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Limit behaviors for dependent Bernoulli variables

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Abstract. In this paper, we consider a class of dependent Bernoulli variables, which has the following form: for $k \geq m$,

$$P(X_{k+1} = 1|\mathcal{F}_k) = \sum_{i=1}^{m} \theta_i X_{k+1-i} + \theta_0 p,$$

where m is a positive integer, $\sum_{i=0}^{m} \theta_i = 1$, $\mathcal{F}_n = \sigma\{X_1, \cdots, X_n\}$, $0 < p < 1$. The convergence rate of the strong law of large numbers and the moderate deviation principle for the model are established. Furthermore, we study some properties of parameter estimation for the model.

§1 Introduction

Drezner and Farnum [4] first proposed a generalized binomial distribution, which allows dependence between trials, nonconstant probabilities of success from trial to trial. The conditional success probability is a linear combination of the empirical mean and the probability of success. The resulting class of distributions includes the binomial, unimodal distributions, and bimodal distributions. Heyde [5] studied its limit theorem by a martingale representation, which was further generalized by James et al. [7], Wu et al. [9] and Miao et al. [8]. Zhang and Zhang [10] generalized the model to a multi-dimensional case, extending some known results.

Based on the Drezner and Farnum's model [4], Zhang and Zhang [11] considered a special model where only the last few trials are involved, because in reality, the last several samples usually have more impact on what will happen. Let $\{X_k, k \geq 1\}$ be a sequence of dependent Bernoulli random variables, which is defined by the following way: the success probability of the trial conditional on all the previous trials is a linear combination of the last few trials and

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the original success probability p. In particular, for $k \geq m$ and some $0 < p < 1$, we assume that

$$P(X_{k+1} = 1|\mathcal{F}_k) = \sum_{i=1}^{m} \theta_i X_{k+1-i} + \theta_0 p \tag{1.1}$$

where X_n denotes the nth trail, $S_n = \sum_{i=1}^{n} X_i$, $\theta_i, i = 0, 1, \cdots, m$, are non-negative parameters satisfying

$$\sum_{i=0}^{m} \theta_i = 1, \quad \mathcal{F}_n = \sigma\{X_1, \cdots, X_n\},$$

and m is a fixed integer. For simplicity, we assume that $X_1, \cdots, X_m$ are i.i.d. Bernoulli random variables with $P(X_1 = 1) = p$. When $\theta_0 = 1$, $\{X_k, k \geq 1\}$ is a sequence of i.i.d. Bernoulli random variables with parameter p. The presence of $\theta_0, \cdots, \theta_m$ allows for overdispersion compared with the traditional Bernoulli sequence and also makes the model more flexible. In the present paper, we consider the case where the last two trials are involved, i.e., for $k \geq 2$,

$$P(X_{k+1} = 1|\mathcal{F}_k) = \theta_1 X_k + \theta_2 X_{k-1} + \theta_0 p. \tag{1.2}$$

In [11], the authors studied the model (1.2) and established some asymptotic results, which included central limit theorems, law of large numbers and law of the iterated logarithm.

The aims of the paper are to establish the convergence rate and the moderate deviation principle for this model. The rest of the paper is organized as follows. The next section is devoted to the descriptions of our main results and their proofs will be given in Section 3. In Section 4, we consider the problems of parameter estimation and the case where θ changes with n.

§2 Main results

The first result is to consider the strong law of large numbers.

Theorem 2.1. *For any $r > 0$, we have*

$$\frac{S_n - np}{\sqrt{n}(\log n)^{(1+r)/2}} \xrightarrow{a.s.} 0.$$

Remark 2.1. *In [11], the authors obtained the following strong law of large numbers: for any $\beta > 1/2$,*

$$\frac{S_n - np}{n^\beta} \xrightarrow{a.s.} 0.$$

So Theorem 2.1 improves the above result.

