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Structured condition numbers and statistical condition estimation for the LDU factorization

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Abstract. In this article, we consider the structured condition numbers for LDU, factorization by using the modified matrix-vector approach and the differential calculus, which can be represented by sets of parameters. By setting the specific norms and weight parameters, we present the expressions of the structured normwise, mixed, componentwise condition numbers and the corresponding results for unstructured ones. In addition, we investigate the statistical estimation of condition numbers of LDU factorization using the probabilistic spectral norm estimator and the small-sample statistical condition estimation method, and devise three algorithms. Finally, we compare the structured condition numbers with the corresponding unstructured ones in numerical experiments.

§1 Introduction

As a real $n \times n$ matrix A whose first $n-1$ leading principal submarines are all nonsingular there exists a unique unit lower triangular matrix L, unit upper triangular matrix U and diagonal matrix D such that $A \in \mathbb{R}^{n\times n}$ have the following unique LDU factorization

$$A = LDU, \tag{1.1}$$

Since L, D and U in (1.1) are uniquely determined by A. where L is a unit lower triangular and U is a unit upper triangular and D is diagonal matrix. The difference between LDU and LU factorizations in upper triangular matrix U , i.e. U is unit upper triangular matrix in LDU factorization.

The LDU factorization is one of the most important matrix factorizations and has many applications, such as solving systems of linear equations, inverting matrices, and computing determinants [1,2]. The componentwise perturbation bounds were first discussed by Galántai [3]. Later, the acquired first-order bounds for LDU factorization were enhanced by Wenjun

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[4]. The obtained bounds [4] are optimal, which leads to the normwise condition numbers for LDU factorization. The structured perturbation theory for the LDU factorization of diagonally dominant matrices was presented by Dopico and Koev [5] and later it was extended by Dailey et. al [6].

It is necessary to mention that the systematic theory for normwise condition number was first given by Rice [7] and the terminologies of mixed and componentwise condition numbers were first introduced by Gohberg and Koltracht [8]. The normwise condition numbers for LU, Cholesky, and QR factorizations can be found in [9-11]. As we know that the normwise condition numbers may overestimate the illness of problem because they ignore the structure of coefficient matrices with respect to sparsity or scaling. To tackle this drawback some researchers have paid attention to the mixed and componentwise condition numbers for the above three matrix factorizations; see [12, 13]. Considering the applications in structured algorithms of matrix factorizations and structured problems involved with matrix factorizations [14], some scholars investigated the structured condition numbers for the above three matrix factorizations; see [14-18] and the references therein.

In this paper, we continue the research of structured condition numbers for LDU factorization. However, to our best knowledge, there is no work on structured condition numbers of LDU factorization so far. Specifically, we will discuss the structured condition numbers for LDU factorization, whose explicit expressions are given in Section 3. Meanwhile, in this section, we also discuss how to recover the expressions of structured normwise, mixed and componentwise condition numbers for LDU factorization and the corresponding results for unstructured ones. Statistical condition estimation is also applied to this factorization which can be figured effectively in Section 4. In addition, Section 2 provides some useful notation and preliminaries and Section 5 presents some numerical examples to show the obtained results.

§2 Notations and preliminaries

Throughout this paper, we let $\mathbb{R}^{m\times n}$ be the set of $m \times n$ real matrices and $\mathbb{R}_r^{m\times n}$ be the subset of $\mathbb{R}^{m\times n}$ consisting of matrices with rank r. Accordingly, $\mathbb{R}^m$ denotes the vector space of dimension m. For the matrix $A = [\alpha_1, \alpha_2, \cdots, \alpha_n] = (a_{ij}) \in \mathbb{R}^{n\times n}$, we denote the vector of the first i entries of α_j by $\alpha_j^{(i)}$ and the vector of the last i entries of α_j by $\alpha_j^{[i]}$. With these, we adopt the following operators defined in [19],

