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Periodic dividends and capital injections for a spectrally negative Lévy risk process under absolute ruin

DONG Hua ZHAO Xiang-hua

Abstract. The spectrally negative Lévy risk model with random observation times is considered in this paper, in which both dividends and capital injections are made at some independent Poisson observation times. Under the absolute ruin, the expected discounted dividends and the expected discounted capital injections are discussed. We also study the joint Laplace transforms including the absolute ruin time and the total dividends or the total capital injections. All the results are expressed in scale functions.

§1 Introduction

Let $X = \{X_t;\, t \geq 0\}$ be a spectrally negative Lévy process (SNLP) with nonmonotone paths defined on the filtered space $(\Omega,\, \mathcal{F},\, \{\mathcal{F}_t : t \geq 0\},\, \mathbb{P})$, i.e. X is a stochastic process which has stationary and independent increments, càdlàg paths that have no positive jump discontinuities. For $x \in \mathbb{R}$, we denote by $\mathbb{P}_x$ the law of X given $X_0 = x$ and write for convenience $\mathbb{P}$ instead of $\mathbb{P}_0$. Accordingly, we write $\mathbb{E}_x$ and $\mathbb{E}$ for the associated expectations. The process X is uniquely characterized by the Laplace exponent

$$\psi(\lambda) = \alpha\,\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty \left(\mathrm{e}^{-\lambda\, z} - 1 + \lambda\, z\mathbf{1}_{(0,1]}(z)\right)\Pi(dz),$$

where $\alpha \in \mathbb{R}$, $\sigma \geq 0$, and the Lévy measure Π is a $\sigma-$finite measure on $(0,\, \infty)$ satisfying

$$\int_0^\infty \left(1 \wedge\, z^2\right)\Pi(dz) < \infty.$$

In classical dividend barrier strategy, dividends are paid immediately to the shareholders once the surplus reaches a fixed barrier $b > 0$, as long as ruin has not occurred. However, the insurer's surplus can not be monitored continuously in practice. Then Albrecher et al. (2011a) first proposed periodic barrier dividend strategy in the Cramér-Lundberg model, in which ruin and dividend can only be observed at some random observation times. After that,

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the risk models with periodic dividends have attracted a lot of attention, see e.g. Albrecher et al. (2011b), Avanzi et al. (2014), Albrecher et al. (2016), Pérez and Yamazaki (2018), Noba et al. (2018), Liu et al. (2015), Peng et al. (2013), Zhang (2014), Zhang and Cheung (2016), Dong et al. (2019) and references therein.

Since ruin is certain under barrier strategy, then it may be profitable to rescue the company by capital injection. For risk model with classical barrier dividends and capital injections reader is referred to Avanzi et al. (2011) and Yao et al. (2011) and references therein. More recently, Zhao et al. (2017) studied a spectrally positive Lévy risk process with periodic dividends and classical capital injection, Avram et al. (2018) investigated a spectrally negative Lévy risk process with classical barrier dividend, periodic capital injection and absolute ruin.

In practice, the capital can not be injected as soon as the surplus drops below zero, which may occur at some discrete time points. Hence, Dong and Zhou (2020) considered the spectrally negative Lévy process with both dividends and capital injections being made at some independent Poisson observation times. In this paper, we continue this topic. The rate of opportunities for dividends and capital injections may be different in insurance practice. But we assume that they can be observed at the same sequence of Poisson times here. Otherwise, different sequence of observation times can make the problem and the mathematical calculation more complex. The modified surplus process is described as follows:

Let $\{T_i\}$ be the sequence of observation times and $T_i - T_{i-1}$ be independently exponential distribution with rate $r > 0$. Denote by $X^r = \{X_t^r; t \geq 0\}$ the Lévy process with periodic capital injections. Define the capital injection times:

$$T_0^-(1) := \inf\{T_i : X_{T_i-} < 0\}, \quad \text{and} \quad T_0^-(n) := \inf\{T_i > T_0^-(n-1) : X^r_{T_i-} < 0\}$$

with convention $\inf \emptyset = \infty$. Then X^r is given by

$$X_t^r = X_t + R_t^r, \qquad t \geq 0,$$

where $R_t^r := \sum_{T_0^-(i) \leq t} |X_{T_0^-(i)} - |$ is the cumulative amount of reflection until time $t \geq 0$.

