Appl. Math. J. Chinese Univ.
2020, 35(4): 379-391

The valuation of multi-counterparties CDS with credit rating migration

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Abstract. In this paper, the pricing of a Credit Default Swap (CDS) contract with multiple counterparties is considered. The pricing model takes into account the credit rating migration risk of the reference. It is a new model established under the reduced form framework, where the intensity rates are assumed to have structural styles. We derive from it a non-linear partial differential equation system where both positive and negative correlations of counterparties and the references are considered via a single factor model. Then, an ADI (Alternating Direction Implicit) difference method is used to solve the partial differential equations by iteration. From the numerical results, the comparison of multi-counterparty CDS contract and the standard one are analyzed respectively. Moreover, the impact of default parameters on value of the contracts are discussed.

§1 Introduction

In the past two decades, a variety of credit derivatives has become increasingly prominent with the rapid development of the credit derivatives market. CDS and other credit derivatives provide credit risk distraction, hedging and innovation for investment strategy optimization. In the same time, especially after the financial crisis, credit risks are getting more and more attentions. In this situation, Basel Committee on Banking Supervision encouraged investors taking a counterparty risk and credit migration into account on the Basel III framework. Now, based on the principle of risk diversification, investors might be interested in credit risk measurement and control: if the counterparty risk is considered, does using more counterparties reduce the risk? What are the effects of the correlation of the reference and the counterparties? Besides the counterparty risk, credit rating migrations cannot also be neglected in the credit risk management. A variety of general bond credit rating structures are given by the rating

Received: 2016-10-14. Revised: 2019-10-23.
MR Subject Classification: 91G40, 35K55.
Keywords: CDS, credit rating migration risk, multi-counterparties, reduced form, structure style.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-020-3503-4.
Supported by the National Natural Science Foundation of China(11671301,12071349).

agencies, such as Standard Poor's, Moody's, etc. These ratings affect directly the pricing and the possible migrations should be taken into consideration.

Jarrow and Yildirim [14] proposed to consider the credit risk associated with conditions based on reduced form, and assumed default intensity to be linear on the risk-free rate, which lead to an explicit CDS pricing formula. Jarrow and Yu [15] dealt with the default correlation with a common factor. Madan and Unal [6] took the risk of counterparty default into account for CDS pricing in the infection model. Brigo et al. [4, 5] used multi-factors model to price CDS with counterparty risks. Bo and Capponi [13] gave an explicit formula for the bilateral counterparty valuation adjustment of a credit default swap portfolio referencing an asymptotically large number of entities. There are also many more research works relative pricing CDS with counterparty risk, see e.x. [1, 2, 8, 19]. In terms of credit rating, Lando [7] assumed that for a risky bond there exists a credit rating transition matrix. Furthermore, Alavian, et al. [16] estimated the credit rating transition matrix with the Bayesian method. Seng and Kwok [18] used public information to estimate the bond's credit rating. Crepey [17], etc. introduced an implicit factor reflecting the macroeconomic cycle to model and forecast the credit transfer matrix. Liang & Li [11] first calculated and analyzed CVA of the multi-counterparties CDS in simple intensities without credit migration risks. [3, 10, 20] used virtual trading at migration time to deal with credit rating migration for corporate bonds. We refer Capponi [1] for a survey on counterparty risk valuation and mitigation up to 2012. For measuring credit rating migration risks by structure model, we refer [3], [9].

In this paper, based on the Liang & Li [11], we consider the pricing of a CDS that involves multi-counterparties defaults and takes into account the credit rating migration risks of the reference. The model is established under the reduced form framework, where the intensities are considered to be stochastic processes in a structural type. It is different from general reduced form models, which are not relative to the firm's assets. We use a new model to join the advantage of the structure model with the reduced form model, so that the changes of the firm's asset are considered. In this way, the model is more practice. The models can be expressed by a nonlinear system of partial differential equations. The key innovation is establishing a model for pricing multi-counterparties under credit rating migration. A numerical scheme is obtained and applied for calculation and analysis.

The rest of paper is organized as follows. Section 2 introduces a multi - counterparties CDS contract, where credit rating migration is taken into account for the reference. The discounted cash flow is analyzed when the reference company changes its credit rating status. For these cases, a model is established for pricing the CDS. In Section 3, the model is developed into a coupled nonlinear partial differential equation system. The positive and negative correlations among the reference and the counterparties are modelled by a single common factor model, which is described by a geometric Brownian motion and an its inverse respectively. Section 4 presents the numerical tests, and the solution is solved by iterating through numerical ADI algorithm. The numerical results are also analyzed from a financial perspective. Section 5 is the conclusion of the paper.