$$\mathrm{suvec}(A) := \begin{bmatrix} \alpha_2^{[1]} \\ \alpha_3^{[2]} \\ \vdots \\ \alpha_n^{[n-1]} \end{bmatrix} \in \mathbb{R}^{\tau_1}, \ \mathrm{slvec}(A) := \begin{bmatrix} \alpha_1^{[n-1]} \\ \alpha_2^{[n-2]} \\ \vdots \\ \alpha_{n-1}^{[1]} \end{bmatrix} \in \mathbb{R}^{\tau_1},$$

$$\mathrm{dgvec}(A) := \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \vdots \\ \alpha_{nn} \end{bmatrix} \in \mathbb{R}^{\tau_2},\ \mathrm{vec}(A) := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^{n^2},$$

$$\mathrm{dg}(A) := \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix},\ \mathrm{ut}(A) := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix},$$

where $\tau_1 = n(n-1)/2$ and $\tau_2 = n(n+1)/2$ and $\mathrm{slt}(A) := A - \mathrm{ut}(A)$, $\mathrm{sut}(A) := \mathrm{slt}(A')'$. Considering the structures of these operators, we have

$$\mathrm{suvec}(A) = M_{\mathrm{suvec}}\mathrm{vec}(A),\ \mathrm{slvec}(A) = M_{\mathrm{slvec}}\mathrm{vec}(A),\ \mathrm{dgvec}(A) = M_{\mathrm{dgvec}}\mathrm{vec}(A) \tag{2.1}$$

and

$$\mathrm{vec}(\mathrm{dg}(A)) = M_{\mathrm{dg}}\mathrm{vec}(A),\ \mathrm{vec}(\mathrm{sut}(A)) = M_{\mathrm{sut}}\mathrm{vec}(A),\ \mathrm{vec}(\mathrm{slt}(A)) = M_{\mathrm{slt}}\mathrm{vec}(A), \tag{2.2}$$

where

$$\begin{aligned}
&M_{\mathrm{suvec}} = [0_{\tau_1\times n}, \mathrm{diag}\,(J_2, J_3 \cdots, J_n)] \in \mathbb{R}^{\tau_1\times n^2},\ J_i = \left[I_{i-1},\ 0_{i-1\times n-(i-1)}\right] \in \mathbb{R}^{(i-1)\times n},\\
&M_{\mathrm{slvec}} = \left[\mathrm{diag}\left(\widetilde{J}_1, \widetilde{J}_2, \cdots, \widetilde{J}_{n-1}\right),\ 0_{\tau_1\times n}\right] \in \mathbb{R}^{\tau_1\times n^2},\ \widetilde{J}_i = \left[0_{(n-i)\times i},\ I_{n-i}\right] \in \mathbb{R}^{(n-i)\times n},\\
&M_{\mathrm{dgvec}} = \left(\mathrm{diag}\left(\widehat{J}_1, \widehat{J}_2, \cdots, \widehat{J}_{n-1}\right),\ I_{n\times n}\right) \in \mathbb{R}^{\tau_2\times n^2},\ \widehat{J}_i = [I_i,\ 0_{i\times i}] \in \mathbb{R}^{i\times 2i},\\
&M_{\mathrm{dg}} = \mathrm{diag}\,(S_1, S_2, \cdots, S_{n-1}, I_{n\times n}) \in \mathbb{R}^{n^2\times n^2},\ S_i = \mathrm{diag}\,(I_i, 0_{i\times i}) \in \mathbb{R}^{n\times n},\\
&M_{\mathrm{sut}} = \mathrm{diag}\left(0_{n\times n}, \widetilde{S}_2, \widetilde{S}_3, \cdots, \widetilde{S}_n\right) \in \mathbb{R}^{n^2\times n^2},\ \widetilde{S}_i = \mathrm{diag}\left(I_i, 0_{(n-i)\times(n-i)}\right) \in \mathbb{R}^{n\times n},\\
&M_{\mathrm{slt}} = \mathrm{diag}\left(\widehat{S}_1, \widehat{S}_2, \cdots, \widehat{S}_{n-1}, 0_{n\times n}\right) \in \mathbb{R}^{n^2\times n^2},\ \widehat{S}_i = \mathrm{diag}(0_{i\times i}, I_{n-i}) \in \mathbb{R}^{n\times n}.
\end{aligned}$$

