Appl. Math. J. Chinese Univ.
2020, 35(4): 392-398

Some questions on partial metric spaces

GE Xun[1] LIN Shou[2,*]

Abstract. We show that the completion of a partial metric space can fail be unique, which answers a question on completions of partial metric spaces. In addition, to this paper discusses metrizability around partial metric spaces.

§1 Introduction

Partial metric spaces were introduced and investigated by S. Matthews in [15] (also see [4]).

Definition 1.1 ([4]). Let X be a non-empty set and $\mathbb{R}^*$ be the set of all nonnegative real numbers. A mapping $p : X \times X \longrightarrow \mathbb{R}^*$ is called a partial metric and (X, p) is called a partial metric space if the following are satisfied for all $x, y, z \in X$:
(1) $x = y \Longleftrightarrow p(x, x) = p(y, y) = p(x, y)$;
(2) $p(x, y) = p(y, x)$;
(3) $p(x, x) \leq p(x, y)$;
(4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

In the past years, partial metric spaces aroused popular attentions and many interesting results were obtained (for example, see [1, 2, 5, 7–12, 16–18, 20]). In [7], it was proved that the completion of every partial metric space is unique under assumption of symmetrical denseness. However, the authors of [7] did not know whenever the completion of every partial metric space is unique without the assumption. So, the following question was posed in [7].

Question 1.2. Are there a partial metric space (X, p) and its completion (X^*, p^*) such that (X, p) is dense in (X^*, p^*), but not symmetrically dense in (X^*, p^*)?

Recently, Dung constructed a complete partial metric space having a non-symmetrically dense and dense subset ([5]), which gives an answer to Question 1.2. However, it can not be

Received: 2017-08-17. Revised: 2020-04-17.
MR Subject Classification: 54E35, 54E50, 54D10.
Keywords: partial metric space, completion, metrizability.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-020-3569-z.
This project is supported by the National Natural Science Foundation of China (11801254, 61472469, 11461005).
*Corresponding Author.

answered that whenever the completion of every partial metric space is unique. In this paper, we construct a partial metric space that has uncountably many completions, which shows that the completion of a partial metric space can fail to be unique and also gives an answer to Question 1.2. In addition, note that a partial metric topological space $(X, \mathscr{T}_p)$ need not to be T_1, and hence it doesn't need to be metrizable. As some relevant investigations, we give a sufficient condition such that $(X, \mathscr{T}_p)$ is metrizable and prove that the sequential coreflection $(X, \sigma(p))$ of (X, p) is metrizable.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}^*$ denote the set of all positive integers and the set of all nonnegative real numbers, respectively.

§2 The Completion of Partial Metric Spaces

Definition 2.1 ([4]). Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in X is called a Cauchy sequence if there is $r \in \mathbb{R}^*$ such that $\lim\limits_{n,m\to\infty} p(x_n, x_m) = r$.

(2) A sequence $\{x_n\}$ in X is called to be convergent in (X, p) if there is $x \in X$ such that $p(x, x) = \lim\limits_{n\to\infty} p(x, x_n) = \lim\limits_{n\to\infty} p(x_n, x_n)$.

(3) (X, p) is called to be complete if every Cauchy sequence in X converges in (X, p).

Remark 2.2 ([4, 7]). Let (X, p) be a partial metric space.

(1) For each $x \in X$ and each $\varepsilon > 0$, put $B(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$. Let $\mathscr{B} = \{B(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. Then $\mathscr{B}$ is a base for some topology on X. The topology is called a partial metric topology induced by the partial metric p and denoted by $\mathscr{T}_p$.

(2) A convergent sequence $\{x_n\}$ in (X, p) is different from that $\{x_n\}$ converges in $(X, \mathscr{T}_p)$. It is also worth noting that if a sequence $\{x_n\}$ converges to x in (X, p), then $\{x_n\}$ converges to x in $(X, \mathscr{T}_p)$, but the other direction only yields $p(x, x) = \lim\limits_{n\to\infty} p(x, x_n)$ and $\limsup\limits_{n\to\infty} p(x_n, x_n) \leq p(x, x)$.