Let $\mathcal{D}[0, \infty)$ be the function space on $[0, \infty)$ consisting of functions which are right continuous and have left limits and ρ denote the uniform metric on $\mathcal{D}[0, \infty)$ (see Billingsley [2]). In [11], the authors show the following weak convergence:

$$\frac{S_{[nt]} - [nt]p}{\sqrt{n}} \xrightarrow{\mathfrak{D}} \sigma W(t)$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion and

$$\sigma^2 := \theta_0^{-2}\left(1-\theta_1^2-\theta_2^2-\frac{2\theta_1^2\theta_2}{1-\theta_2}\right)p-\theta_0^{-1}\left(\theta_0+2\theta_1+2\theta_2+\frac{2\theta_1\theta_2}{1-\theta_2}\right)p^2. \tag{2.1}$$

In particular, it is easy to get the central limit theorem:

$$\frac{S_n-np}{\sqrt{n}} \xrightarrow{\mathfrak{D}} N(0,\sigma^2).$$

Our second result is to give the asymptotic behavior of the functional associated with the above weak convergence:

$$Z_n(t) := \frac{S_{[nt]}-[nt]p}{b_n}, \quad t\in[0,1], \tag{2.2}$$

in $\mathcal{D}[0,1]$ equipped with the Skorohod topology and with the Borel σ-field $\mathbb{B}$. Here, the sequence $\{b_n, n\geq 1\}$ is increasing such that

$$\frac{b_n}{\sqrt{n}}\to\infty, \quad \frac{b_n}{n}\to 0. \tag{2.3}$$

Theorem 2.2. *$Z_n(\cdot)$ satisfies the moderate deviation principle in $\mathcal{D}[0,1]$ (equipped with the Skorohod topology), with speed b_n^2/n and the good rate function*

$$I(\phi)=\begin{cases}\frac{1}{2\sigma^2}\int_0^1|\phi'(t)|dt, & \text{if } \phi\in\mathcal{AC}_0([0,1])\\ +\infty, & \text{otherwise,}\end{cases} \tag{2.4}$$

where

$$\mathcal{AC}_0([0,1])=\{\phi:[0,1]\to\mathbb{R} \text{ is absolutely continuous with } \phi(0)=0\}.$$

More precisely, for any Borel-measurable subset $A\subset\mathcal{D}[0,1]$, we have that

$$-\inf_{\phi\in A^o} I(\phi)\leq \liminf_{n\to\infty}\frac{n}{b_n^2}\log P(Z_n(\cdot)\in A)$$
$$\leq \limsup_{n\to\infty}\frac{n}{b_n^2}\log P(Z_n(\cdot)\in A)\leq -\inf_{\phi\in\bar{A}} I(\phi)$$

where A^o and $\bar{A}$ denote the interior and the closure of A, respectively.

From Theorem 2.2, we have the following moderate deviation principle for S_n.

Corollary 2.1. *$(S_n-np)/b_n$ satisfies the moderate deviation principle with speed b_n^2/n and the good rate function $I(x)=x^2/2\sigma^2$, namely, for any $r>0$, we have*

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\frac{S_n-np}{b_n}\right|>r\right)=-\frac{r^2}{2\sigma^2}.$$

The next corollary deals with the moderate deviation of a self-normalized result for our model.

Corollary 2.2. *For any $r>0$, we have*

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\frac{\sqrt{n}}{b_n}\left|\frac{S_n-np}{\sqrt{\sum_{k=1}^n(X_i-p)^2}}\right|>r\right)=-\frac{r^2}{2\sigma_1^2},$$

where

$$\sigma_1^2=\frac{\sigma^2}{p(1-p)}.$$

§3 Proofs for main results

Using the decomposition in Zhang and Zhang [11], we define

$$D_1 = X_1 - p, \quad D_2 = X_2 - p, \quad D_n = X_n - \theta_1 X_{n-1} - \theta_2 X_{n-2} - \theta_0 p$$

for $n \geq 3$. Let $\mathcal{F}_0$ be the trivial σ-algebra, then it is easy to see that $\{D_k, \mathcal{F}_k; k \geq 1\}$ is a sequence of bounded martingale differences. If we define $M_n = \sum_{k=1}^n D_k$, we have the relationship between S_n and M_n:

$$\begin{aligned} M_n =& \underbrace{(1-\theta_1-\theta_2)S_n - \theta_0 np} + \underbrace{(\theta_1+\theta_2)X_n + \theta_2 X_{n-1} + \theta_1 X_1 - 2(1-\theta_0)p} \\ =:& \theta_0(S_n - np) + N_n \end{aligned} \tag{3.1}$$