Here, I_r denotes the identity matrix of order r and $0_{s\times t}$ is the $s \times t$ zero matrix. It is easy to verify that

$$M_{\mathrm{suvec}}M_{\mathrm{suvec}}^T = I_{\tau_1},\ M_{\mathrm{slvec}}M_{\mathrm{slvec}}^T = I_{\tau_1},\ M_{\mathrm{dgvec}}M_{\mathrm{dgvec}}^T = I_{\tau_2} \tag{2.3}$$

and

$$M_{\mathrm{suvec}}^T M_{\mathrm{suvec}} = M_{\mathrm{sut}},\ M_{\mathrm{slvec}}^T M_{\mathrm{slvec}} = M_{\mathrm{slt}},\ M_{\mathrm{dgvec}}^T M_{\mathrm{dgvec}} = M_{\mathrm{dg}}. \tag{2.4}$$

As a result,

$$\begin{aligned}
\mathrm{vec}(\mathrm{sut}(A)) &= M_{\mathrm{suvec}}^T\mathrm{suvec}(A),\\
\mathrm{vec}(\mathrm{slt}(A)) &= M_{\mathrm{slvec}}^T\mathrm{slvec}(A),\\
\mathrm{vec}(\mathrm{dg}(A)) &= M_{\mathrm{dgvec}}^T\mathrm{dgvec}(A).
\end{aligned} \tag{2.5}$$

For the vectors $\alpha \in \mathbb{R}^p$ and $\beta = [b_1, b_2, \cdots, b_p]^T \in \mathbb{R}^p$, following [20], we define the entry-wise division between α and β by

$$\frac{\alpha}{\beta} = \mathrm{diag}^{\ddagger}(\beta)\alpha, \tag{2.6}$$

where $\text{diag}^{\ddagger}(\beta)$ is diagonal with diagonal entries $b_1^{\ddagger}, b_2^{\ddagger}, \cdots, b_p^{\ddagger}$. Here, for a number $c \in \mathbb{R}$, $c^{\ddagger}$ is defined by

$$c^{\ddagger} = \begin{cases} \frac{1}{c}, & c \neq 0, \\ 1, & c = 0 \ . \end{cases}$$

Using (2.6), we now define a new condition number.

Definition 2.1. *Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous mapping defined on an open set $Dom(F) \in \mathbb{R}^p$, the domain of definition of F. Then the condition number of F at $x \in Dom(F)$ is defined by*

$$\kappa_F(x) \quad = \quad \lim_{\delta \to 0} \sup_{0 < \left\| \frac{\Delta x}{\beta} \right\|_\mu \leq \delta} \frac{\left\| \frac{F(x+\Delta x) - F(x)}{\xi} \right\|_\nu}{\left\| \frac{\Delta x}{\beta} \right\|_\mu},$$

where $\| \cdot \|_\mu$ and $\| \cdot \|_\nu$ are the vector norms defined on $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, and $\beta \in \mathbb{R}^p$ and $\xi \in \mathbb{R}^q$ are parameters with a requirement that if some entry of β is zero, then the corresponding entry of Δx must be zero.

When the mapping F in Definition 2.1 is Fréchet differentiable, the following lemma gives an easily computable form of the unified condition number $\kappa_F(x)$.