(3) For every $x, y \in X$, put $d(x, y) = 2p(x, y) - p(x, x) - p(y, y)$. Then (X, d) is a metric space induced by (X, p). The metric topology is denoted by $\mathscr{T}_d$ with a base $\mathscr{B}_d = \{B_d(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for every $x \in X$ and every $\varepsilon > 0$.

(4) A convergent sequence in (X, d) is also called to be properly convergent in (X, p) by S. O'Neill in [18].

Lemma 2.3 ([15]). Let (X, p) be a partial metric space and (X, d) be the metric space induced by (X, p). Then the following holds.

(1) Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ converges to x in (X, p) if and only if $\{x_n\}$ converges to x in $(X, \mathscr{T}_d)$.

(2) (X, p) is complete if and only if (X, d) is complete.

Definition 2.4 ([7]). Let (X, p) be a partial metric space and Y be a subset of X.

(1) Y is called to be dense in (X,p) if for any $x \in X$ and any $\varepsilon > 0$, there is $y \in Y$ such that $y \in B(x,\varepsilon)$, i.e., Y is dense in $(X, \mathscr{T}_p)$.

(2) Y is called to be symmetrically dense in (X,p) if for any $x \in X$ and any $\varepsilon > 0$, there is $y \in Y$ such that $y \in B(x,\varepsilon)$ and $x \in B(y,\varepsilon)$.

Remark 2.5 ([7])**.** It is clear that symmetrical denseness and denseness are equivalent in metric spaces, and symmetrical denseness implies denseness in partial metric spaces.

Definition 2.6 ([7])**.** Let (X,p) and (Y,q) be partial metric spaces. A mapping $f : X \longrightarrow Y$ is called to be an isometry if $q(f(x), f(x')) = p(x,x')$ for all $x, x' \in X$.

Definition 2.7. Let (X,p) be a partial metric space. A complete partial metric space (X^*, p^*) is called a completion of (X,p) if there is an isometry $f : X \longrightarrow X^*$ such that $f(X)$ is dense in $(X^*, \mathscr{T}_{p^*})$.

Now we give the main result of this section, which answers Question 1.2.

Example 2.8. Let $X = \{1/n : n \in \mathbb{N}\}$, $X^* = X \cup \{0\}$ and $\widetilde{X} = X^* \cup X_0$, where X_0 is a closed subset of the interval $[1, +\infty)$. Define $\widetilde{p} : \widetilde{X} \times \widetilde{X} \longrightarrow \mathbb{R}$ as follows. For every $x, y \in \widetilde{X}$, put $\widetilde{p}(x,y) = \max\{x, y\}$. That is,

$$\widetilde{p}(x,y) = (|x-y| + x + y)/2.$$

Let p and p^* be the restrictions of $\widetilde{p}$ on $X \times X$ and $X^* \times X^*$, respectively. Then (X,p), (X,p^*) and $(\widetilde{X}, \widetilde{p})$ are partial metric spaces, and the following hold:

(1) (X^*, p^*) is the completion of (X,p), where X is a symmetrical dense subset of (X^*, p^*);

(2) $(\widetilde{X}, \widetilde{p})$ is a completion of (X,p), where X is a dense subset of $(\widetilde{X}, \widetilde{p})$.

Proof. It is not difficult to check that (X,p), (X,p^*) and $(\widetilde{X}, \widetilde{p})$ are partial metric spaces. Define $\widetilde{d} : \widetilde{X} \times \widetilde{X} \longrightarrow \mathbb{R}$ as follows. For every $x, y \in \widetilde{X}$, put

$$\widetilde{d}(x,y) = 2\widetilde{p}(x,y) - \widetilde{p}(x,x) - \widetilde{p}(y,y).$$

Then $\widetilde{d}(x,y) = |x-y|$. Let d and d^* be the restrictions of $\widetilde{d}$ on $X \times X$ and $X^* \times X^*$, respectively. Then (X,d), (X^*,d^*) and $(\widetilde{X}, \widetilde{d})$ are metric spaces induced by (X,p), (X^*,p^*) and $(\widetilde{X}, \widetilde{p})$, respectively. Note that (X^*,d^*) and $(\widetilde{X}, \widetilde{d})$ are closed subspaces of the ordinary real space. So (X^*,d^*) and $(\widetilde{X}, \widetilde{d})$ are complete metric spaces. By Lemma 2.3(2), (X^*,p^*) and $(\widetilde{X}, \widetilde{p})$ are complete. In order to complete our proof, it suffices to prove the following two claims.