which implies

$$S_n - np = \theta_0^{-1}(M_n - N_n). \tag{3.2}$$

3.1 Proof of Theorem 2.1

Let $c_n = \sqrt{n}(\log n)^{(1+r)/2}$, then by the relation (3.1) and the fact that N_n is a bounded random variable, it is enough to prove that $M_n/c_n \xrightarrow{a.s.} 0$. Since $\{D_k, \mathcal{F}_k, k \geq 1\}$ is a sequence of bounded martingale difference with

$$|D_k| \leq 1, \ a.s. \ \text{ for all } \ k \geq 1,$$

by the Hoeffding's inequality (see [1, 6]) of martingale, for any $\varepsilon > 0$, we have

$$P\left(|M_n| \geq \varepsilon c_n\right) \leq 2\exp\left(-\frac{1}{2n}\varepsilon^2 c_n^2\right) = 2\exp\left(-\frac{1}{2}\varepsilon^2 (\log n)^{1+r}\right)$$

which, by the Borel-Cantelli lemma, yields that

$$M_n/c_n \xrightarrow{a.s.} 0.$$

3.2 Proof of Theorem 2.2

Let $\{D_k, \mathcal{F}_k; k \geq 1\}$ be a sequence of martingale differences and $M_0 = 0, M_n = \sum_{k=1}^n D_k$ for $n \geq 1$. We denote by $\langle M \rangle_n$ the quadratic variation process of the martingale $\{M_n, \mathcal{F}_n; n \geq 1\}$ given by

$$\langle M \rangle_n = \sum_{k=1}^n E(D_k^2 | \mathcal{F}_{k-1}).$$

Now we recall the result of Djellout [3, Theorem 1].

Proposition 3.1. *[3, Theorem 1] Let $\{b_n, n \geq 1\}$ be a sequence satisfying (2.3), such that $c(n) := n/b_n$ is non-decreasing, and define the reciprocal function $c^{-1}(t)$ by*

$$c^{-1}(t) := \inf\{n : c(n) \geq t\}.$$

Assume that the following conditions hold:

(H1) there exists a constant σ^2 such that for any $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\left|\frac{\langle M\rangle_n}{n} - \sigma^2\right| > \varepsilon\right) = -\infty;$$

(H2)

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \left[n \operatorname{ess\,sup}_{1\le k\le c^{-1}(b_{n+1})} P\left(|D_k| > b_n|\mathcal{F}_{k-1}\right)\right] = -\infty;$$

(H3) for any $a > 0$ and $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{1}{n}\sum_{k=1}^{n} E\left(|D_k|^2 1_{\{|D_k|\ge a\frac{n}{b_n}\}}|\mathcal{F}_{k-1}\right) \ge \varepsilon\right) = -\infty,$$

then $Z_n(\cdot)$ satisfies the moderate deviation principle in $\mathcal{D}[0,1]$ (equipped with the Skorohod topology), with speed b_n^2/n and the good rate function

$$I(\phi) = \begin{cases} \frac{1}{2\sigma^2}\int_0^1 |\phi^{'}(t)|dt, & \text{if } \phi \in \mathcal{AC}_0([0,1]); \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.3}$$

Next we shall use Proposition 3.1 to prove Theorem 2.2. By the relation (3.1), we have

$$Z_n(t) = \theta_0^{-1}\left(M_{[nt]} - N_{[nt]}\right)/b_n,$$

and from the fact that $N_{[nt]}$ is a bounded random variable, it is easy to see that $N_{[nt]}/b_n$ can be neglected in the sense of moderate deviation principle. So it is enough to show that $\theta_0^{-1}M_{[nt]}/b_n$ satisfies the moderate deviation principle. Since $M_n = \sum_{k=1}^n D_k$ is a martingale with bounded martingale difference sequence,

$$|D_k| \le 1, \ a.s. \ \text{ for all } \ k \ge 1,$$

then it is easy to check that the conditions $(H2)$ and $(H3)$ in Proposition 3.1 hold.

