(\widehat{\mu}, \widehat{\phi}, \widehat{\psi})$, where $(\widehat{\mu}, \widehat{\phi}, \widehat{\psi}) \in \mathbb{R} \times (Y \pm \{0, \widehat{\mu}\})$.

By Lemma 3.5 and Theorem 3.6, we known that $\mu_* < \mu < \bar{\mu}$, where $\bar{\mu}$ satisfies $(\bar{\mu}(\lambda), \lambda)$ on the curve S_3. Therefore, we known that the alternative (1) is impossible. In view of Theorem 4.3, we find that the alternative (2) does not occur. Thus, the alternative (3) must hold. Furthermore, by Theorem 4.1, we obtain that Γ^M ends at some point $(\mu_*, \lambda, 0)$ on Γ_u for some μ_*. Thus, the proof of Theorem 4.4 is complete. □

§5 Dependence of the coexistence region on r and Ω_0

Combining Theorem 3.6, Theorem 4.2 and Theorem 4.4, we can draw the coexistence region and non-existence regions of (3) in the $\lambda - \mu$ plane (see Fig. 1). If (λ, μ) lies in the region surrounded by S_1 and S_2, (3) admits at least one positive solution.

From (4), we can see that both S_1 and S_2 depend on r. In order to study the dependence of S_1 on r, we denote the curve $\mu_* = \mu_*(\lambda)$ by $\mu_* = \mu_*(\lambda, r)$. Obviously, for any fixed $\lambda > 0$, $\mu_*(\lambda, r)$ is strictly monotone decreasing with respect to r, and $\lim_{r\to\infty} \mu_*(\lambda, r) = -c$.

The next result yields the monotone and limiting behavior of S_2 with respect to r. Let the inverse function of $\mu^* = \mu^*(\lambda)$ be $\lambda^* = \lambda^*(\mu)$, and also denote the curve $\lambda^* = \lambda^*(\mu)$ by $\lambda^* = \lambda^*(\mu, r)$.

***Proposition* 1.** *For any fixed $\mu \in (0, \infty)$, $\lambda^* = \lambda^*(\mu, r)$ is strictly monotone increasing with respect to r, and* $\lim_{r\to\infty} \lambda^*(\mu, r) = \lambda_1^D(\Omega_0)$.

The proof of Proposition 1 is essentially the same as that of Theorem 2.1 in [8], so we omit it here.

Remark 1. The monotonicity of S_1 depending on r implies that the coexistence region in the lower half plane spreads as r increases. On the other hand, Proposition 1 implies that the coexistence region in the upper half plane narrows as r increases, see Fig. 2 (i).

Next, we consider the dependence of the coexistence region of (3) on Ω_0. A glance of (4) shows that the set S_1 is independent of Ω_0, but the set S_2 depends on Ω_0. We write $\lambda^* = \lambda^*(\mu, \Omega_0)$ instead of $\lambda^* = \lambda^*(\mu)$ to express the dependence on Ω_0 explicitly. We can obtain the following proposition.

***Proposition* 2.** *Suppose that $\mu > 0$. If $\Omega_0^1 \subset \Omega_0^2 \subset \Omega$, then $\lambda^*(\mu, \Omega_0^1) > \lambda^*(\mu, \Omega_0^2)$. Moreover, $\lambda^*(\mu, \Omega_0) \le \dfrac{r\mu|\Omega \setminus \Omega_0|}{|\Omega|}$.*

Proof. Note that $\lambda^*(\mu, \Omega_0) = \lambda_1^N(b(x)r\mu, \Omega)$. From the monotone increasing property of $\lambda_1^N(q, \Omega)$ with respect to q and the assumption for $b(x)$, we can see that $\lambda^*(\mu, \Omega_0^1) > \lambda^*(\mu, \Omega_0^2)$ if $\Omega_0^1 \subset \Omega_0^2 \subset \Omega$.

For any $\mu > 0$, let ϕ be a unique positive solution of

$$\begin{cases} -\triangle\phi + [-\lambda^*(\mu, \Omega_0) + b(x)r\mu]\phi = 0, & x \in \Omega, \\ \partial_\nu \phi = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega \phi^2 dx = 1 \end{cases} \tag{22}$$

For any $\psi \in H^1(\Omega)$, we have

$$\int_\Omega \nabla\phi\nabla\psi dx + r\mu \int_{\Omega\setminus\Omega_0} \phi\psi dx - \lambda^*(\mu, \Omega_0) \int_\Omega \phi\psi dx = 0.$$

Namely, ϕ is a weak solution of

$$\begin{cases} -\triangle\phi + r\mu\chi_{\Omega\setminus\Omega_0}\phi - \lambda^*(\mu, \Omega_0)\chi_\Omega\phi = 0, & x \in \Omega, \\ \partial_\nu\phi = 0, & x \in \partial\Omega. \end{cases}$$

Since $\phi \ge 0$ on $\bar{\Omega}$ and $\int_\Omega \phi^2 dx = 1$, we see $\phi > 0$ on $\bar{\Omega}$ by the strong maximum principle. This means that $\lambda^*(\mu, \Omega_0)$ is the first eigenvalue of

$$\begin{cases} -\triangle\phi + r\mu\chi_{\Omega\setminus\Omega_0}\phi = \eta\chi_\Omega\phi, & x \in \Omega, \\ \partial_\nu\phi = 0, & x \in \partial\Omega. \end{cases}$$

Therefore, by the variational characterization of the first eigenvalue, we have

$$\lambda^*(\mu, \Omega_0) = \inf_{\phi\in\Theta'} \frac{\int_\Omega |\nabla\phi|^2 dx + r\mu \int_{\Omega\setminus\Omega_0} \phi^2 dx}{\int_\Omega \phi^2 dx}. \tag{23}$$

By setting $\phi \equiv 1$ on $\bar{\Omega}$ in (23), we have $\lambda^*(\mu, \Omega_0) \le \dfrac{r\mu|\Omega \setminus \Omega_0|}{|\Omega|}$. □

Remark 2. Proposition 2 means that the coexistence region becomes larger as Ω_0 spreads to Ω, and $\lambda^*(\mu, \Omega_0)$ decreases to 0 as Ω_0 is enlarged to the entire Ω, see Fig. 2 (ii). It is shown that the protection zone for prey has beneficial effects on the coexistence of the two species when the growth rate of predator is positive.

§6 Conclusions

This paper is devoted to study the effect of a protection zone on a diffusion predator-prey model with Ivlev-type functional response. Applying the bifurcation theory, we obtain some sufficient conditions of coefficients for the existence of positive steady state solutions and draw the coexistence region and non-existence regions of (3) in the $\lambda - \mu$ plane. Through examining the effects of the predator capture of the prey r and the protection zone Ω_0 on the coexistence region, we find that the coexistence region in the lower half plane spreads as r increases while the coexistence region in the upper half plane narrows as r increases. Moreover, the coexistence region becomes larger as Ω_0 spreads to Ω. Therefore, the protection zone for prey has beneficial effects on the coexistence of the two species.

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