For $b > 0$, consider an extension of X_t^r with additional periodic dividends, which we denote by $X_t^{r,b}$. Similar procedure to the definition for X_r, we have

$$X_t^{r,b} = X_t^r, \qquad 0 \leq t \leq T_b^+(1),$$

where $T_b^+(1) := \inf\{T_i : X^r_{T_i} > b\}$ with convention $\inf \emptyset = \infty$. Obviously, $X^{(r,b)}_{T_b^+(1)} = b$. For $T_b^+(1) \leq t < T_b^+(2) := \inf\{T_i > T_b^+(1) : X^r_{T_i} > b\}$,

$$X_t^{r,b} = X_t^r - (X^r_{T_b^+(1)-} - b).$$

Repeating the procedure above, we have

$$X_t^{r,b} = X_t - L_t^{r,b} + R_t^{r,b},$$

where $L_t^{r,b}$ and $R_t^{r,b}$ are the cumulative amounts of periodic dividends and capital injections until time $t \geq 0$, respectively.

The remainder of the paper is organized as follows. Section 2 is the preliminaries. The spectrally negative Lévy process and some fluctuation results and identities on scale functions are listed. In Section 3, the expected discounted dividends and the expected discounted capital injections before absolute ruin are discussed. Some joint Laplace transforms involving the absolute ruin time and the total periodic dividends or the total periodic capital injections are also investigated.

§2 Preliminaries

In this section, we first present the definition of the scale functions. For $q \geq 0$, the q-scale function $W^{(q)}(y)$ is defined as the continuous function on $[0, \infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda y} W^{(q)}(y) dy = \frac{1}{\psi(\lambda) - q}, \qquad \lambda > \Phi_q, \tag{1}$$

where $\Phi_q = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$. This function is unique, positive and strictly increasing for $x \geq 0$. We extend $W^{(q)}$ to the whole real line by setting $W^{(q)}(x) = 0$ for $x < 0$. When X has unbounded variation or Π has no atoms, by Kyprianou et al. (2010), $W^{(q)} \in C^1(0, \infty)$.

We also define the following notations:

$$\overline{W}^{(q)}(x) := \int_0^x W^{(q)}(z) dz, \quad \overline{\overline{W}}^{(q)}(x) := \int_0^x \overline{W}^{(q)}(z) dz, \quad Z^{(q)}(x) := 1 + q\overline{W}^{(q)}(x),$$

for $x > 0$, and $\overline{W}^{(q)}(x) := 0$, $\overline{\overline{W}}^{(q)}(x) := 0$, $Z^{(q)}(x) := 1$ for $x < 0$. For convenience, we write W and Z instead of $W^{(0)}$ and $Z^{(0)}$, respectively.

Furthermore, denote by

$$Z_p(x, \theta) := e^{\theta x}\left(1 + (p - \psi(\theta)) \int_0^x e^{-\theta y} W^{(p)}(y) dy\right), \qquad x \geq 0; \tag{2}$$

and $Z_p(x, \theta) := e^{\theta x}$ for $x < 0$. Obviously, $Z_p(x, 0) = Z^{(p)}(x)$.

Given $p, q \geq 0$, for $0 \leq a \leq x$, by Li and Zhou (2014), we know

$$\begin{aligned} W_a^{(p,q)}(x) : &= W^{(p)}(x) + (q - p) \int_a^x W^{(q)}(x - y) W^{(p)}(y) dy \\ &= W^{(q)}(x) + (p - q) \int_0^a W^{(q)}(x - y) W^{(p)}(y) dy \end{aligned} \tag{3}$$

and

$$\begin{aligned} Z_a^{(p,q)}(x) : &= Z^{(p)}(x) + (q - p) \int_a^x W^{(q)}(x - y) Z^{(p)}(y) dy \\ &= Z^{(q)}(x) + (p - q) \int_0^a W^{(q)}(x - y) Z^{(p)}(y) dy \end{aligned} \tag{4}$$

with the conventions of $W_a^{(p,q)}(x) := W^{(p)}(x)$ and $Z_a^{(p,q)}(x) := Z^{(p)}(x)$ for $x < a$.

Let

$$\tau_b^+ = \inf\{t \geq 0, X_t > b\} \text{ and } \tau_b^- = \inf\{t \geq 0, X_t < b\}, \quad b \in \mathbb{R},$$

be the first passage times. Since X_t has no positive jumps, $X_{\tau_b^+} = b$. Then it follows from Kyprianou (2014) that

$$\mathbb{E}_u\left(\mathrm{e}^{-q\,\tau_b^+}; \tau_b^+ < \tau_0^-\right) = \frac{W^{(q)}(u)}{W^{(q)}(b)}, \tag{5}$$

$$\mathbb{E}_u\left(\mathrm{e}^{-q\,\tau_0^-}; \tau_0^- < \infty\right) = Z^{(q)}(u) - \frac{qW^{(q)}(u)}{\Phi_q}, \tag{6}$$

for $0 < u < b$ and $q \geq 0$.