Noting that for $k \ge 3$

$$\begin{aligned} E(D_k^2|\mathcal{F}_{k-1}) =&(1 - \theta_1 X_{k-1} - \theta_2 X_{k-2} - \theta_0 p)(\theta_1 X_{k-1} + \theta_2 X_{k-2} + \theta_0 p) \\ =&\theta_0 p - \theta_0^2 p^2 + (\theta_1 - \theta_1^2 - 2\theta_0\theta_1 p)X_{k-1} + (\theta_2 - \theta_2^2 - 2\theta_0\theta_2 p)X_{k-2} \\ &- 2\theta_1\theta_2 X_{k-1}X_{k-2}, \end{aligned}$$

then we have

$$\begin{aligned} \frac{\langle M\rangle_n}{n} =&\frac{1}{n}\sum_{k=1}^{n} E(D_k^2|\mathcal{F}_{k-1}) = (\theta_0 p - \theta_0^2 p^2) - \frac{2\theta_1\theta_2}{n}\sum_{k=1}^{n} X_k X_{k+1} \\ &+ \frac{(\theta_1 - \theta_1^2 + \theta_2 - \theta_2^2 - 2\theta_0\theta_1 p - 2\theta_0\theta_2 p)}{n} S_n + \frac{1}{n} Q_n \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} Q_n =&2\theta_1\theta_2 X_n(X_{n-1} + X_{n+1}) + 2\theta_1\theta_2 X_2(X_1 + X_3) - (\theta_1 - \theta_1^2 - 2\theta_0\theta_1 p)(X_1 + X_n) \\ &- (\theta_2 - \theta_2^2 - 2\theta_0\theta_2 p)(X_{n-1} + X_n) + 2p(1-p). \end{aligned}$$

Because Q_n is a bounded random variable, for any $\varepsilon > 0$, we can get

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(|Q_n| > \varepsilon n\right) = -\infty. \tag{3.5}$$

Furthermore, by Hoeffding's inequality (see the proof of Theorem 2.1), for any $\varepsilon > 0$, there exists a positive constant c, such that

$$P\left(\left|\frac{S_n}{n}-p\right|>\varepsilon\right)=P\left(\left|\frac{\theta_0^{-1}(M_n-N_n)}{n}\right|>\varepsilon\right)\leq 2e^{-cn}$$

which implies

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\frac{S_n}{n}-p\right|>\varepsilon\right)=-\infty. \tag{3.6}$$

Now let us define

$$\hat{D}_1=X_1-p,\quad \hat{D}_2=X_1(X_2-p),\quad \hat{D}_n=X_{n-1}(X_n-\theta_1X_{n-1}-\theta_2X_{n-2}-\theta_0p)$$

for $n\geq 3$, then $\{\hat{D}_k,\mathcal{F}_k;k\geq 1\}$ is a sequence of bounded martingale differences and it is not difficult to show

$$\hat{M}_n:=\sum_{k=1}^n\hat{D}_k=(1-\theta_2)\sum_{k=1}^nX_kX_{k+1}-(\theta_1+\theta_0p)S_n+\hat{Q}_n$$

where

$$\hat{Q}_n=\theta_2X_nX_{n-1}-(1-\theta_2)X_nX_{n+1}+(\theta_1+\theta_0p)(X_1+X_n)+(1-p)X_1-p.$$

Based on the above proofs, we know that for any $\varepsilon>0$,

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\frac{\hat{M}_n}{n}\right|>\varepsilon\right)=-\infty$$

and

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\frac{\hat{Q}_n}{n}\right|>\varepsilon\right)=-\infty,$$

which imply that $(1-\theta_2)\sum_{k=1}^nX_kX_{k+1}$ and $(\theta_1+\theta_0p)S_n$ are exponentially equivalent in the sense of moderate deviation principle. And because of the limit (3.6), we get

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\frac{(1-\theta_2)}{n}\sum_{k=1}^nX_kX_{k+1}-(\theta_1+\theta_0p)p\right|>\varepsilon\right)=-\infty. \tag{3.7}$$

So the result of Theorem 2.2 is obtained.

3.3 Proof of Corollary 2.2

From Corollary 2.1, it is enough to prove that for any $r>0$

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\frac{\sum_{k=1}^n(X_k-p)^2}{n}-p(1-p)\right|>r\right)=-\infty. \tag{3.8}$$

Since

$$\sum_{k=1}^n(X_k-p)^2=(1-2p)\sum_{k=1}^nX_k+np^2=(1-2p)S_n+np^2,$$

from the decomposition (3.2), we have

$$(1-2p)S_n+np^2=(1-2p)\theta_0^{-1}(M_n-N_n)+np(1-p).$$

By the Hoeffding inequality (see the proof of Theorem 2.1), for any $r>0$, we have

$$P\left(\left|(1-2p)\theta_0^{-1}(M_n-N_n)\right|\geq nr\right)\leq ce^{-cnr^2},$$

where c is a positive constant, which implies (3.8).