Lemma 2.2. *(see [18]) Assume that the mapping F in Definition 2.1 is Fréchet differentiable. Then*

$$\kappa_F(x) \quad = \quad \left\| \text{diag}^{\ddagger}(\xi) DF(x) \text{diag}(\beta) \right\|_{\mu,\nu}, \tag{2.7}$$

where $DF(x)$ is the Fréchet derivative of F at x, $\text{diag}(\beta)$ is a diagonal matrix with entries b_i on the diagonal, and $\| \cdot \|_{\mu,\nu}$ is the induced matrix norm by the vector norms $\| \cdot \|_\mu$ and $\| \cdot \|_\nu$.

To obtain the Fréchet derivative, we need the well-known Kronecker product [21] which is denoted by $A \otimes B$ with $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$. From [21], we have the following equalities

$$\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X), \tag{2.8}$$

$$\text{vec}(A^T) = \Pi_{mn}\text{vec}(A), \tag{2.9}$$

$$\Pi_{pm}(A \otimes B) = (B \otimes A)\Pi_{qn}, \tag{2.10}$$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \tag{2.11}$$

where $X \in \mathbb{R}^{n \times p}$, $\Pi_{st} \in \mathbb{R}^{st \times st}$ is the vec-permutation matrix depending only on the dimensions s and t, and the matrices C and D are of suitable orders. In addition, from [21], we also have that if A and B are nonsingular, then $A \otimes B$ is also nonsingular and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{2.12}$$

§3 Structured condition numbers for LDU factorization

In this section, we assume that the entries of the matrix A are the differentiable functions of a set parameters $\Omega = [\omega_1, \omega_2, \cdots, \omega_s]^T \in \mathbb{R}^s$ and denote the matrix by $A(\Omega)$. For LDU

factorization (1.1), we first define the following mapping

$$\varphi_L : \Omega \to \mathrm{slvec}(L), \quad \varphi_D : \Omega \to \mathrm{dgvec}(D), \quad \varphi_U : \Omega \to \mathrm{suvec}(U).$$

In the following, we present the Fréchet derivatives of φ_L, φ_D and φ_U at Ω, from which we can obtain the unified structured condition numbers for LDU factorization.

Theorem 3.1. *Let the unique LDU factorization of $A(\Omega) \in \mathbb{R}_n^{n\times n}$ be as in* (1.1)*. Then the Fréchet derivatives of φ_L, φ_D and φ_U at Ω are given respectively by*

$$D\varphi_L(\Omega) = M_L \frac{\partial A(\Omega)}{\partial \Omega}, \ D\varphi_D(\Omega) = M_D \frac{\partial A(\Omega)}{\partial \Omega}, \ D\varphi_U(\Omega) = M_U \frac{\partial A(\Omega)}{\partial \Omega}. \tag{3.1}$$

where

$$\begin{aligned} M_L &= M_{\mathrm{slvec}}(D^{-T} \otimes L) M_{\mathrm{slt}}(U^{-T} \otimes L^{-1}), \\ M_D &= M_{\mathrm{dgvec}}(I^T \otimes I) M_{\mathrm{dg}}(U^{-T} \otimes L^{-1}), \\ M_U &= M_{\mathrm{suvec}}(U^{-T} \otimes D) M_{\mathrm{sut}}(U^{-T} \otimes L^{-1}). \end{aligned} \tag{3.2}$$

Proof. Differentiating the equation (1.1) with respect to ω_i $(1 \leqslant i \leqslant s)$ gives

$$\frac{\partial A(\Omega)}{\partial \omega_i} = \frac{\partial L}{\partial \omega_i} DU + L \frac{\partial D}{\partial \omega_i} U + LD \frac{\partial U}{\partial \omega_i}.$$

Premultiplying the above equation by L^{-1} and postmultiplying it by U^{-1}, we have

$$L^{-1} \frac{\partial L}{\partial \omega_i} D + \frac{\partial D}{\partial \omega_i} + D \frac{\partial U}{\partial \omega_i} U^{-1} = L^{-1} \frac{\partial A(\Omega)}{\partial \omega_i} U^{-1}.$$