For any $q \geq 0$ and $0 \leq x, y \leq b$, the q-potential measure of a spectrally negative Lévy process killed on exiting $[0, b]$ has a density function

$$u^{(q)}(b, x, y) := \frac{W^{(q)}(x) W^{(q)}(b - y)}{W^{(q)}(b)} - W^{(q)}(x - y). \tag{7}$$

Let e_r be an independent exponential random variable with parameter r, then

$$\mathbb{P}_x(X_{e_r} \in dy, e_r < \tau_0^-) = r\left(e^{-\Phi_r y}W^{(r)}(x) - W^{(r)}(x-y)\right)dy, \qquad x, y \geq 0, \tag{8}$$

which is given in Section 8.4 of Kyprianou (2006).

For $q \geq 0$, $0 \leq x \leq b$,

$$\mathbb{E}_x[e^{-qe_r}; e_r < \tau_b^+ \wedge \tau_0^-] = r\left(\frac{W^{(q+r)}(x)}{W^{(q+r)}(b)}\overline{W}^{(q+r)}(b) - \overline{W}^{(q+r)}(x)\right), \tag{9}$$

see (5.1) of Perez and Yamazaki (2018).

§3 Main results

For any $a \in \mathbb{R}$, define

$$\tau_a^+(r) := \inf\{t \geq 0; X_t^r > a\} \text{ and } \tau_a^-(r) := \inf\{t \geq 0; X_t^r < a\}$$

and

$$\hat{\tau}_a^+(r) := \inf\{t \geq 0; X_t^{r,b} > a\} \text{ and } \hat{\tau}_a^-(r) := \inf\{t \geq 0; X_t^{r,b} < a\}.$$

For $a < 0$, $\hat{\tau}_a^-(r)$ may be seen as the absolute ruin time for $X_t^{r,b}$.

We then give some notations that are used throughout this paper. For $a < 0 < b$, $q, r, \theta > 0$, $x > a$, the following results can be derived directly by (8) and some calculations:

$$\begin{aligned}
A_{a,b}^{(q,r)}(x,\theta) &:= \mathbb{E}_x[e^{-qe_r - \theta(X_{e_r} - b)}; e_r < \tau_a^-, X_{e_r} > b] \\
&= \frac{r}{\theta + \Phi_{q+r}}W^{(q+r)}(x-a)e^{-\Phi_{q+r}(b-a)} - re^{-\theta(x-b)}\int_0^{x-b} e^{\theta t}W^{(q+r)}(t)dt, \\
B_a^{(q,r)}(x,\theta) &:= \mathbb{E}_x[e^{-qe_r + \theta X_{e_r}}; e_r < \tau_a^-, a < X_{e_r} < 0] \\
&= \frac{rW^{(q+r)}(x-a)}{\theta - \Phi_{q+r}}\left(e^{\Phi_{q+r}a} - e^{\theta a}\right) \\
&\quad + \frac{r}{q + r - \psi(\theta)}\left(Z_{q+r}(x,\theta) - e^{\theta a}Z_{q+r}(x-a,\theta)\right).
\end{aligned}$$

For $a < 0 < b$, $q, r, \theta_1, \theta_2 > 0$, $x > a$, we also define

$$\begin{aligned}
H_a^{(q,r)}(x) &:= \frac{1 + r\overline{W}^{(q+r)}(-a)}{W^{(q+r)}(-a)}W_{-a}^{(q+r,q)}(x-a) \\
&\quad - \frac{r}{q+r}Z_{-a}^{(q+r,q)}(x-a) + \frac{r}{q+r}Z^{(q)}(x),
\end{aligned}$$

with $H_a^{(q,r)}(0) = 1$.