§4 Further discussions

In order to reveal some properties for the model (1.1), we consider the case where only the last trial is involved in the dependent structure, i.e., the model (1.1) is simplified with only one parameter and we can write the model as

$$P(X_{k+1}=1|\mathcal{F}_k)=\theta X_k+(1-\theta)p. \tag{4.1}$$

with $k\geq 1$ and $\theta\in[0,1]$. By letting $\theta_2=0$ in Section 2, we can obtain the following results.

Corollary 4.1. *For model (4.1), we have for any $r>0$,*

$$\frac{S_n-np}{\sqrt{n}(\log n)^{(1+r)/2}}\xrightarrow{a.s.}0$$

and

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\frac{S_n-np}{b_n}\right|>r\right)=-\frac{r^2(1-\theta)}{2p(1-p)(1+\theta)}$$

where the sequence $\{b_n,n\geq 1\}$ satisfies the conditions in (2.3).

4.1 Statistical estimate for the parameters

In this subsection we shall consider some statistical problems for the model (4.1). As in [11], the authors stated that the estimator $\hat{p}:=S_n/p$ is unbiased, strongly consistent and

$$\sqrt{n}(\hat{p}-p)\xrightarrow{d}N\left(0,\frac{p(1-p)(1+\theta)}{1-\theta}\right).$$

Similarly, from Corollary 4.1, for any $r>0$, we have

$$\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}(\hat{p}-p)\xrightarrow{a.s.}0$$

and

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\frac{n}{b_n}\left|\hat{p}-p\right|>r\right)=-\frac{r^2(1-\theta)}{2p(1-p)(1+\theta)}.$$

Naturally there is a problem to estimate θ. Let us define $Y_k=(X_k,X_{k+1})$, $k=1,2,\cdots,n-1$, then Y_k takes values in $\{(0,1),(0,0),(1,1),(1,0)\}$. Let m_1 be the number for $(0,1)$ in Y_k, i.e., $m_1=cards\{k:Y_k=(0,1)\},m_2$ for $(1,1),n_1$ for the total number of $(0,1)$ and $(0,0),n_2$ for the total number of $(1,1)$ and $(1,0)$.

As we can see, the appearance of θ increases (resp., decreases) the success probability for the next trial if the last trial succeeds (resp. fails) and the difference between the two probabilities is exactly θ. So a natural estimator arises as $\hat{\theta}=\frac{m_2}{n_2}-\frac{m_1}{n_1}$. Zhang and Zhang [11] studied the strong consistency of the estimator $\hat{\theta}$, namely, $\hat{\theta}\to\theta$ almost sure.

The following result is to obtain the convergence rate of $\hat{\theta}-\theta$, which improves the above work.

Theorem 4.1. *For the estimator $\hat{\theta}=\frac{m_2}{n_2}-\frac{m_1}{n_1}$, we have*

$$\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}(\hat{\theta}-\theta)\xrightarrow{a.s.}0\quad \textit{for any}\quad r>0.$$

Proof. Note that $n_1 = n-1-S_{n-1}, n_2 = S_{n-1}$, and $m_1 = \sum_{k=1}^{n-1}(1-X_k)X_{k+1}, m_2 = \sum_{k=1}^{n-1} X_k X_{k+1}$. Now we partition $\hat{\theta}-\theta$ into the following four terms:

$$\begin{aligned}
&\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}(\hat{\theta}-\theta)\\
=&\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}\frac{\left(\frac{m_2}{n}-p(\theta+p(1-\theta))\right)}{n_2/n}\\
&+\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}\frac{(\theta+p(1-\theta))\left(p-\frac{n_2}{n}\right)}{n_2/n}\\
&+\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}\frac{\frac{m_1}{n}-p(1-p)(1-\theta)}{n_1/n}\\
&+\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}\frac{p(1-\theta)(1-p-\frac{n_1}{n})}{n_1/n}\\
=:&A_{n1}+A_{n2}+A_{n3}+A_{n4}.
\end{aligned}$$