Note that the diagonal entries of $\frac{\partial L}{\partial \omega_i}$ are zero, and so are the diagonal entries of $L^{-1} \frac{\partial L}{\partial \omega_i}$. Thus, using the operators 'slt' ,'sut' and 'dg' defined in Section 2, we obtain

$$L^{-1} \frac{\partial L}{\partial \omega_i} D = \mathrm{slt}\left(L^{-1} \frac{\partial A(\Omega)}{\partial \omega_i} U^{-1} \right), \tag{3.3}$$

$$\frac{\partial D}{\partial \omega_i} = \mathrm{dg}\left(L^{-1} \frac{\partial A(\Omega)}{\partial \omega_i} U^{-1} \right), \tag{3.4}$$

$$D \frac{\partial U}{\partial \omega_i} U^{-1} = \mathrm{sut}\left(L^{-1} \frac{\partial A(\Omega)}{\partial \omega_i} U^{-1} \right). \tag{3.5}$$

Applying the operator 'vec' to (3.3), (3.4) and (3.5) and using (2.2) and (2.8) implies

$$(D^T \otimes L^{-1}) \mathrm{vec}\left(\frac{\partial L}{\partial \omega_i} \right) = M_{\mathrm{slt}}(U^{-T} \otimes L^{-1}) \mathrm{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i} \right), \tag{3.6}$$

$$(I^T \otimes I) \mathrm{vec}\left(\frac{\partial D}{\partial \omega_i} \right) = M_{\mathrm{dg}}(U^{-T} \otimes L^{-1}) \mathrm{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i} \right), \tag{3.7}$$

$$(U^{-T} \otimes D) \mathrm{vec}\left(\frac{\partial U}{\partial \omega_i} \right) = M_{\mathrm{sut}}(U^{-T} \otimes L^{-1}) \mathrm{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i} \right). \tag{3.8}$$

Thus, multiplying (3.6), (3.7) and (3.8) from the left side by $D^{-T} \otimes L$, $I^T \otimes I$ and $U^T \otimes D$, respectively, and noting (2.11) and (2.12) leads to

$$\operatorname{vec}\left(\frac{\partial L}{\partial \omega_i}\right) = (D^{-T}\otimes L)M_{\mathrm{slt}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right), \tag{3.9}$$

$$\operatorname{vec}\left(\frac{\partial D}{\partial \omega_i}\right) = (I^{T}\otimes I)M_{\mathrm{dg}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right), \tag{3.10}$$

$$\operatorname{vec}\left(\frac{\partial U}{\partial \omega_i}\right) = (U^{T}\otimes D)M_{\mathrm{sut}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right). \tag{3.11}$$

Considering (2.5) and (2.3), we have

$$\operatorname{slvec}\left(\frac{\partial L}{\partial \omega_i}\right) = M_{\mathrm{slvec}}(D^{-T}\otimes L)M_{\mathrm{slt}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right), \tag{3.12}$$

$$\operatorname{dgvec}\left(\frac{\partial D}{\partial \omega_i}\right) = M_{\mathrm{dgvec}}(I^{T}\otimes I)M_{\mathrm{dg}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right), \tag{3.13}$$

$$\operatorname{suvec}\left(\frac{\partial U}{\partial \omega_i}\right) = M_{\mathrm{suvec}}(U^{T}\otimes D)M_{\mathrm{sut}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right). \tag{3.14}$$

Further, premultiplying (3.12), (3.13) and (3.14) by M_{slvec}^T, M_{dgvec}^T and M_{suvec}^T, respectively, and using (2.5) and (2.4) yields

$$\operatorname{vec}\left(\frac{\partial L}{\partial \omega_i}\right) = M_{\mathrm{slt}}(D^{-T}\otimes L)M_{\mathrm{slt}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right),$$

$$\operatorname{vec}\left(\frac{\partial D}{\partial \omega_i}\right) = M_{\mathrm{dg}}(I^{T}\otimes I)M_{\mathrm{dg}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right),$$

$$\operatorname{vec}\left(\frac{\partial U}{\partial \omega_i}\right) = M_{\mathrm{sut}}(U^{T}\otimes D)M_{\mathrm{sut}}(U^{-T}\otimes L^{-1})\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right).$$