$$\begin{aligned}
\gamma_a^{(q+r)}(x,y) &:= r\left(e^{-\Phi_{q+r}(y-a)}W^{(q+r)}(x-a) - W^{(q+r)}(x-y)\right), \\
\mathcal{M}_{a,b}^{(q,r)}(x,\theta_1,\theta_2) &:= H_a^{(q,r)}(b)[1 - A_{(a,b)}^{q,r}(x,\theta_1)] \\
&\quad - \int_0^b \gamma_a^{(q+r)}(x,y)H_a^{(q,r)}(y)dy - B_a^{(q,r)}(x,\theta_2), \\
\mathcal{K}_{a,b}^{(q,r)}(x,\theta_1) &:= -\int_0^b \gamma_a^{(q+r)}(x,y)K_a^{(q,r)}(y)dy + K_a^{(q,r)}(b)(1 - A_{a,b}^{(q,r)}(b,\theta_1)), \\
Q_{a,b}^{(q,r)}(x,\theta_1) &:= Z^{(q+r)}(x-a) - \frac{q+r}{\Phi_{q+r}}W^{(q+r)}(x-a) - \mathcal{K}_{a,b}^{(q,r)}(x,\theta_1),
\end{aligned}$$

and also define, for any measurable function $f:\mathbb{R}\to\mathbb{R}$,

$$\mathcal{L}f(x,\theta_1)=\int_0^b \gamma_a^{(q+r)}(x,y)f(y)dy+B_a^{(q,r)}(x,\theta_1)f(0).$$

To prove Theorem 1, the following Lemma will be needed:

Lemma 1. *For any $p,q\ge 0$ and $0\le a\le x\le b$, we have*

$$\begin{aligned}\mathbb{E}_x[e^{-q\tau_a^-}W^{(p)}(X_{\tau_a}^-);\tau_a^-<\tau_b^+] &= W_a^{(p,q)}(x)-\frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}W_a^{(p,q)}(b),\\ \mathbb{E}_x[e^{-q\tau_a^-}Z^{(p)}(X_{\tau_a}^-);\tau_a^-<\tau_b^+] &= Z_a^{(p,q)}(x)-\frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}Z_a^{(p,q)}(b),\end{aligned}$$

see Loeffen et al. (2014) for details.

Theorem 1. *Given $a<0<b$, for $q\ge 0$ and $x>a$, denote by $V(x;a,b)=\mathbb{E}_x[\int_0^{\hat{\tau}_a^-(r)}e^{-qt}dL_t^{r,b}]$. Then we have*

$$\begin{aligned}V(x;a,b) &= \frac{rW^{(q+r)}(b-a)\left(H_a^{(q,r)}(b)-\mathcal{M}_{a,b}^{(q,r)}(x,0,0)\right)}{\Phi_{q+r}^2\mathcal{M}_{a,b}^{(q,r)}(b,0,0)}e^{-\Phi_{q+r}(b-a)}\\ &\quad+\frac{rW^{(q+r)}(x-a)}{\Phi_{q+r}^2}e^{-\Phi_{q+r}(b-a)}.\end{aligned}\tag{10}$$

Proof. For $a<x<0$, by the strong Markov property, (5) and (9), we have

$$\begin{aligned}V(x;a,b) &= \mathbb{E}_x[e^{-q\tau_0^+};\tau_0^+<e_r\wedge\tau_a^-]V(0;a,b)+\mathbb{E}_x[e^{-qe_r};e_r<\tau_0^+\wedge\tau_a^-]V(0;a,b)\\ &= \mathbb{E}_x[e^{-(q+r)\tau_0^+};\tau_0^+<\tau_a^-]V(0;a,b)+\mathbb{E}_x[e^{-qe_r};e_r<\tau_0^+\wedge\tau_a^-]V(0;a,b)\\ &= \frac{W^{(q+r)}(x-a)}{W^{(q+r)}(-a)}V(0;a,b)\\ &\quad+r\left(\frac{W^{(q+r)}(x-a)}{W^{(q+r)}(-a)}\overline{W}^{(q+r)}(-a)-\overline{W}^{(q+r)}(x-a)\right)V(0;a,b)\\ &= \frac{1+r\overline{W}^{(q+r)}(-a)}{W^{(q+r)}(-a)}W^{(q+r)}(x-a)V(0;a,b)\\ &\quad-r\overline{W}^{(q+r)}(x-a)V(0;a,b).\end{aligned}\tag{11}$$