From Theorem 2.1, we get

$$\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}\left(\frac{n_1}{n}-(1-p)\right)\xrightarrow{a.s.}0,\quad \frac{\sqrt{n}}{(\log n)^{(1+r)/2}}\left(\frac{n_2}{n}-p\right)\xrightarrow{a.s.}0 \tag{4.2}$$

which implies $A_{n2}\xrightarrow{a.s.}0$ and $A_{n4}\xrightarrow{a.s.}0$. For the term A_{n1}, since $n_2/n\xrightarrow{a.s.}p$ and $n_1/n\xrightarrow{a.s.}1-p$, it is enough to show

$$\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}\left(\frac{1}{n}\sum_{k=1}^{n-1}X_kX_{k+1}-p(\theta+p(1-\theta))\right)\xrightarrow{a.s.}0$$

and

$$\frac{\sqrt{n}}{(\log n)^{(1+r)/2}}\left(\frac{1}{n}\sum_{k=1}^{n-1}(1-X_k)X_{k+1}-p(1-p)(1-\theta)\right)\xrightarrow{a.s.}0.$$

By the similar proof of Theorem 2.2, the two limits can be obtained easily. □

4.2 The case when θ changes by n

In this subsection, we consider the case when θ changes by n, i.e.,

$$P(X_{n+1}=1|\mathcal{F}_n)=\theta_nX_n+(1-\theta_n)p \tag{4.3}$$

where $\theta_n\in[0,1]$ and the initial distribution is $P(X_1=1)=p$. Zhang and Zhang [11] obtained the strong law of large numbers for S_n,

$$\frac{S_n}{n}\xrightarrow{a.s.}p.$$

The following result gives the convergence rate of $S_n/n-p$.

Theorem 4.2. *If there exists $\theta\in(0,1)$ such that $|\theta_n-\theta|=O(n^{-1/2})$, then for any $r>0$, we have*

$$\frac{S_n-np}{\sqrt{n}(\log n)^{(1+r)/2}}\xrightarrow{a.s.}0.$$

Proof. First, we define

$$\xi_1 = X_1 - p, \quad \xi_n = X_n - \theta_{n-1}X_{n-1} - (1-\theta_{n-1})p$$

for $n \geq 2$. Then it is easy to see that $\{\xi_k, \mathcal{F}_k; k \geq 1\}$ is a sequence of bounded martingale differences and

$$\sum_{k=1}^{n} \xi_k = (1-\theta)(S_n - np) + \sum_{k=1}^{n}(\theta - \theta_k)(X_k - p) + \theta_n(X_n - p)$$

which implies

$$S_n - np = (1-\theta)^{-1}\left(\sum_{k=1}^{n}\xi_k - \sum_{k=1}^{n}(\theta-\theta_k)(X_k - p) - \theta_n(X_n - p)\right). \tag{4.4}$$

From the similar proof of Theorem 2.1, we have

$$\frac{1}{\sqrt{n}(\log n)^{(1+r)/2}}\sum_{k=1}^{n}\xi_k \xrightarrow{a.s.} 0 \quad \text{and} \quad \frac{\theta_n(X_n - p)}{\sqrt{n}(\log n)^{(1+r)/2}} \xrightarrow{a.s.} 0.$$

Furthermore, since $|\theta_n - \theta| = O(n^{-1/2})$, we have

$$\frac{1}{\sqrt{n}(\log n)^{(1+r)/2}}\sum_{k=1}^{n}(\theta - \theta_k)(X_k - p) \leq \frac{C}{\sqrt{n}(\log n)^{(1+r)/2}}\sum_{k=1}^{n}k^{-1/2} \to 0, \tag{4.5}$$

where C is a positive constant. Thus from the above discussion, the desired result can be obtained. □

Remark 4.1. *In [11], the authors assume that $|\theta_n - \theta| = O(n^{-\beta})$ for some $\beta > 1/2$, then $S_n/n \xrightarrow{a.s.} p$. So Theorem 4.2 improves their works.*

Remark 4.2. *From the inequality in (4.5), we can weaken the condition $|\theta_n - \theta| = O(n^{-1/2})$ to*

$$|\theta_n - \theta| = o\left(\frac{(\log n)^{(1+r)/2}}{n^{1/2}}\right).$$

The following result is the moderate deviation principle for the case when θ changes by n.