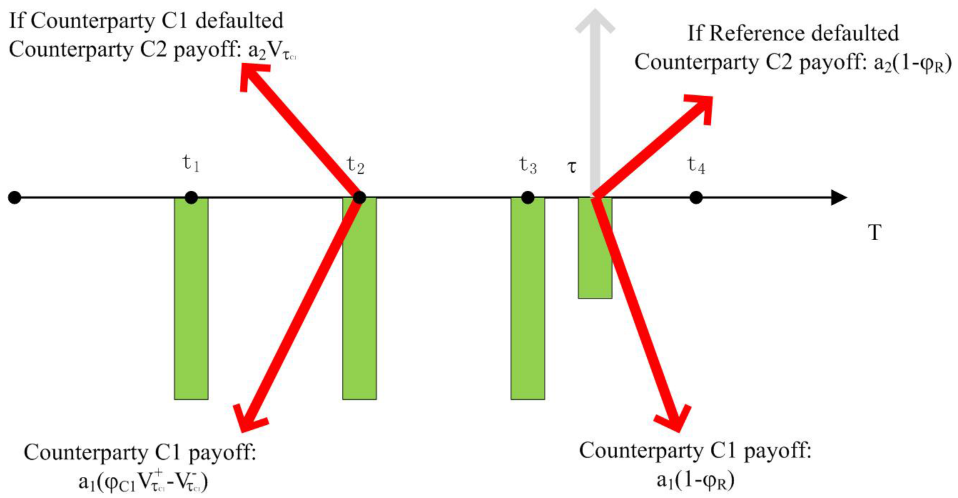

Figure 1: Multi-counterparties CDS structure.

§2 Modelling

2.1 Contract

Consider a multi-counterparties CDS contract, where the possibility of credit rating migration of the reference company is taken into account. Let us denote by T the maturity and k the premium rate which is paid continuously. The face value of the reference bond is assumed to be 1. We refer to the counterparties and the reference respectively by $(i = C1, C2, R)$. The default recovery rates are $\varphi_i, 0 < \varphi_i \leq 1,\ i = C1, C2, R$, which are assumed to be constants.

If a counterparty defaults before the reference company, it clears the corresponding proportion of the contract, and the other part of the contract will also be cleared immediately. If the reference company default event occur before the counterparties, then the counterparties would compensate respectively $a_1(1-\varphi_R)$ and $a_2(1-\varphi_R),\ (a_1 + a_2 = 1)$ for the investor.

2.2 Assumptions

Let $(\Omega, G_t, \{G_t\}_{t\geq 0}, Q)$ be a filtered probability space. The filtration $G = \{G_t\}_{t\geq 0}$ is the information flow of the market, including credit events, Q is the risk neutral measure. This space is endowed as right-continuous and complete, the sub-filtration represents all the observable market quantities but the default event, $\{F_t\}_{t\geq 0}$ is the right-continuous filtration generated by the credit events, either of the reference or counterparties.

For the reference company with credit rating migration, the following assumption are made:

1. There exist credit rating migration risks for the reference, the credit states are represented by a set of three: $\kappa_R = \{0, 1, 2\}$, where $1, 2$ indicate the low and the high credit rating states respectively, and 0 indicates the default state which is an absorbing one.

2. The counterparties have only two states: default or survival, which are expressed by $\kappa_{C1} = \kappa_{C2} = \{0, 1\}$ respectively.

3. The credit events of the counterparties and the reference do not occur simultaneously. Also, for the simplicity, the reference company does not directly default when it is in the highest rating.

4. At time t, the credit state of the contract is denoted by M_t. It contains three variables, the credit state, the state of $C1$, the state of $C2$. According to the assumptions above, for any $t \in [0, T)$, M_t has the following possible value:
$$\kappa \triangleq \{M_t\} = \left\{ \begin{array}{cccccc} (2,1,1), & (2,1,0), & (1,1,1), & (1,1,0), & (0,1,1), & (0,1,0), \\ (2,0,1), & (2,0,0), & (1,0,1), & (1,0,0), & (0,0,1), & (0,0,0) \end{array} \right\}.$$

5. M_0 is the credit state of the contract at the initial time. $C1$, $C2$ are survive at $t = 0$, which means $\kappa_{C1} = 1$, $\kappa_{C2} = 1$. τ_i^j, $i = C1, C2, R$, $j = 0, 1, 2$, are random variables defined in the probability space $(\Omega, G_t, \{G_t\}_{t \geq 0}, Q)$, that correspond to the first passage time of the reference company default, credit rating raising and credit rating declining respectively, i. e.
$$\begin{aligned} \tau_R^0 &= \inf\{t > 0 | M_0 = (1,1,1), M_t = (0,1,1)\}, \\ \tau_R^1 &= \inf\{t > 0 | M_0 = (1,1,1), M_t = (2,1,1)\}, \\ \tau_R^2 &= \inf\{t > 0 | M_0 = (2,1,1), M_t = (1,1,1)\}, \\ \tau_{C1}^j &= \inf\{t > 0 | M_0 = (j,1,1), M_t = (j,0,1)\},\ j = 1, 2, \\ \tau_{C2}^j &= \inf\{t > 0 | M_0 = (j,1,1), M_t = (j,1,0)\},\ j = 1, 2. \end{aligned}$$

Remark The default times of counterparties are independent from the rating states of the reference. However, for the convenience of the analysis, we use different notations depending on their rating states.

