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The spike layer solution to singular perturbation Robin boundary value problem for higher order elliptic equation

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Abstract. The singularly perturbed Robin boundary value problem for the higher order elliptic equation is considered. Under suitable conditions, the existence and asymptotic behavior of solution to the boundary value problems are studied. The uniform validity of its asymptotic expansion is proved by using the fixed point theorem.

§1 Introduction

Nonlinear differential equations are very important in applied mathematics, mathematical physics and engineering mathematics. Singular perturbation theory has been studied in nonlinear problems in natural sciences. Many scholars have studied the problems[1−14] and some singularly perturbed problems have also been discussed[15−22].

Consider the following singular perturbation Robin boundary value problem for higher order elliptic equation

$$\varepsilon^{2m} L^m w = f(x, \varepsilon, w), \quad x \in \Omega, \tag{1}$$

$$B_j w = g_j(x), \quad x \in \partial\Omega, \quad j = 0, 1, \cdots, m-1, \tag{2}$$

where L^m denotes $2m$-order elliptic operator, and L means second-order elliptic operator which are expressed as the following

$$L \equiv \sum_{i,j=1}^{n} \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \beta_i(x) \frac{\partial}{\partial x_i}, \quad \sum_{i,j=1}^{n} \alpha_{ij}(x) \zeta_i \zeta_j \geq \lambda \sum_{i=1}^{n} {\zeta_i}^2, \quad \forall \zeta_i \in R, \quad \lambda > 0,$$

$$B_j = \varepsilon^j a_j \frac{\partial^j}{\partial n^j} + b_j, \quad j = 0, 1, \cdots, m-1.$$

where ε ($\varepsilon > 0$) is a small parameter, $x = (x_1, x_2, \cdots x_n) \in \Omega$, Ω is bounded convex domain in R^n, $\partial\Omega$ is smooth boundary of Ω, $\dfrac{\partial}{\partial n}$ is an outer normal derivative on $\partial\Omega$, $a_j(x) > 0$, $b_j(x) \geqslant$

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$b_0 > 0$. This paper is concerned with the solution for boundary layer and inner spike layers. And the uniform efficiency of the asymptotic solution is proved by using the fixed point theorem.

With the following hypotheses we may find the solutions:

$[H_1]$ Coefficients of operators L, $a_j(x)$, $b_j(x)$, f and g_j in the corresponding region, except $x_0 \in \Omega$, for corresponding variable is a sufficiently smooth function.

$[H_2]$ There is a positive constant δ, such as

$$\frac{\partial f}{\partial w}(x, w, \varepsilon) > \delta, \quad \forall x \in \overline{\Omega}, \quad (x \neq x_0), \quad \forall w \in R.$$

Let $W(x, \varepsilon)$ be the outer solution to the problem (1)(2), and let

$$W(x, \varepsilon) = \sum_{i=0}^{\infty} W_i(x)\varepsilon^i. \tag{3}$$

Substituting Eq.(3) into Eq.(1), developing f in ε, combining the coefficients in ε, and let the coefficients equate to zero, for ε^0 we obtain

$$f(x, W_0, 0) = 0. \tag{4}$$

From the hypotheses, there is a solution $W_0(x)$ to Eq.(4). And for the coefficient of the terms ε^i $(i = 1, 2, \cdots)$, we obtain

$$W_i(x) = \frac{1}{f_w(x, W_0, 0)} \Big[L^m U_{2m-i} + \Big[\frac{1}{i!}\frac{\partial^j}{\partial \varepsilon^j} f\Big(x, \sum_{j=0}^{\infty} W_j(x)\varepsilon^j, \varepsilon\Big)\Big]_{\varepsilon=0} \Big], \quad i = 1, 2, \cdots.$$

Substituting $W_i(x)$ $(i = 0, 1, \cdots)$ into Eq.(3), we can obtain the outer solution $W(x, \varepsilon)$ to the original problem (1)(2). But it may not be satisfied in $x_0 \in \Omega$ and also it may not satisfy the boundary condition (2), so we need construct the corrective terms to the inner spike layer solution and the boundary layer solution respectively in neighborhood of $x_0 \in \Omega$ and $\partial\Omega$.

The rest of this paper is organized as follows. Section 2 we construct non-singular local coordinate system and lead into the coordinate of multi-scales, and obtain the spike layer shock behavior corrective term. Section 3 we construct local coordinate system by using the singular perturbation method respectively, from the multi-scales variables, we have the boundary corrective term. Then the asymptotic solution of the whole region is obtained by using the complex method. Finally, a brief conclusion is given in section 4, by using the fixed point theorem, we can prove the formal asymptotic solution is uniformly valid.