Claim 1. X is a symmetrically dense subset of (X^*, p^*).

We only need to prove that for any $\varepsilon > 0$, there is $x \in X$ such that $x \in B(0,\varepsilon)$ and $0 \in B(x,\varepsilon)$. In fact, if $\varepsilon > 0$, then there is $x \in X$ such that $x < \varepsilon$. It is clear that $p^*(0,x) = x$, $p^*(0,0) = 0$ and $p^*(x,x) = x$. It follows that $p^*(0,x) = x < p^*(0,0) + \varepsilon$ and $p^*(x,0) = x < p^*(x,x) + \varepsilon$. So $x \in B(0,\varepsilon)$ and $0 \in B(x,\varepsilon)$.

Claim 2. X is a dense subset of $(\widetilde{X}, \widetilde{p})$.

We only need to prove that for any $x_0 \in X_0$ and any $\varepsilon > 0$, there is $x \in X$ such that $x \in B(x_0, \varepsilon)$. In fact, let $x_0 \in X_0$ and $\varepsilon > 0$. Pick $x \in X$, then $x_0 \geq x$. It follows that $\widetilde{p}(x_0, x) = x_0 = \widetilde{p}(x_0, x_0) < \widetilde{p}(x_0, x_0) + \varepsilon$. So $x \in B(x_0, \varepsilon)$. □

Remark 2.9. Dung constructed a complete partial metric space having a dense and non-symmetrically dense subset ([5, Example 12]), which gives an answer to Questions 1.2. However, it can not be answered that whenever the completion of every partial metric space is unique. Example 2.8 answers this question, which shows that the completion of a partial metric space can fail be unique and also gives an answer to Questions 1.2. In fact, if (X, p) is the partial metric space described in Example 2.8, then there are uncountably many completions of (X, p).

§3 The Metrizability around Partial Metric Topological Spaces

Let (X, p) be a partial metric space. We give a base for the metric topology $\mathscr{T}_d$ induced by p. As an application of this base, we give a sufficient condition such that $(X, \mathscr{T}_p)$ is metrizable.

Proposition 3.1. Let (X, d) be the metric space induced by a partial metric space (X, p). For each $x \in X$ and each $\varepsilon > 0$, put $B'(x, \varepsilon) = \{y \in X : p(x, y) < \min\{p(x, x), p(y, y)\} + \varepsilon\}$. Let $\mathscr{B}' = \{B'(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. Then $\mathscr{B}'$ is a base for the metric topology $\mathscr{T}_d$.

Proof. Let $x \in X$ and $\varepsilon > 0$. It suffices to prove that $B_d(x, \varepsilon) \subseteq B'(x, \varepsilon) \subseteq B_d(x, 2\varepsilon)$. Let $y \in B_d(x, \varepsilon)$. Then $(p(x, y) - p(x, x)) + (p(x, y) - p(y, y)) = d(x, y) < \varepsilon$. It follows that $p(x, y) - p(x, x) < \varepsilon$ and $p(x, y) - p(y, y) < \varepsilon$, i.e., $p(x, y) < p(x, x) + \varepsilon$ and $p(x, y) < p(y, y) + \varepsilon$. So $p(x, y) < \min\{p(x, x), p(y, y)\} + \varepsilon$. Hence, $y \in B'(x, \varepsilon)$. This proves that $B_d(x, \varepsilon) \subseteq B'(x, \varepsilon)$. On the other hand, let $y \in B'(x, \varepsilon)$. Then $p(x, y) < \min\{p(x, x), p(y, y)\} + \varepsilon$. So $p(x, y) < p(x, x) + \varepsilon$ and $p(x, y) < p(y, y) + \varepsilon$, i.e., $p(x, y) - p(x, x) < \varepsilon$ and $p(x, y) - p(y, y) < \varepsilon$. It follows that $d(x, y) = (p(x, y) - p(x, x)) + (p(x, y) - p(y, y)) < 2\varepsilon$. Hence, $y \in B_d(x, 2\varepsilon)$. This proves that $B'(x, \varepsilon) \subseteq B_d(x, 2\varepsilon)$. Consequently, $B_d(x, \varepsilon) \subseteq B'(x, \varepsilon) \subseteq B_d(x, 2\varepsilon)$. □