2.3 Credit transfer intensities and their correlations

The traditional reduced form framework are mostly used for modelling default. The idea of the intensity can be extended to the credit rating migration one, i.e. we introduce transfer intensity. In this way, the reduced form framework can also be used for modelling credit rating migration. However, the traditional intensity is assumed to be either constant or stochastic process, the credit rating usually depends on the assets values, which is more suitable for the structural framework. In our model, using the structural framework idea (see also [2]), we assume that the reference's credit rating transfer intensities depend not only on the outside economic factor but also on its asset value.

Let λ_R^j , $j = 0, 1, 2$, be respectively, the reference company's intensities of the default, the downgrade and upgrade of the credit rating . The idea is that the asset value X of a company is divided into three parts: $\Omega_0 = \{0 < X < b_R\}$, $\Omega_1 = \{b_R < X < B_R\}$ and $\Omega_2 = \{X > B_R\}$.
$$\lambda_R^j = \left\{ \begin{array}{ll} P_i, & \text{in } \Omega_i \\ 0, & \text{otherwise,} \end{array} \right. \quad i = 0, 1, 2.$$

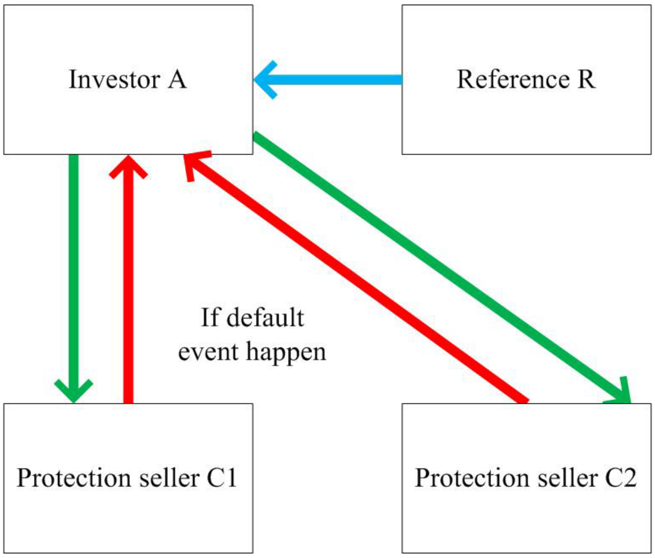

Figure 2: Cash flow of multi-counterparty CDS product.

That is,

$$\lambda_R^0 = P_0 H(b_R - X_{Rt}), \quad \lambda_R^1 = P_1 H(X_{Rt} - B_R), \quad \lambda_R^2 = P_2 H(X_{Rt} - b_R) H(B_R - X_{Rt}). \tag{1}$$

Similarly, the default intensities of the counterparties are $\lambda_i,\ i = C1, C2$:

$$\lambda_{C1} = P_3 H(b_{C1} - X_{C1t}), \quad \lambda_{C2} = P_4 H(b_{C2} - X_{C2t}), \tag{2}$$

where the function $H(\cdot)$ is the Heaviside function, i.e. $H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{otherwise.} \end{cases}$, B_R and $b_i,\ i = R, C1, C2$, are the boundaries where the intensities change states; $P_j,\ j = 0, 1, 2, 3, 4$, are non-negative constants that are determined by an outside systematic risk factor.

In the above, X_{it} is the asset values of company i, $i = R, C1, C2$. We assume that they are correlated and following stochastic processes. A single-factor model is set to describe them. Let y_t represent a common systemic factor, like the GDP process, β_{it} represent the specific factor of the company i, $i = R, C1, C2$:

$$\frac{dy_t}{y_t} = \mu_0 dt + \sigma_0 dW_{0t}, \qquad \frac{d\beta_{it}}{\beta_{it}} = \mu_i dt + \sigma_i dW_{it}, \tag{3}$$

where μ_0, μ_i are the expect returns; σ_0, σ_i are the volatilities, all of them are positive constants. W_{0t}, W_{it} are standard brownian motions, which are independent of each other. $i = R, C1, C2$.

It is assumed that the asset values are characterized in the following in terms of the common factor, see also Liang and Wang [12]:

Case 1 the asset of company i is positively correlated with the common factor:

$$dX_{it} = \rho_i dy_t + l_i d\beta_{it}, \quad i = R, C1, C2; \tag{4}$$

Case 2 the asset of company i is negatively correlated with the common factor, considering the asset should be positive:

$$dX_{it} = \rho_i dz_t + l_i d\beta_{it}, \quad i = R, C1, C2, \tag{5}$$

where $z_t = 1/y_t$, satisfying

$$\frac{dz_t}{z_t} = ({\sigma_0}^2 - \mu_0)dt - \sigma_0 dW_{0t},$$

where ρ_i, l_i are positive constants. We call ρ_i/ρ_j the correlation coefficient of company i to j, $i = R, C1, C2$.

2.4 Cash flow analysis for the CDS

While pricing a financial contract, it is necessary to understand the cash flows of the contract. However, involving credit rating migration, traditional cash flow analysis tends to be more difficult. When the credit rating transfers, there is no cash flow of transaction. It is different from the default, where the liquidation happens. To deal with this problem, we introduce an approach called "virtual substitute termination" [3]. Therefore, in the model, each credit rating is mapped to a corresponding virtual contract. We assume that the values of the contract in the high and the low credit ratings are U_t and V_t respectively. If one counterparty, say $C1$, defaults, there is two possibilities. If the value of the CDS contract is positive at the default time, the investor obtains the corresponding proportion of the recovery. But, if it is negative, he pays the full corresponding proportion to the counterparty. At the same time, for counterparty $C2$, under the terms of the contract, the contract is stopped and the investor will receive (the value of the CDS is positive to the investor) or pay (the value of the CDS is negative to the investor) the full corresponding proportion to the counterparty $C2$.