From the structures of M_{slt} and M_{sut}, we can verify that $M_{\mathrm{slt}}(D\otimes L)M_{\mathrm{slt}} = (D\otimes L)M_{\mathrm{slt}}$ and $M_{\mathrm{sut}}(U^T\otimes D)M_{\mathrm{sut}} = (U^T\otimes D)M_{\mathrm{sut}}$. Consequently, (3.9) is equivalent to (3.12), (3.10) is equivalent to (3.13) and (3.11) is equivalent to (3.14).

Note that

$$\operatorname{slvec}(\Delta L) = \operatorname{slvec}(L(\Omega+\Delta\Omega)-L(\Omega)) = \sum_{i=1}^{s}\operatorname{slvec}\left(\frac{\partial L}{\partial \omega_i}\right)\delta\omega_i + (\text{h.o.t}),$$

$$\operatorname{dgvec}(\Delta D) = \operatorname{dgvec}(D(\Omega+\Delta\Omega)-D(\Omega)) = \sum_{i=1}^{s}\operatorname{dgvec}\left(\frac{\partial D}{\partial \omega_i}\right)\delta\omega_i + (\text{h.o.t}),$$

$$\operatorname{suvec}(\Delta U) = \operatorname{suvec}(U(\Omega+\Delta\Omega)-U(\Omega)) = \sum_{i=1}^{s}\operatorname{suvec}\left(\frac{\partial U}{\partial \omega_i}\right)\delta\omega_i + (\text{h.o.t}),$$

where $\Delta\Omega = [\delta\omega_1, \delta\omega_2, \cdots, \delta\omega_s]^T$ and (h.o.t) is the abbreviation of 'higher order terms'. Then

$$\operatorname{slvec}(\Delta L) = M_{\mathrm{slvec}}(D^{-T}\otimes L)M_{\mathrm{slt}}(U^{-T}\otimes L^{-1})\sum_{i=1}^{s}\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right)\delta\omega_i + (\text{h.o.t}),$$

$$\operatorname{dgvec}(\Delta D) = M_{\mathrm{dgvec}}(I^{T}\otimes I)M_{\mathrm{dg}}(U^{-T}\otimes L^{-1})\sum_{i=1}^{s}\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right)\delta\omega_i + (\text{h.o.t}),$$

$$\operatorname{suvec}(\Delta U) = M_{\mathrm{suvec}}(U^{T}\otimes D)M_{\mathrm{sut}}(U^{-T}\otimes L^{-1})\sum_{i=1}^{s}\operatorname{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_i}\right)\delta\omega_i + (\text{h.o.t}).$$

Set

$$\frac{\partial A(\Omega)}{\partial \Omega} = \left[\mathrm{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_1}\right), \mathrm{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_2}\right), \cdots, \mathrm{vec}\left(\frac{\partial A(\Omega)}{\partial \omega_s}\right)\right].$$

Thus,

$$\mathrm{slvec}\,(\Delta L) = M_{\mathrm{slvec}}(D^{-T} \otimes L) M_{\mathrm{slt}}(U^{-T} \otimes L^{-1}) \frac{\partial A(\Omega)}{\partial \Omega} \Delta\Omega + (\mathrm{h.o.t}), \tag{3.15}$$

$$\mathrm{dgvec}\,(\Delta D) = M_{\mathrm{dgvec}}(I^{T} \otimes I) M_{\mathrm{dg}}(U^{-T} \otimes L^{-1}) \frac{\partial A(\Omega)}{\partial \Omega} \Delta\Omega + (\mathrm{h.o.t}), \tag{3.16}$$

$$\mathrm{suvec}\,(\Delta U) = M_{\mathrm{suvec}}(U^{T} \otimes D) M_{\mathrm{sut}}(U^{-T} \otimes L^{-1}) \frac{\partial A(\Omega)}{\partial \Omega} \Delta\Omega + (\mathrm{h.o.t}). \tag{3.17}$$

From (3.15), (3.16) and (3.17) and the definitions of the mappings φ_L, φ_D and φ_U and the Fréchet derivative, we have desired results. □

Now we present the unified structured condition numbers for LDU factorization.