For a partial metric space (X, p), if $p(x, x) = p(y, y)$ for all $x, y \in X$, then $B(x, \varepsilon) = B'(x, \varepsilon)$ for all $x \in X$ and all $\varepsilon > 0$, and so $\mathscr{B} = \mathscr{B}'$. It follows that $\mathscr{T}_p = \mathscr{T}_d$ from Proposition 3.1. Thus, we have the following corollary as a straight result of Proposition 3.1.

Corollary 3.2. Let (X, p) be a partial metric space. If $p(x, x) = p(y, y)$ for all $x, y \in X$, then $(X, \mathscr{T}_p)$ is metrizable.

Remark 3.3. Corollary 3.2 can not be reverted. In fact, put $X_1 = \{a, b\}$ and define a function $p_1 : X_1 \times X_1 \longrightarrow \mathbb{R}$, where $p_1(a, a) = p_1(b, b) = 1$ and $p_1(a, b) = 2$; put $X_2 = \{c, d\}$ and define a function $p_2 : X_2 \times X_2 \longrightarrow \mathbb{R}$, where $p_2(c, c) = p_2(d, d) = 3$ and $p_1(c, d) = 4$. Then (X_1, p_1) and (X_2, p_2) are partial metric spaces. By Corollary 3.2, $(X_1, \mathscr{T}_{p_1})$ and $(X_2, \mathscr{T}_{p_2})$ are metrizable. It follows that the topological sum $(X_1, \mathscr{T}_{p_1}) \oplus (X_2, \mathscr{T}_{p_2})$ of $(X_1, \mathscr{T}_{p_1})$ and $(X_2, \mathscr{T}_{p_2})$ is metrizable.

Question 3.4. For a partial metric space (X, p), can one give a sufficient and necessary condition such that $(X, \mathscr{T}_p)$ is metrizable?

Now we discuss the sequential coreflection of (X, p). The sequential coreflection of a topological space was introduced and investigated by S. Leader and S. Baron in 1966 [13], where a sequential coreflection was called a sequential closure topology. In the past years, sequential coreflections had been investigated further by S. P. Franklin in [6], S. Lin in [14], T. Banakh, V. Bogachev and A. Kolesnikov in [3], L. Peng and Z. Guo in [19], and so on. Replacing topological spaces by partial metric spaces, we give the concept of sequential coreflections of partial metric spaces (refer to [14]).

Definition 3.5. Let (X, p) be a partial metric space.

(1) Let $x \in U \subseteq X$. U is called a sequential barrier at x in (X, p) if, whenever $\{x_n\}$ is a sequence converging to x in (X, p), then $\{x_n\}$ is eventually in U, i.e., $\{x_n : n \geq k\} \subseteq U$ for some $k \in \mathbb{N}$.

(2) Let $U \subseteq X$. U is called to be sequentially open in (X, p) if U is a sequential barrier at each of its points in (X, p).

(3) Put $\sigma(p) = \{U : U \text{ is sequentially open in } (X, p)\}$. Then $\sigma(p)$ is a topology on X, and $(X, \sigma(p))$ is called the sequential coreflection of (X, p).

Theorem 3.6. *Let (X, d) be the metric space induced by a partial metric space (X, p). Then $\sigma(p) = \mathscr{T}_d$, and so $(X, \sigma(p))$ is metrizable.*

Proof. We only need to prove the following two claims.

Claim 1: $\sigma(p) \subseteq \mathscr{T}_d$.

Let $x \in U \in \sigma(p)$. It suffices to prove that there is $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq U$. In fact, if not, then $B_d(x, 1/n) \nsubseteq U$ for every $n \in \mathbb{N}$. Pick $x_n \in B_d(x, 1/n) \setminus U$ for every $n \in \mathbb{N}$. Then $\{x_n\}$ is a sequence converging to x in (X, d). By Lemma 2.3(1), $\{x_n\}$ converges to x in (X, p). However, $x_n \notin U$ for every $n \in \mathbb{N}$. So U is not a sequential barrier at x in (X, p). Hence, U is not sequentially open in (X, p), i,e,, $U \notin \sigma(p)$. This is a contradiction.