Thus the investor's cash flow at time t is:

$$1_{\tau^2_{C1}<\tau^2_R\wedge\tau^2_{C2}} 1_{t<\tau^2_{C1}<T}\left[a_1(\varphi_{C1}V^+_{\tau^2_{C1}} - V^-_{\tau^2_{C1}}) + a_2 V_{\tau^2_{C1}}\right] e^{-r(\tau^2_{C1}-t)},$$

where r is the risk free interest rate, $V^+ = \max\{V, 0\}$ and $V^- = \max\{-V, 0\}$. The other cases of cash flow can be analyzed similarly. Putting all cases together, under the risk-neutral measure Q, the values of the contract respect to the investor can be described as

1. At time t, if the value of the CDS contract is in the high credit rating:

$$\begin{aligned} V_t \;=\; & E^Q\big[-\int_t^T 1_{\tau^2_R>u}1_{\tau^2_{C1}>u}1_{\tau^2_{C2}>u} k e^{-r(u-t)}du \\ & +1_{\tau^2_{C1}<\tau^2_R\wedge\tau^2_{C2}\wedge T}[a_1(\varphi_{C1}V^+_{\tau^2_{C1}} - V^-_{\tau^2_{C1}}) + a_2V_{\tau^2_{C1}}]e^{-r(\tau^2_{C1}-t)} \\ & +1_{\tau^2_{C2}<\tau^2_R\wedge\tau^2_{C1}\wedge T}[a_1V_{\tau^2_{C2}} + a_2(\varphi_{C2}V^+_{\tau^2_{C2}} - V^-_{\tau^2_{C2}})]e^{-r(\tau^2_{C2}-t)} \\ & +1_{\tau^2_R<\tau^2_{C1}\wedge\tau^2_{C2}\wedge T}U_{\tau^2_R}e^{-r(\tau^2_R-t)}|G_t\big]. \end{aligned}$$

2. At time t, if the value of the CDS contract is in the low credit rating:

$$U_t \;=\; E^Q\big[-\int_t^T 1_{\tau^0_R>u}1_{\tau^1_R>u}1_{\tau^1_{C1}>u}1_{\tau^1_{C2}>u} k e^{-r(u-t)}du$$

$$+1_{\tau_{C1}^1<\tau_R^0\wedge\tau_R^1\wedge\tau_{C2}^1\wedge T}[a_1(\varphi_{C1}U^+_{\tau_{C1}^1}-U^-_{\tau_{C1}^1})+a_2U_{\tau_{C1}^1}]e^{-r(\tau_{C1}^1-t)}$$
$$+1_{\tau_{C2}^1<\tau_R^0\wedge\tau_R^1\wedge\tau_{C1}^1\wedge T}[a_1U_{\tau_{C2}^1}+a_2(\varphi_{C2}U^+_{\tau_{C2}^1}-U^-_{\tau_{C2}^1})]e^{-r(\tau_{C2}^1-t)}$$
$$+1_{\tau_R^1<\tau_R^0\wedge\tau_{C1}^1\wedge\tau_{C2}^1\wedge T}V_{\tau_R^1}e^{-r(\tau_R^1-t)}$$
$$+1_{\tau_R^0<\tau_R^1\wedge\tau_{C1}^1\wedge\tau_{C2}^1\wedge T}(1-\varphi_R)e^{-r(\tau_R^0-t)}|G_t\big].$$

By the standard process of reduced form analysis (e.x. see [12]), we can obtain

$$V_t=1_{\tau_R^2>t}1_{\tau_{C1}^2>t}1_{\tau_{C2}^2>t}E^Q\Big[\int_t^T e^{-\int_t^s(r+\lambda_R^2+\lambda_{C1}^2+\lambda_{C2}^2)d\theta}\Big(-k+\lambda_R^2U_s$$
$$+\lambda_{C1}^2[a_1(\varphi_{C1}V_s^+-V_s^-)+a_2V_s]+\lambda_{C2}^2[a_2(\varphi_{C2}V_s^+-V_s^-)+a_1V_s]\Big)ds\Big|F_t\Big].$$
$$U_t=1_{\tau_R^0>t}1_{\tau_R^1>t}1_{\tau_{C1}^1>t}1_{\tau_{C2}^1>t}E^Q\Big[\int_t^T e^{-\int_t^s(r+\lambda_R^0+{\lambda_R}^1+\lambda_{C1}^1+\lambda_{C2}^1)d\theta}(-k+\lambda_R^1V_s$$
$$+\lambda_R^0(1-\varphi_R)+\lambda_{C1}^1[a_1(\varphi_{C1}{U_s}^+-{U_s}^-)+a_2U_s]$$
$$+\lambda_{C2}^1[a_2(\varphi_{C2}U_s^+-U_s^-)+a_1U_s])ds\Big|F_t\Big].$$