§2 Inner Spike Layer Term

Non-singular local coordinate system (r, θ) is constructed in the neighborhood of $x_0 \in \Omega$. Define the coordinate r $(r \leqslant r_0)$ of each point Q as the distance of the point Q to x_0 in neighborhood of $x_0 \in \Omega$, r_0 is an appropriately small positive constant, while $\theta = (\theta_1, \theta_2, \cdots, \theta_{n-1})$ is a non-singular coordinate $(n-1)-$dimensional manifold $(r = r_0)$, the coordinate θ of Q as the coordinate θ of P, where P is an intersection which through Q following the internal normal line direction on the boundary $r = r_0$.

In the neighborhood of $x_0 \in \Omega$: $0 \leqslant r \leqslant r_0$, we have

$$L = a_{nn}\frac{\partial^2}{\partial r^2} + \sum_{i=1}^{n-1} a_{ni}\frac{\partial^2}{\partial r \partial \theta_i} + \sum_{i,j=1}^{n-1} a_{ij}\frac{\partial^2}{\partial \theta_i \partial \theta_j} + b_n\frac{\partial}{\partial r} + \sum_{i=1}^{n-1} b_i\frac{\partial}{\partial \theta_i}, \tag{5}$$

where

$$a_{nn}=\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k},\quad a_{ni}=2\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial r}{\partial x_j}\frac{\partial \theta_i}{\partial x_k},\quad a_{ij}=\sum_{k,l=1}^{n}\alpha_{kl}\frac{\partial \theta_i}{\partial x_k}\frac{\partial \theta_j}{\partial x_l},$$

$$b_n=\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial^2 r}{\partial x_j\partial x_k},\quad b_i=\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial^2 \theta_i}{\partial x_j\partial x_k}.$$

In the neighborhood of x_0: $0\leqslant r\leqslant t_0$, by leading into the coordinate of multi-scales[1], we have $\tau=\dfrac{h(r,\theta)}{\varepsilon}$, $\widetilde{r}=r$, $\widetilde{\theta}=\theta$, where $h(r,\theta)$ is determined later. For convenience, we use r, θ instead of $\widetilde{r},\widetilde{\theta}$. From (5), we have

$$L=\frac{1}{\varepsilon^2}S_0+\frac{1}{\varepsilon}S_1+S_2, \tag{6}$$

while S_0, S_1 and S_2 are the decomposition expressions of the second-order elliptic operator L (Eq.(5)), their expressions are as follows

$$S_0=a_{nn}h_r^2\frac{\partial^2}{\partial\tau^2},$$

$$S_1=2a_{nn}h_r\frac{\partial^2}{\partial\tau\partial r}+\sum_{i=1}^{n-1}a_{ni}h_r\frac{\partial^2}{\partial\tau\partial\theta_i}+(a_{nn}h_{rr}+b_nh_r)\frac{\partial}{\partial\tau},$$

$$S_2=a_{nn}\frac{\partial^2}{\partial r^2}+\sum_{i=1}^{n-1}a_{ni}\frac{\partial^2}{\partial r\partial\theta_i}+\sum_{i,j=1}^{n-1}a_{ij}\frac{\partial^2}{\partial\theta_i\partial\theta_j}+b_n\frac{\partial}{\partial r}+\sum_{i=1}^{n-1}b_i\frac{\partial}{\partial\theta_i}.$$

Let $h_r=\sqrt{1/a_{nn}}$, and w is outer solution to the problem (1),(2) for higher order elliptic equation

$$w=W(x,\varepsilon)+Y(\tau,r,\theta), \tag{7}$$

where Y is the corrective spike layer term

$$Y=\sum_{i=0}^{\infty}Y_i(\tau,r,\theta)\varepsilon^i. \tag{8}$$

Substituting Eqs.(7),(8) into Eq.(1), developing nonlinear term in ε, combining the same coefficients of ε^i $(i=0,1,\cdots)$, we obtain

$$S_0^mY_0=0,\quad (0\leqslant r\leqslant r_0), \tag{9}$$

$$\left.\frac{\partial^jY_0}{\partial r^j}\right|_{r=0}=-\left.\frac{\partial^jW_0}{\partial r^j}\right|_{r=0},\quad j=0,1,\cdots,m-1, \tag{10}$$