For $0\le x<b$, by the strong Markov property, (5) and (11), we have

$$\begin{aligned}V(x;a,b) &= \mathbb{E}_x[e^{-q\tau_b^+};\tau_b^+<\tau_0^-]V(b;a,b)+\mathbb{E}_x[e^{-q\tau_0^-}V(X_{\tau_0^-};b);\tau_0^-<\tau_b^+]\\ &= \frac{W^{(q)}(x)}{W^{(q)}(b)}V(b;a,b)+\mathbb{E}_x[e^{-q\tau_0^-}\frac{W^{(q+r)}(X_{\tau_0^-}-a)}{W^{(q+r)}(-a)};\tau_0^-<\tau_b^+]V(0;a,b)\\ &\quad+r\mathbb{E}_x[e^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}[e^{-qe_r};e_r<\tau_0^+\wedge\tau_a^-];\tau_0^-<\tau_b^+]V(0;a,b)\\ &= \frac{W^{(q)}(x)}{W^{(q)}(b)}V(b;a,b)\\ &\quad+\frac{1+r\overline{W}^{(q+r)}(-a)}{W^{(q+r)}(-a)}\mathbb{E}_x[e^{-q\tau_0^-}W^{(q+r)}(X_{\tau_0^-}-a);\tau_0^-<\tau_b^+]V(0;a,b)\\ &\quad-\frac{r}{q+r}\mathbb{E}_x[e^{-q\tau_0^-}Z_{q+r}(X_{\tau_0^-}-a);\tau_0^-<\tau_b^+]V(0;a,b)\end{aligned}$$

$$
\begin{aligned}
&+\frac{r}{q+r}\mathbb{E}_x[e^{-q\tau_0^-};\tau_0^-<\tau_b^+]V(0;a,b)\\
=\;&\frac{1+r\overline{W}^{(q+r)}(-a)}{W^{(q+r)}(-a)}\left[W_{-a}^{(q+r,q)}(x-a)-W_{-a}^{(q+r,q)}(b-a)\frac{W^{(q)}(x)}{W^{(q)}(b)}\right]V(0;a,b)\\
&-\frac{r}{q+r}\left[Z_{-a}^{(q+r,q)}(x-a)-Z_{-a}^{(q+r,q)}(b-a)\frac{W^{(q)}(x)}{W^{(q)}(b)}\right]V(0;a,b)\\
&+\frac{r}{q+r}\left[Z^{(q)}(x)-Z^{(q)}(b)\frac{W^{(q)}(x)}{W^{(q)}(b)}\right]V(0;a,b)+\frac{W^{(q)}(x)}{W^{(q)}(b)}V(b;a,b)\\
=\;&\frac{W^{(q)}(x)}{W^{(q)}(b)}V(b;a,b)+\left(H_a^{(q,r)}(x)-H_a^{(q,r)}(b)\frac{W^{(q)}(x)}{W^{(q)}(b)}\right)V(0;a,b), \qquad (12)
\end{aligned}
$$

where we have used Lemma 1 in the fourth equality.

If X_t has bounded variation, letting $x=0$ in (12), we have

$$
H_a^{(q,r)}(b)V(0;a,b)=V(b;a,b). \qquad (13)
$$

For the unbounded variation case, we use the perturbation approach, which was proposed by Li et al. (2014). For $0\le x\le b$, it is easy to show that

$$
\begin{aligned}
V(0;a,b)\;=\;&\mathbb{E}[e^{-q\tau_x^+};\tau_x^+<\tau_a^-\wedge e_r]V(x;a,b)\\
&+\int_a^x\mathbb{E}[e^{-qe_r};e_r<\tau_a^-\wedge\tau_x^+,X_{e_r}\in dy]V(y;a,b)\\
=\;&\mathbb{E}[e^{-q\tau_x^+};\tau_x^+<\tau_a^-\wedge e_r]V(x;a,b)\\
&+\int_0^x\mathbb{E}[e^{-qe_r};e_r<\tau_a^-\wedge\tau_x^+,X_{e_r}\in dy]V(y;a,b)\\
&+\int_a^0\mathbb{E}[e^{-qe_r};e_r<\tau_a^-\wedge\tau_x^+,X_{e_r}\in dy]V(0;a,b)\\
=\;&\frac{W^{(q+r)}(-a)}{W^{(q+r)}(x-a)}V(x;a,b)+\Delta_2(x)+\Delta_3^{(q,r)}(x)V(0;a,b), \qquad (14)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_2(x)&=\int_0^x\mathbb{E}[e^{-qe_r};e_r<\tau_a^-\wedge\tau_x^+,X_{e_r}\in dy]V(y;a,b),\\
\Delta_3^{(q,r)}(x)&=\int_a^0\mathbb{E}[e^{-qe_r};e_r<\tau_a^-\wedge\tau_x^+,X_{e_r}\in dy].
\end{aligned}
$$