Using the Feynman-Kac formula, V_t and U_t satisfy the following partial differential equations:

$$\begin{cases} LV=(r+\lambda_R^2+\lambda_{C1}+\lambda_{C2})V+k-\lambda_R^2U-\lambda_{C1}[a_1(\varphi_{C1}V^+-V^-)+a_2V] \\ \quad +\lambda_{C2}[a_2(\varphi_{C2}V^+-V^-)+a_1V], \qquad y,\beta_i\in(0,\infty),\ t\in(0,T),\ i=R,C1,C2, \\ LU=(r+\lambda_R^0+\lambda_R^1+\lambda_{C1}+\lambda_{C2})U+k-\lambda_R^1V \\ \quad -{\lambda_R}^0(1-\varphi_R)-\lambda_{C1}[a_1(\varphi_{C1}U^+-U^-)+a_2U] \\ \quad +\lambda_{C2}[a_2(\varphi_{C2}U^+-U^-)+a_1U], \qquad y,\beta_i\in(0,\infty),\ t\in(0,T),\ i=R,C1,C2, \\ V(y,\beta_i,T)=U(y,\beta_i,T)=0, \qquad y,\beta_i\in(0,\infty),\ i=R,C1,C2, \end{cases} \tag{6}$$

where $\lambda_i^j=\lambda_i^j(y,\beta_i)$, $i=R,C1,C2$, $j=0,1,2$, are defined in (1), (2), (4) (or (5) depending on the correlation to be positive or negative). The differential operator L is defined by:

$$L=\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2y^2\frac{\partial^2}{\partial y^2}+\mu y\frac{\partial}{\partial y}+\sum_{i\in\{R,C1,C2\}}\Big(\frac{1}{2}{\sigma_i}^2{\beta_i}^2\frac{\partial^2}{\partial{\beta_i}^2}+\mu_i\beta_i\frac{\partial}{\partial\beta_i}\Big).$$

§3 Numerical analysis

Now, we are able to calculate the value of CDS contract with multi-counterparties under different credit ratings by the above PDE problem, which is a nonlinear PDE system. So far, there is no closed form solution. Thus, we have to use numerical methods to solve the problem. An iteration is used with the following procedure:

1. First, solve the system without considering the counterparty default and reference rating transfer risks, i.e. solve the following linear uncoupled PDE system:

$$\begin{cases} LV_1=(r+\lambda_R^2+\lambda_{C1}+\lambda_{C2})V_1+k, \qquad y,\beta_i\in(0,\infty),\ t\in(0,T),\ i=R,C1,C2, \\ LU_1=(r+\lambda_R^0+\lambda_R^1+\lambda_{C1}+\lambda_{C2})U_1 \\ \quad +k-{\lambda_R}^0(1-\varphi_R), \qquad y,\beta_i\in(0,\infty),\ t\in(0,T),\ i=R,C1,C2, \\ V_1(y,\beta_i,T)=U_1(y,\beta_i,T)=0, \qquad y,\beta_i\in(0,\infty),\ i=R,C1,C2. \end{cases}$$

In the iteration steps, we apply finite difference method to solve the equations above in a truncated finite domain $(t, y, \beta_R, \beta_{C1}, \beta_{C2}) \in (0,T)\times(0,H_y]\times(0,H_R]\times(0,H_{C1}]\times(0,H_{C2}]$. We denote H_i, $i \in y, R, C1, C2$, as a upper bound, which is sufficiently large to make the default probability of each party approaching 0. If reference company has no default risk, the value of the contract is 0, too. So we use $V_1 = U_1 = 0$ as the boundary condition at $y = H_y$ and $\beta_R = H_R$.

2. Once V_l, U_l are obtained, we solve the following linear uncoupled PDE system:

$$\begin{cases} LV_{l+1} = (r + \lambda_R^2 + \lambda_{C1} + \lambda_{C2})V_{l+1} + k - \lambda_R^2 U_l - \lambda_{C1}[a_1(\varphi_{C1}V_l^+ - V_l^-) + a_2 V_l] \\ \qquad +\lambda_{C2}[a_2(\varphi_{C2}V_l^+ - V_l^-) + a_1 V_l], \qquad y, \beta_i \in (0,\infty),\ t \in (0,T),\ i = R, C1, C2, \\ LU_{l+1} = (r + \lambda_R^0 + \lambda_R^1 + \lambda_{C1} + \lambda_{C2})U_{l+1} + k - \lambda_R^1 V_l - {\lambda_R}^0(1-\varphi_R) \\ \qquad -\lambda_{C1}[a_1(\varphi_{C1}U_l^+ - U_l^-) + a_2 U_l] + \lambda_{C2}[a_2(\varphi_{C2}U_l^+ - U_l^-) + a_1 U_l] \\ \qquad +\lambda_{C2}[a_2(\varphi_{C2}U_l^+ - U_l^-) + a_1 U_l], \qquad y, \beta_i \in (0,\infty),\ t \in (0,T),\ i = R, C1, C2, \\ V_{l+1}(y, \beta_i, T) = U_{l+1}(y, \beta_i, T) = 0, \qquad y, \beta_i \in (0,\infty),\ i = R, C1, C2. \end{cases}$$

3. Then, by solving the linear equations with ADI difference form, we obtain a sequence $\{V_l, U_l\}$, $l = 1, 2, \ldots$. We continue the iterations until the difference of the last two solutions is less than a preset constant. The iteration is compressed, so that it converges. This can be proved similarly by typical PDE technique as the one by W. Wei [20], where the interest rate swap with single counterparty risk without credit rating migration is considered.