Claim 2: $\mathscr{T}_d \subseteq \sigma(p)$.

Let $x \in X$ and $\varepsilon > 0$. It suffices to prove that $B_d(x, \varepsilon)$ is a sequential barrier at x in (X, p). In fact, if $\{x_n\}$ is a sequence converging to x in (X, p), then $\{x_n\}$ converges to x in (X, d) from Lemma 2.3(1). Since $B_d(x, \varepsilon) \in \mathscr{T}_d$, $\{x_n\}$ is eventually in $B'(x, \varepsilon)$. So $B'(x, \varepsilon)$ is a sequential barrier at x in (X, p). $\square$

References

[1] T Abdeljawad, E Karapinar, K Tas. *A generalized contraction principle with control functions on partial metric spaces*, Computers and Mathematics with Applications, 2012, 63(3): 716-719.

[2] H Aydi, M Abbas, C Vetro. *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology and its Applications, 2012, 159(4): 3234-3242.

[3] T Banakh, V Bogachev, A Kolesnikov. *k^*-metrizable spaces and their applications*, Journal of Mathematical Sciences, 2008, 155(4): 475-522.

[4] M Bukatin, R Kopperman, S Matthews, H Pajoohesh. *Partial metric spaces*, American Mathematical Monthly, 2009, 116(8): 708-718.

[5] N V Dung. *On the completion of partial metric spaces*, Quaestiones Mathematicae, 2017, 40(5): 589-597.

[6] S P Franklin. *Spaces in which sequences suffice, II*, Fundamenta Mathematicae, 1967, 61: 51-56.

[7] X Ge, S Lin. *Completions of partial metric spaces*, Topology and its Applications, 2015, 182: 16-23.

[8] S Han, J Wu, D Zhang. *Properties and principles on partial metric spaces*, Topology and its Applications, 2017, 230: 77-98.

[9] D Ilic, V Pavlovic, V Rakocevic. *Extensions of the Zamfirescu theorem to partial metric spaces*, Mathematical and Computer Modelling, 2012, 55(3-4): 801-809.

[10] E Karapinar, I Erhan. *Fixed point theorems for operators on partial metric spaces*, Applied Mathematics Letters, 2011, 24(11): 1894-1899.

[11] R Kopperman, S Matthews, H Pajoohesh. *Completions of partial metrics into value lattices*, Topology and its Applications, 2009, 156(8): 1534-1544.

[12] H P Künzi, H Pajoohesh, M Schellekens. *Partial quasi-metrics*, Theoretical Computer Science, 2006, 365(3): 237-246.

[13] S Leader, S Baron. *Sequential topologies*, American Mathematical Monthly, 1966, 73: 677-678.

[14] S Lin. *A note on the Arens' space and sequential fan*, Topology and its Applications, 1997, 81(3): 185-196.

[15] S Matthews. *Partial metric topology*, in General Topology and Applications: Eighth Summer Conference at Queens College 728(1992), G Itzkowitz et al, eds, Annals of the New York Academy of Sciences, New York, 1994: 183-197.

[16] S Moshokoa. *On the 0-Cauchy completion of a partial metric space*, Turkish Journal of Mathematics and Computer Science, 2016, 4: 10-15

[17] S Oltra, S Romaguera, E Sánchez-Pérez. *Bicompleting weightable quasi-metric spaces and partial metric spaces*, Rendiconti del Circolo Matematico di Palermo, 2002, 51: 151-162.

[18] S O'Neill. *Two Topologies are Better Than One*, University of Warwick, Coventry, UK, 1995.

[19] L Peng, Z Guo. *A note on joint metrizability of spaces on families subspaces*, Topology and its Applications, 2015, 188: 1-15.

[20] S Shukla. *Partial b-metric spaces and fixed point theorems*, Mediterranean Journal of Mathematics, 2014, 11: 703-711.

[1]School of Mathematical Sciences, Soochow University, Suzhou 215006, China.
Email: gexun@suda.edu.cn

[2]Department of Mathematics, Ningde Normal University, Fujian 352100, China.
Email: shoulin60@163.com