Theorem 4.3. *For model (4.3), if there exists $\theta \in (0,1)$ such that $\theta_n \to \theta$ and $\sum_{k=1}^{n}|\theta - \theta_k| = o(b_n)$, then for any $r > 0$,we have*

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\frac{S_n - np}{b_n}\right| > r\right) = -\frac{r^2(1-\theta)}{2p(1-p)(1+\theta)}$$

where the sequence $\{b_n, n \geq 1\}$ satisfies the conditions in (2.3).

Proof. From the condition $\sum_{k=1}^{n}|\theta - \theta_k| = o(b_n)$, we know that there is a positive constant C, such that

$$\frac{\left|\sum_{k=1}^{n}(\theta-\theta_k)(X_k - p) + \theta_n(X_n - p)\right|}{b_n(1-\theta)} \leq \frac{C}{b_n}\sum_{k=1}^{n}|\theta - \theta_k| \to 0, \tag{4.6}$$

thus from the equality (4.4), it is enough to show

$$\frac{n}{b_n^2}\log P\left(\frac{1}{b_n(1-\theta)}\left|\sum_{k=1}^{n}\xi_k\right| > r\right) \to -\frac{r^2(1-\theta)}{2p(1-p)(1+\theta)}.$$

Note that $\{\xi_k, \mathcal{F}_k; k \geq 1\}$ is a sequence of bounded martingale differences, then from the similar proof of Theorem 2.2, the conditions $(H2)$ and $(H3)$ in Proposition 3.1 hold. It is enough to show

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\left|\frac{1}{n}\sum_{k=1}^{n} \frac{E(\xi_k^2|\mathcal{F}_{k-1})}{(1-\theta)^2} - \frac{p(1-p)(1+\theta)}{1-\theta}\right| > r\right) = -\infty. \tag{4.7}$$

It is easy to see that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} E(\xi_k^2|\mathcal{F}_{k-1})\\
=&\frac{p(1-p)}{n} + \frac{1}{n}\sum_{k=2}^{n} (\theta_{k-1}X_{k-1} + (1-\theta_{k-1})p)\left(1-\theta_{k-1}X_{k-1} - (1-\theta_{k-1})p\right) \qquad (4.8)\\
=&\frac{1}{n}\sum_{k=1}^{n-1} \theta_k(1-\theta_k)(1-2p)X_k + \frac{1}{n}\sum_{k=1}^{n-1} p(1-\theta_k)(1-p+p\theta_k) + \frac{p(1-p)}{n}.
\end{aligned}$$

Since $\theta_n \to \theta$, then we have

$$\frac{1}{n}\sum_{k=1}^{n-1} p(1-\theta_k)(1-p+p\theta_k) \to p(1-\theta)(1-p+\theta p)$$

and

$$\frac{p(1-p)}{n} \to 0.$$

Hence the claim (4.7) holds, if we show

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\left|\frac{1}{n}\sum_{k=1}^{n-1} \frac{\theta_k(1-\theta_k)(1-2p)X_k}{(1-\theta)^2} - \frac{\theta p(1-2p)}{1-\theta}\right| > r\right) = -\infty. \tag{4.9}$$

From the condition $\theta_n \to \theta$, we can get

$$\frac{1}{n}\left|\sum_{k=1}^{n-1} \left(\theta_k(1-\theta_k) - \theta(1-\theta)\right) X_k\right| \leq \frac{1}{n}\sum_{k=1}^{n-1} |\theta_k(1-\theta_k) - \theta(1-\theta)| \to 0.$$

So in order to obtain (4.9), it is enough to show

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{\theta(1-2p)}{(1-\theta)}\left|\frac{1}{n}\sum_{k=1}^{n-1} X_k - p\right| > r\right) = -\infty. \tag{4.10}$$

It's worth noting that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n-1} \theta_k(1-\theta_k)(1-2p)X_k\\
=&\frac{1-2p}{n}\sum_{k=1}^{n-1} \left(\theta_k(1-\theta_k) - \theta(1-\theta)\right) X_k + \frac{(1-2p)\theta(1-\theta)}{n}\sum_{k=1}^{n-1} X_k.
\end{aligned}$$

From the equality (4.4), the condition $b_n/n \to 0$ and (4.6), it is enough to show

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\left|\frac{1}{n}\sum_{k=1}^{n-1} \xi_k\right| > r\right) = -\infty.$$

By using the similar proof to get the limit (3.6), the claim (4.10) holds. □

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