4. In this way, we obtain the approximating solution of the problem (6),

After the numerical tests and results, we analyze the parameters of the model and choose the following:

$$T = 5(year), r = 0.025, k = 100(bp), \Delta t = 0.5, S_t = 200(bp), a_1 = 0.5, a_2 = 0.5,$$
$$x_i = 0.031, \varphi_1 = 0.4, \mu_i = 0.02, \sigma_i = 0.1, {\beta_0}^i = 0.031, \quad (i = R, C1, C2).$$

Figure 3 shows that the value of CDS (U, similar for V) has almost no change when the number of iterations $n \geq 3$ indicating a very fast convergence. With this pricing approach, calculating the value of CDS has a high efficiency.

In Figure 4, we can find the relationship between the value of CDS contract and the asset of the reference company. The figure shows that the value of the contract decreases when the reference company asset increases. In fact, the default probability of the reference company decreases as the reference company asset increases.

Figure 5 shows the relationship between the value of CDS and the asset of the counterparty C1. The value of the contract increases with the counterparty asset. The default probability of the counterparty decreases when the asset goes up, so that the value of the CDS contract increases correspondingly. For C2, the figure is similar.

The interesting results that we reveal in this paper are the impact of the correlation on the CDS value.

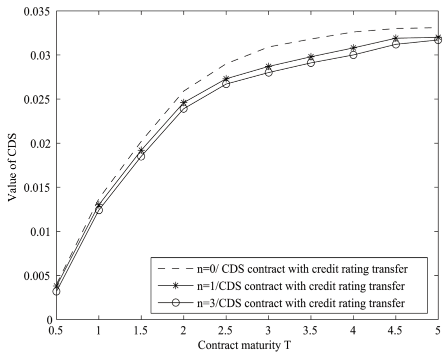

Figure 3: The value of CDS with different number of iterations.

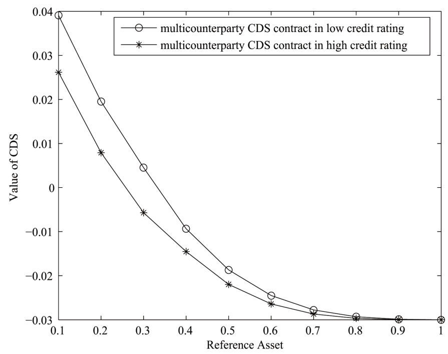

Figure 4: The value of CDS with different asset value of reference company.

Figure 6 indicates the value of the CDS as well as the positive correlation between the counterparties and the reference. The value of the contract decreases when the positive correlation coefficient increases. We can see that as this coefficient increases, the default probability of counterparties increases with reference defaults, which results in decreasing the value of the contract.

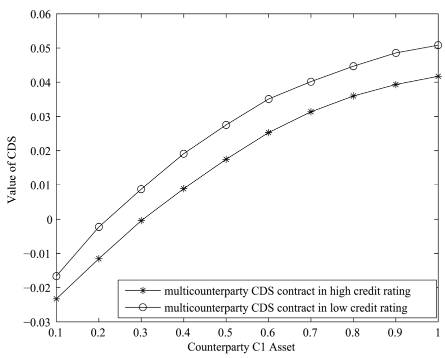

Figure 5: The value of CDS with different asset value of C1.

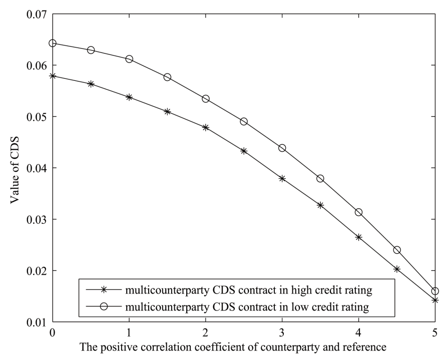

Figure 6: The value of CDS with different positive correlation coefficient of counterparty and reference.

Figure 7 indicates the value of the CDS contract and the negative correlation between the counterparties and the reference. The value of the contract increases when the absolute of negative correlation coefficient goes larger. In fact, as the absolute of negative correlation coefficient increases, the counterparty's default probability decreases with respect to the reference default. Thus, the contract value goes up.

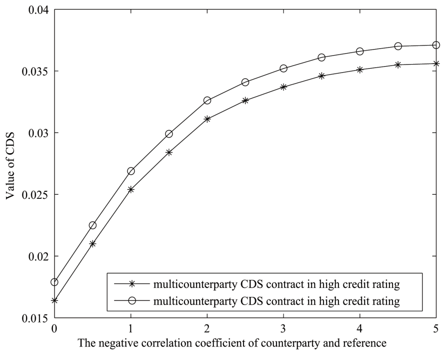

Figure 7: The value of CDS with different negative correlation coefficient of counterparty and reference.