For $x\to 0^+$, by (7), we have the following results:

$$
\begin{aligned}
\Delta_2(x)&=r\int_0^x\left(\frac{W^{(q+r)}(-a)W^{(q+r)}(x-y)}{W^{(q+r)}(x-a)}-W^{(q+r)}(-y)\right)V(y;a,b)dy\\
&=\frac{rW^{(q+r)}(-a)}{W^{(q+r)}(x-a)}\int_0^x W^{(q+r)}(y)V(y;a,b)dy=o(W^{(q+r)}(x)).\\
\Delta_3^{(q,r)}(x)&=r\int_a^0\left(\frac{W^{(q+r)}(-a)W^{(q+r)}(x-y)}{W^{(q+r)}(x-a)}-W^{(q+r)}(-y)\right)dy\\
&=r\int_a^0\left(1-\frac{W^{(q+r)\prime}(-a)}{W^{(q+r)}(-a)}x\right)W^{(q+r)}(x-y)dy-r\overline{W}^{(q+r)}(-a)\\
&+\;o(W^{(q+r)}(x))
\end{aligned}
$$

$$
\begin{aligned}
&= r\left(1-\frac{W^{(q+r)\prime}(-a)}{W^{(q+r)}(-a)}x\right)\left(\overline{W}^{(q+r)}(x-a)-\overline{W}^{(q+r)}(x)\right)-r\overline{W}^{(q+r)}(-a)\\
&+ o(W^{(q+r)}(x))\\
&= r\left(1-\frac{W^{(q+r)\prime}(-a)}{W^{(q+r)}(-a)}x\right)\left(xW^{(q+r)}(-a)+\overline{W}^{(q+r)}(-a)\right)-r\overline{W}^{(q+r)}(-a)\\
&+ o(W^{(q+r)}(x))\\
&= rW^{(q+r)}(-a)x-r\frac{W^{(q+r)\prime}(-a)}{W^{(q+r)}(-a)}\overline{W}^{(q+r)}(-a)x+o(W^{(q+r)}(x));\\
H_a^{(q,r)}(x) &= 1+\left(\frac{1+r\overline{W}^{(q+r)}(-a)}{W^{(q+r)}(-a)}W^{(q+r)\prime}(-a)-rW^{(q+r)}(-a)\right)x+o(W^{(q+r)}(x)),
\end{aligned}
$$

and

$$
\frac{W^{(q+r)}(-a)}{W^{(q+r)}(x-a)}=1-\frac{W^{(q+r)\prime}(-a)}{W^{(q+r)}(-a)}x+o(W^{(q+r)}(x)).
$$

Substituting (12) into (14) and letting $x\to 0$, we get (13) again.

For $x>a$, conditioning on the first observation time, we have

$$
\begin{aligned}
V(x;a,b) &= \mathbb{E}_x[e^{-qe_r}(X_{e_r}-b);e_r<\tau_a^-,X_{e_r}>b]\\
&+\mathbb{E}_x[e^{-qe_r};e_r<\tau_a^-,X_{e_r}>b]V(b;a,b)\\
&+\mathbb{E}_x[e^{-qe_r}V(X_{e_r};a,b);e_r<\tau_a^-,0<X_{e_r}<b]\\
&+\mathbb{E}_x[e^{-qe_r};e_r<\tau_a^-,a<X_{e_r}<0]V(0;a,b)\\
&= \frac{r}{\Phi_{q+r}^2}W^{(q+r)}(x-a)e^{-\Phi_{q+r}(b-a)}+A_{a,b}^{(q,r)}(x,0)V(b;a,b)\\
&+\int_0^b \gamma^{(q+r)}(x-a,y-a)\frac{H_a^{(q,r)}(y)}{H_a^{(q,r)}(b)}dyV(b;a,b)\\
&+B_a^{(q,r)}(x,0)V(0;a,b).
\end{aligned}
\tag{15}
$$

Letting $x=b$ in (15) and using (14), we have

$$
V(b;a,b)=\frac{rW^{(q+r)}(b-a)H_a^{(q,r)}(b)e^{-\Phi_{q+r}(b-a)}}{\Phi_{q+r}^2\mathcal{M}_{a,b}^{(q,r)}(b,0,0)}. \tag{16}
$$

Then (10) can be obtained from (13), (15) and (16).