$$S_0^mY_i=G_i,\quad (0\leqslant r\leqslant r_0),\quad i=1,2,\cdots, \tag{11}$$

$$\left.\frac{\partial^jY_i}{\partial r^j}\right|_{r=0}=-\left.\frac{\partial^jW_i}{\partial r^j}\right|_{r=0},\quad j=0,1,\cdots,m-1, \tag{12}$$

where S_0^m is the m-power of $S_0=a_{nn}h_r^2\dfrac{\partial^2}{\partial\tau^2}$, G_i $(i=1,2,\cdots)$ are known functions successively. From Eqs.(9)(10), we can obtain Y_0. From Eqs.(11)(12), we can obtain solutions Y_i $(i=1,2,\cdots)$ successively.

Observe from the hypotheses, it is not difficult to see that Y_i $(i=1,2,\cdots)$ possesses spike layer shock wave behavior

$$Y_i=O\Big(exp(-\delta_i\frac{r}{\varepsilon})\Big),\quad 0<\varepsilon\ll 1,\quad i=0,1,\cdots, \tag{13}$$

where $\delta_i>0$ $(i=0,1,\cdots)$ are constants.

Let $\overline{Y_i} = \psi(r)Y_i$, where $\psi(r)$ is a sufficiently smooth function in $0 \leqslant r \leqslant r_0$, satisfies

$$\psi(r) = \begin{cases} 1, & 0 \leqslant r \leqslant \dfrac{1}{3}r_0, \\ 0. & r \geqslant \dfrac{2}{3}r_0. \end{cases}$$

we still use Y_i instead of $\overline{Y}_i$ below. Then we can obtain the spike layer shock behavior corrective term Y in the neighborhood of x_0 $(0 \leqslant r \leqslant r_0)$.

§3 Boundary Layer Term

In the neighborhood of the boundary $\partial\Omega$, we construct local coordinate system $(\overline{r}, \overline{\theta})$. Define $\overline{Q}(\overline{r}, \overline{\theta})$ in the neighborhood of $\partial\Omega$, where the coordinate $\overline{r}$ $(r \leqslant \overline{r}_0)$ is the distance between $\overline{Q}$ and the boundary of $\partial\Omega$, $\overline{r}_0$ is a appropriate small positive constant.The $\overline{\theta} = (\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_{n-1})$ is a non-singular coordinate in $(n-1)-$dimensional manifold $(\overline{r} = 0)$. The coordinate $\overline{\theta}$ of $\overline{Q}$ as the coordinate $\overline{\theta}$ of $\overline{P}$, where $\overline{P}$ is an intersection which through $\overline{Q}$ following the internal normal line direction and the boundary of $\overline{r} = 0$. Then

$$L = \overline{a}_{nn}\frac{\partial^2}{\partial \overline{r}^2} + \sum_{i=1}^{n-1} \overline{a}_{ni}\frac{\partial^2}{\partial \overline{r}\partial \overline{\theta}_i} + \sum_{i,j=1}^{n-1} \overline{a}_{ij}\frac{\partial^2}{\partial \theta_i \partial \theta_j} + \overline{b}_n\frac{\partial}{\partial \overline{r}} + \sum_{i=1}^{n-1} \overline{b}_i\frac{\partial}{\partial \theta_i}, \tag{14}$$

where

$$\overline{a}_{nn} = \sum_{j,k=1}^{n} \alpha_{jk}\frac{\partial \overline{r}}{\partial x_j}\frac{\partial \overline{r}}{\partial x_k}, \quad \overline{a}_{ni} = 2\sum_{j,k=1}^{n} \alpha_{jk}\frac{\partial \overline{r}}{\partial x_j}\frac{\partial \overline{\theta}_i}{\partial x_k}, \quad \overline{a}_{ij} = \sum_{k,l=1}^{n} \alpha_{kl}\frac{\partial \overline{\theta}_i}{\partial x_k}\frac{\partial \overline{\theta}_j}{\partial x_l},$$

$$\overline{b}_n = \sum_{j,k=1}^{n} \alpha_{jk}\frac{\partial^2 \overline{r}}{\partial x_j \partial x_k}, \quad \overline{b}_i = \sum_{j,k=1}^{n} \alpha_{jk}\frac{\partial^2 \overline{\theta}_j}{\partial x_j \partial x_k}.$$