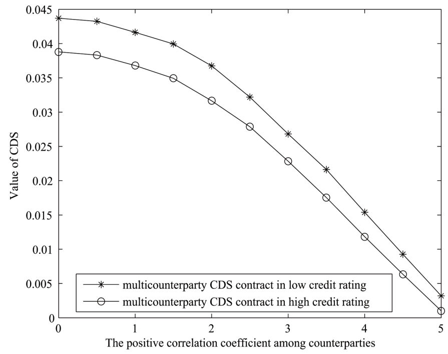

Figure 8: The value of CDS with different positive correlation coefficient among counterparties.

Figure 8 indicates the relationship between the value of the CDS contract and the positive correlation between the counterparties. The value of the contract decreases as the positive correlation coefficient increases. In fact, if one of the counterparties defaults, the other counterparty is also more likely to default. Thus, the contract looses value.

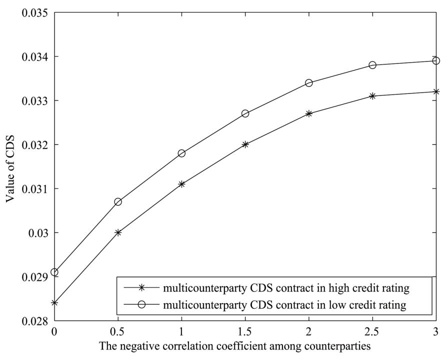

Figure 9: The value of CDS with different negative correlation coefficient among counterparties.

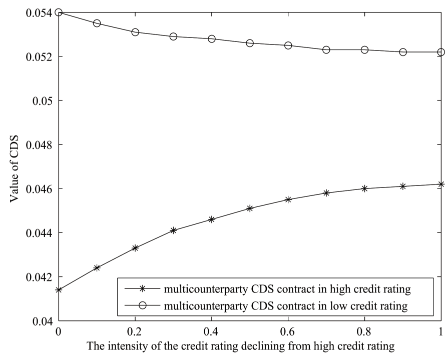

Figure 10: The value of CDS with different intensity of the credit rating declining from high credit rating.

Figure 9 indicates the relationship between the value of the CDS contract and the negative correlation between the counterparties in certain parameters. The value of the contract increases when the absolute negative correlation increases. Once default risk of one counterparty rose, the one of the other counterparty would decline, then the contract value may be added.

Figure 10 indicates the relationship between the value of the CDS contract and the intensity of the credit rating downgrades from the high credit rating. The value of the contract increases when the downgrading intensity of credit rating increases. Once the downgraded credit rating intensity increases, the asset of the reference company decreases correspondingly and the reference company default becomes easier. So the value of the contract in the high credit rating increases. On the other hand, the value of the contract in the low credit rating has almost no change.

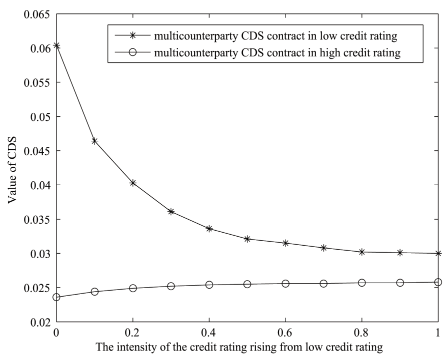

Figure 11: The value of CDS with different intensity of the credit rating rising from low credit rating.

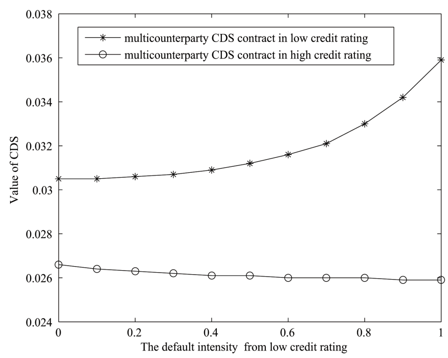

Figure 12: The value of CDS with different default intensity in low credit rating.

Figure 11 shows the relationship between the value of the CDS contract and the intensity of credit rating upgrading form the low credit rating. The value of the contract decreases when the upgraded credit rating intensity increases. Once the upgraded credit rating intensity increases, the value of the reference asset increases correspondingly and the default probability of the reference company will be smaller. So, the value of the contract in the low credit rating decreases while the value of the contract in the high credit rating has almost no change.

Figure 12 shows the relationship between the value of the CDS contract and the default intensity in the low credit rating. The value of the contract in low credit rating does not change dramatically when the intensity is relatively small, but it changes obviously when the intensity becomes larger. In such processes, when the default intensity increases, the reference default is more likely to happen, so the value of the contract increases quickly. At the same time, the value of the contract in the high credit rating has almost no change.

§4 Conclusion

We consider the pricing of a multi-counterparty CDS contract with the presence of credit rating migration risks in the reference. A reduced form model with structural type intensities is established. The credit rating migration and default intensities are assumed to be functions of the companies' assets. From the model, The value of the contract satisfies a nonlinear system of partial differential equations. By solving the problem numerically by iteration, the impacts of the correlation among the reference and counterparties are analyzed correspondingly.