□

Remark 1. For $a\to 0$, (10) is identical to (3.9) in Zhang and Chueng (2018).

Using the same method as that in Theorem 1, we can derive the following result:

Theorem 2. *For $q,\theta\geq 0$, $a<0<b$, and $x>a$, denote by $U(x)=\mathbb{E}_x[e^{-q\hat{\tau}_a^-(r)-\theta L^{r,b}_{\hat{\tau}_a^-(r)}}]$. Then we have*

$$
\begin{aligned}
U(x) &= \frac{H_a^{(q,r)}(b)-\mathcal{M}_{a,b}^{(q,r)}(x,\theta,0)}{\mathcal{M}_{a,b}^{(q,r)}(b,\theta,0)}Q_{a,b}^{(q,r)}(b,\theta)\\
&+Q_{a,b}^{(q,r)}(x,\theta)+\mathcal{I}_a^{(q,r)}(b).
\end{aligned}
$$

The following Lemma is given in Theorem 3.1 and Theorem 3.2 of Avram et al. (2018).

Lemma 2. *For $a < 0 < b$, q, $\theta \ge 0$ and $x \le b$,*

$$f(x;a,b) = \mathbb{E}_x\left(\int_0^{\tau_a^-(r)\wedge\tau_b^+(r)} e^{-qt}dR_t^r\right) = \frac{H_a^{(q,r)}(x)}{H_a^{(q,r)}(b)}N_a^{(q+r,q)}(b) - N_a^{(q+r,q)}(x).$$

$$h(x;a,b) = \mathbb{E}_x[e^{-q\tau_a^-(r)-\theta R_r(\tau_a^-(r))}; \tau_a^-(r) < \tau_b^+(r)] = \mathcal{I}_a^{(q,r)}(x) - \frac{H_a^{(0,r)}(x)}{H_a^{(0,r)}(b)}\mathcal{I}_a^{(q,r)}(b),$$

and

$$\mathbb{E}_x[e^{-q\tau_b^+(r)}; \tau_b^+(r) < \tau_a^-(r)] = \frac{H_a^{(q,r)}(x)}{H_a^{(q,r)}(b)},$$

where

$$\begin{aligned}\mathcal{I}_a^{(q,r)}(x) &= Z_{-a}^{(q+r,q)}(x-a) - W_{-a}^{(q+r,q)}(x-a)\frac{Z^{(q+r)}(-a)}{W^{(q+r)}(-a)},\\ N_a^{(q,r)}(x) &= \frac{ra\overline{W}^{(q+r)}(-a) + r\int_0^{-a}\overline{W}^{(q+r)}(y)dy}{W^{(q+r)}(-a)}W_{-a}^{(q+r,q)}(x-a)\\ &\quad -a\frac{r}{q+r}Z_{-a}^{(q+r,q)}(x-a) - \frac{r}{q+r}\int_a^0 Z_{-y}^{(q+r,q)}(x-y)dy,\end{aligned}$$

with $N_a^{(q,r)}(0) = 0$.

Theorem 3. *For $q \ge 0$, $x > a, a < 0 < b$, denote by $g(x;a,b) = \mathbb{E}_x[\int_0^{\hat{\tau}_a^-} e^{-qt}dR_t^{r,b}]$. Then for $x > a$,*

$$\begin{aligned}g(x;a,b) &= \frac{\mathcal{L}f(b,0) - B_a^{(q,r)\prime}(b,0)}{\mathcal{M}_{a,b}^{(q,r)}(b,0,0)}\left(H_a^{(q,r)}(b) - \mathcal{M}_{a,b}^{(q,r)}(x,0,0)\right)\\ &\quad +\mathcal{L}f(x,0) - B_a^{(q,r)\prime}(x,0).\end{aligned} \tag{17}$$

Proof. For $0 \le x < b$, by the strong Markov property and Lemma 2, one obtains

$$\begin{aligned}g(x;a,b) &= f(x;a,b) + \mathbb{E}_x[e^{-q\tau_b^+(r)}; \tau_b^+(r) < \tau_a^-(r)]g(b;a,b)\\ &= f(x;a,b) + \frac{H_a^{(q,r)}(x)}{H_a^{(q,r)}(b)}g(b;a,b),\end{aligned} \tag{18}$$