By using of multi-scales[1] variables in $0 \leqslant \overline{r} \leqslant \overline{r}_0$. we have $\overline{\tau} = \dfrac{\overline{h}(\overline{r}, \overline{\theta})}{C}$, $\widetilde{r} = \overline{r}$, $\widetilde{\theta} = \overline{\theta}$, where $\overline{h}(\overline{r}, \overline{\theta})$ is a function to be determined. For convenience, we substitute $\overline{r}$, $\overline{\theta}$ for $\widetilde{r}$, $\widetilde{\theta}$ as follows. From (14), we have

$$L = \frac{1}{\varepsilon^2}T_0 + \frac{1}{\varepsilon}T_1 + T_2, \tag{15}$$

while $T_0 = \overline{a}_{nn}\overline{h}^2_{\overline{r}^2}\dfrac{\partial^2}{\partial \overline{\tau}^2}$ and T_1, T_2 are known operators, therefore the constructions are omitted too.

Let w be the solution of the original problems (1)(2)

$$w = W + V, \tag{16}$$

where

$$V = \sum_{i=0}^{\infty} v_i(\overline{r}, \overline{\theta})\varepsilon^i, \tag{17}$$

and V is the boundary layer corrective function. Set

$$\overline{h}(\overline{r}, \overline{\theta}) = \int_0^{\overline{r}} \frac{1}{\sqrt{\overline{a}_{nn}}} d\overline{\rho}.$$

Substituting Eqs.(16)(17) into Eqs.(1)(2), expanding nonlinear terms in ε, combining the coefficients of ε^i $(i = 0, 1, \cdots)$, then we have

$$T_0 v_0 = 0, \quad 0 \leqslant \overline{r} \leqslant \overline{r}_0, \tag{18}$$

$$\left[a_j\frac{\partial^j v_0}{\partial \overline{r}^j}+b_jv_0\right]\Big|_{\overline{r}=0}=g_j-B_jw_0\Big|_{\overline{r}=0},\quad j=1,2,\cdots,m-1, \tag{19}$$

$$T_0v_j=\overline{G}_i,\quad 0\leqslant\overline{r}\leqslant r_0,\quad i=1,2,\cdots, \tag{20}$$

$$\left[a_j\frac{\partial^j v_i}{\partial r^j}+b_jv_i\right]\Big|_{\overline{r}=0}=-w_j\Big|_{\overline{r}=0},\quad i=1,2,\cdots,\quad j=1,2,\cdots,m-1, \tag{21}$$

where $\overline{G}_i$ $(i=1,2,\cdots)$ are known function successively, whose constructions are also omitted too. From the problems Eqs.(18)(19), we can obtain v_0. From v_0 and Eqs.(20)(21), we can obtain solutions v_i $(i=1,2\cdots)$ successively.

From the above hypotheses, it is easy to see that v_i $(i=0,1\cdots)$ possesses boundary layer behavior

$$v_i=O\Big(exp(-\overline{\delta}_i\frac{\overline{r}}{\varepsilon})\Big),\quad 0<\varepsilon\ll 1,\quad i=0,1,\cdots, \tag{22}$$

where $\overline{\delta}_i>0$ $(i=0,1,\cdots)$ are constants.

Let $\overline{v}_i=\overline{\psi}(\overline{r})v_i$, where $\overline{\psi}(\overline{r})$ is a sufficiently smooth function in $0\leqslant\overline{r}\leqslant\overline{r}_0$, and satisfies

$$\overline{\psi}(\overline{r})=\begin{cases}1, & 0\leqslant\overline{r}\leqslant\frac{1}{3}\overline{r}_0,\\ 0, & \overline{r}\geqslant\frac{2}{3}\overline{r}_0.\end{cases}$$

For convenience, we still use v_i instead of $\overline{v}_i$. Then from Eq.(17) we have the boundary corrective term V near the boundary $\partial\Omega$ $(0\leqslant\overline{r}\leqslant\overline{r}_0)$.