1. When correlations of the reference with both counterparties are positive, the value of the CDS contract increases with respect to the correlation. Thus, the positive correlation increases the risk. This is the case of "wrong way risk". However, the more counterparties we use, the less risk we have.

2. When the correlation of the reference with both counterparties are negative, the value of the CDS contract decreases with respect to the correlation. Thus, the negative correlation does decrease the risk, and the more counterparties we use, the less risk we have.

3. When the default correlation among the counterparties is negative, it is uncertain to reduce the risk by adding a counterparty. In fact, compared to the reference, adding a negatively correlated counterpa

   rty will reduce the risk, while adding a positively correlated counterparty will increase the risk.

4. For a CDS contract, it is more risky when the reference is in the low credit rating than when it is in the high credit rating. So the upgrade in the reference's credit rating status will decrease the risk of the contract.

Remark The model and method of calculation are able to extend to the case of multi-counterparty with more than two counterparties under credit rating migration risks. It also can be used to the case where there are more than two credit grades.

References

[1] Agostino Capponi. *Pricing and Mitigation of Counterparty Credit Exposures*, Handbook of Systemic Risk, Cambridge University Press, 2013.

[2] Bei Hu, Lishang Jiang, Jin Liang, Wei Wei. *A fully non-linear PDE problem from pricing CDS with counterparty risk*, Discrete and Continuous Dynamical Systems, 2012, 17(6): 2001-2016.

[3] Bei Hu, Jin Liang, Yuan Wu. *A free boundary problem for corporate bond with credit rating migration*, Journal of Mathematical Analysis & Applications, 2015, 428(2): 896-909.

[4] Damiano Brigo, Andrea Pallavicini. *Counterparty Risk Pricing under Correlation between Default and Interest Rates*, Numerical Methods for Finance,Chapman & Hall/CRC, 2007, DOI: 10.1201/9781584889267.ch4.

[5] Damiano Brigo, Andrea Pallavicini. *Counterparty risk and CCDSs under correlation*, Risk, 2008.

[6] Dilip B. Madan, Haluk Unal. *Pricing the risks of default*, Review of Derivatives Research, 1998, 2(2-3): 121-160.

[7] David Lando. *Credit risk modeling: theory and applications*, Princeton University Press, 2009.

[8] Giovanni Cesari, John Aquilina, Niels Charpillon, Zlatko Filipovic, Gordon Lee, Ion Manda. *Modelling, pricing, and hedging counterparty credit exposure: A technical guide*, Springer-Verlag Berlin Heidelberg, 2009.

[9] Jin Liang, Yuan Wu, Bei Hu. *Asymptotic Traveling Wave Solution for a Credit Rating Migration Problem*, Journal of Differential Equations, 2016, 261(2): 1017-1045.

[10] Jin Liang, Yuejuan Zhao, Xudan Zhang. *Utility indifference valuation of corporate bond with credit rating migration by structure approach*, Economic Modelling, 2016, 54: 339-346.

[11] Jin Liang, Wenyi Li. *Counterparty Valuation Adjustment Calculation Model of Multi-counterparties Credit Default Swap*, Journal of Tongji University, 2014, 42(1): 144-150.

[12] Jin Liang, Tao Wang. *Valuation of Loan-only Credit Default Swap with Negatively Correlated Default and Prepayment Intensities*, International Journal of Computer Mathematics, 2012, 89(9): 1255-1268.

[13] Lijun Bo, Agostino Capponi. *Bilateral credit valuation adjustment for large credit derivatives portfolios*, Finance and Stochastics, 2013, 18(2): 431-482.

[14] Robert Jarrow, Yildiray Yildirim. *Pricing Treasury Inflation Protected Securities and Related Derivatives using an HJM Model*, Journal of Financial and Quantitative Analysis, 2003, 38(2): 337-358.

[15] Robert Alan Jarrow, Fan yu. *Counterparty Risk and the Pricing of Defaultable Securities*, The Journal of Finance, 2001, 56(5): 1765-1799.

[16] Shahram Alavianz, Jie Dingx, Peter Whitehead, Leonardo Laudicinak. *Counterparty Valuation Adjustment(CVA)*, Available at SSRN, 2008.

[17] S Crpey, M Jeanblanc, B Zargari. *Counterparty Risk on a CDS in a Markov Chain Copula Model with Joint Defaults*, Working paper, 2009.

[18] Seng Yuen Leung, Yue Kuen Kwok. *Credit default swap valuation with counterparty risk*, The Kyoto Economic Review, 2005, 74(1): 25-45.

[19] Tomasz R Bielecki, Stphane Crpey, Monique Jeanblanc, B Zargari. *Valuation and hedging of CDS counterparty exposure in a Markov copula model*, International Journal of Theoretical and Applied Finance, 2012, 15(15).

[20] Wei Wei. *Counterparty Credit Risk on a Standard Swap in "Risky Closeout"*, International Journal of Financial Research, 2011, 2(2): 40-51.

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