Let $x \to 0^+$ in (18), then

$$g(0;a,b) = f(0;a,b) + \frac{g(b;a,b)}{H_a^{(q,r)}(b)}. \tag{19}$$

For $x > a$, conditioning on the first observation time, one obtains

$$\begin{aligned}g(x;a,b) &= \mathbb{E}_x[e^{-qe_r}; e_r < \tau_a^-, X_{e_r} > b]g(b;a,b)\\ &\quad + \int_0^b \mathbb{E}_x[e^{-qe_r}; e_r < \tau_a^-, X_{e_r} \in dy]\left(f(y;a,b) + \frac{H_a^{(q,r)}(y,1)}{H_a^{(q,r)}(b)}g(b;a,b)\right)\\ &\quad +\mathbb{E}_x[e^{-qe_r}(-X_{e_r} + g(0;a,b)); e_r < \tau_a^-, a < X_{e_r} < 0]\\ &= A_{a,b}^{(q,r)}(x,0)g(b;a,b)\\ &\quad + \int_0^b \gamma_a^{(q+r)}(x,y)\left(f(y;a,b) + \frac{H_a^{(q,r)}(y,1)}{H_a^{(q,r)}(b)}g(b;a,b)\right)dy\\ &\quad -B_a^{(q,r)\prime}(x,0) + B_a^{(q,r)}(x,0)g(0;a,b).\end{aligned} \tag{20}$$

Letting $x=b$ in (20), then substituting (19) into (20), we have

$$g(b;a,b)=\frac{H_a^{(q,r)}(b)\left(\mathcal{L}f(b,0)-B_a^{(q,r)'}(b,0)\right)}{\mathcal{M}_{a,b}^{(q,r)}(b,0,0,1,1)}. \tag{21}$$

Substituting (21) into (20) leads to (17).

□

Remark 3. If $b\to\infty$, then we have $g(x;a,\infty)=f(x;a,\infty)$.

Theorem 4. *For $a<0<b, x>a$, let $\hat{h}(x;a,b)=\mathbb{E}_x[e^{-q\hat{\tau}_a^-(r)-\theta R_r(\hat{\tau}_a^-(r))}]$. Then*

$$\begin{aligned}\hat{h}(x;a,b) \quad = \quad & \frac{\mathcal{L}h(b,\theta)+Z^{(q+r)}(b-a)-\frac{q+r}{\Phi_{q+r}}W^{(q+r)}(b-a)}{\mathcal{M}_{a,b}^{(q,r)}(b,0,\theta)}(H_a^{(q,r)}(b)-\mathcal{M}_{a,b}^{q,r}(x,0,\theta)) \\ & +\mathcal{L}h(x,\theta)+Z^{(q+r)}(x-a)-\frac{q+r}{\Phi_{q+r}}W^{(q+r)}(x-a). \end{aligned} \tag{22}$$

Proof. For $a<x<b$, by the strong Markov property, we have

$$\hat{h}(x;a,b)=h(x;a,b)+\frac{H_a^{(q,r)}(x)}{H_a^{(q,r)}(b)}\hat{h}(b;a,b). \tag{23}$$

Specially,

$$\hat{h}(0;a,b)=h(0;a,b)+\frac{1}{H_a^{(q,r)}(b)}\hat{h}(b;a,b). \tag{24}$$

For $x>a$, conditioning on the first observation time, we have

$$\begin{aligned}\hat{h}(x;a,b) \quad = \quad & \mathbb{E}_x[e^{qe_r};e_r<\tau_a^-,X_{e_r}>b]\hat{h}(b;a,b) \\ & +\int_0^b \gamma_a^{(q+r)}(x,y)[h(y;a,b)+\frac{H_a^{(q,r)}(y)}{H_a^{(q,r)}(b)}\hat{h}(b;a,b)]dy \\ & +\int_a^0 e^{\theta y}\gamma_a^{(q+r)}(x,y)\hat{h}(0;a,b)dy+\mathbb{E}_x[e^{-(q+r)\tau_a^-}] \end{aligned} \tag{25}$$

Inserting (24) into (25), using (6) and letting $x=b$ in (25), one obtains

$$\hat{h}(b;a,b)=\frac{\mathcal{L}h(b,\theta)+Z^{(q+r)}(b-a)-\frac{q+r}{\Phi_{q+r}}W^{(q+r)}(b-a)}{\mathcal{M}_{a,b}^{(q,r)}(b,0,\theta)}H_a^{(q,r)}(b). \tag{26}$$

By (24), (25) and (26), (22) is obtained.

□

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School of Statistics, Qufu Normal University, Qufu 273165, China.
Email: sddh1978@126.com, qfzxh@163.com