§4 Uniform Validity of the Asymptotic Solution

From Eqs.(3)(8)(17), we obtain the formal asymptotic expansion of solution $w(x,\varepsilon)$ for the singular perturbation Robin boundary value problems (1)(2)

$$w(x,\varepsilon)=\sum_{i=0}^{\infty}\Big[W_i(x)+Y_i(x)+v_i(x)\Big]\varepsilon^i,\quad x\in\overline{\Omega},\quad 0<\varepsilon\ll 1$$

Now from the fixed point theorem we can prove the above formal asymptotic solution for ε is uniformly valid in the following theorem:

Theorem *Under the hypotheses* $[H_1],[H_2]$, *then there exists a solution* $w(x,\varepsilon)$ *to the singular perturbation Robin boundary value problem for higher order elliptic equations* (1)(2), *there is an uniformly valid asymptotic expansion for* ε *in* $\overline{\Omega}$.

$$w(x,\varepsilon)=\sum_{i=0}^{M}\Big[W_i(x)+Y_i(x)+v_i(x)\Big]\varepsilon^i+O(\varepsilon^{M+1}),\quad x\in\overline{\Omega},\quad 0<\varepsilon\ll 1. \tag{23}$$

Proof From the above boundary problems (1)(2), we obtain the formal asymptotic solution $w=(x,\varepsilon)$ and Eqs.(13)(22), let $w(x,\varepsilon)=\overline{w}(x,\varepsilon)+R(x,\varepsilon)$

where

$$\overline{w}(x,\varepsilon)\equiv\sum_{i=0}^{M}(W_i+Y_i+v_i)\varepsilon^i. \tag{24}$$

Using Eqs.(13)(24) we obtain $F(R):=\varepsilon^{2m}L^mR-f(x,\overline{w}+R,\varepsilon)+f(x,\overline{w},\varepsilon)=O(\varepsilon^{m+1})$, $x\in\Omega$, and $B_jR=0$, $x\in\partial\Omega$, $j=1,2,\cdots,m-1$.

Let L_1 is the corresponding linearized differential operator, $L_1[p]=\varepsilon^{2m}L^m[p]-f_w(x,\overline{w})p$, and we have $\Psi[p]=F[p]-L_1[p]=f(x,\overline{w},\varepsilon)-f(x,\overline{w}+p,\varepsilon)=f_w(x,\overline{w}+\theta p,\varepsilon)p$, $p\in\Omega$, $0<$

$\theta < 1$, for fixed ε , the linear space N is chosen as

$$N = \left\{ p \Big| p(x) \in C(\Omega), \quad \Big[B_j[p(x)]\Big]\Big|_{x\in\partial\Omega} = g_j(x) \right\}, \quad j = 1, 2, \cdots, m-1,$$

with norm

$$\big\|p(x,\varepsilon)\big\| = \max_{x\in\Omega} \big|p(x)\big|,$$

and the Banach space B is chosen as

$$B = \left\{ q \Big| q(x) \in C(\Omega) \right\}$$

with norm

$$\big\|q(x)\big\| = \max_{x\in\Omega} \big|q(x)\big|.$$

From the hypotheses we may show that $\big\|L_1^{-1}[g]\big\| \leqslant l^{-1}\big\|g\big\|, \quad \forall g \in B$, where l^{-1} is independent of ε, L_1^{-1} is continuous inverse operator. The Lipschitz condition of the fixed point theorem yields

$$\begin{aligned} \Big\|\Psi[p_2] - \Psi[p_1]\Big\| &= \max_{x\in\Omega} \left| \frac{\partial f}{\partial \overline{w}}(x, \overline{w} + \theta_2 p_2, \varepsilon) p_2 - \frac{\partial f}{\partial \overline{w}}(x, \overline{w} + \theta_1 p_1, \varepsilon) p_1 \right| \\ &= \max_{x\in\Omega} \left| \frac{\partial f}{\partial \overline{w}}(x, \overline{w} + \theta_2 p_2, \varepsilon)(p_2 - p_1) + \left\{ \frac{\partial f}{\partial \overline{w}}(x, \overline{w} + \theta_2 p_2, \varepsilon) - \frac{\partial f}{\partial \overline{w}}(x, \overline{w} + \theta_1 p_1, \varepsilon) \right\} p_1 \right| \\ &< Cr \Big\| p_2 - p_1 \Big\|, \end{aligned}$$

where C is a constant independent of ε and this inequality is valid for all $p_1,\ p_2$ in a ball $\Omega_N(r)$ ($\|r\| \leqslant 1$). Finally, from the fixed point theorem[1,2], we obtain the result that the remainder term, moreover

$$\max_{x\in\Omega} \Big| R(x,\varepsilon) \Big| = O(\varepsilon^{m+1}).$$

The proof of the problem Eq.(23) is uniformly completed for ε in $\overline{\Omega}$.

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