Theorem 3.2. *Under the same assumptions of Theorem* 3.1*, we have*

$$\kappa_L(\Omega) = \left\| \mathrm{diag}^{\ddagger}(\xi) M_L \frac{\partial A(\Omega)}{\partial \Omega} \mathrm{diag}(\beta) \right\|_{\mu,\nu}, \tag{3.18}$$

$$\kappa_D(\Omega) = \left\| \mathrm{diag}^{\ddagger}(\xi) M_D \frac{\partial A(\Omega)}{\partial \Omega} \mathrm{diag}(\beta) \right\|_{\mu,\nu}, \tag{3.19}$$

$$\kappa_U(\Omega) = \left\| \mathrm{diag}^{\ddagger}(\xi) M_U \frac{\partial A(\Omega)}{\partial \Omega} \mathrm{diag}(\beta) \right\|_{\mu,\nu}, \tag{3.20}$$

where ξ and β are parameter vectors with suitable dimensions and β has a requirement like the one in Definition 2.1.

Proof. The proof is straightforward by considering Lemma 2.2 and Theorem 3.1. □

Remark 3.3. When we set the parameters in Ω to be the entries of A, we can deduce the unstructured unified condition numbers for LDU factorization:

$$\kappa_L(A) = \left\| \mathrm{diag}^{\ddagger}(\xi) M_L \mathrm{diag}(\beta) \right\|_{\mu,\nu}, \tag{3.21}$$

$$\kappa_D(A) = \left\| \mathrm{diag}^{\ddagger}(\xi) M_D \mathrm{diag}(\beta) \right\|_{\mu,\nu}, \tag{3.22}$$

$$\kappa_U(A) = \left\| \mathrm{diag}^{\ddagger}(\xi) M_U \mathrm{diag}(\beta) \right\|_{\mu,\nu}, \tag{3.23}$$

because, in this case, it is easy to check that $\frac{\partial A(\Omega)}{\partial \Omega} = I_{n^2}$.

Remark 3.4. Setting $\mu = \nu = 2$, and $\beta = [\|\Omega\|_2, \cdots, \|\Omega\|_2]^T \in \mathbb{R}^s$ with $\Omega \neq 0$ and

$$\xi = [\|\mathrm{slvec}(L)\|_2, \cdots, \|\mathrm{slvec}(L)\|_2]^T = [\|L\|_F, \cdots, \|L\|_F]^T \in \mathbb{R}^{\tau_1},$$

$$\xi = [\|\mathrm{dgvec}(D)\|_2, \cdots, \|\mathrm{dgvec}(D)\|_2]^T = [\|D\|_F, \cdots, \|D\|_F]^T \in \mathbb{R}^{\tau_2},$$

$$\xi = [\|\mathrm{suvec}(U)\|_2, \cdots, \|\mathrm{suvec}(U)\|_2]^T = [\|U\|_F, \cdots, \|U\|_F]^T \in \mathbb{R}^{\tau_1}.$$

We obtain the structured normwise condition number for the LDU factors L, D and U;

$$\kappa_{2L}(\Omega) = \left\| M_L \frac{\partial A(\Omega)}{\partial \Omega} \right\|_2 \frac{\|\Omega\|_2}{\|L\|_F},$$
$$\kappa_{2D}(\Omega) = \left\| M_D \frac{\partial A(\Omega)}{\partial \Omega} \right\|_2 \frac{\|\Omega\|_2}{\|D\|_F},$$
$$\kappa_{2U}(\Omega) = \left\| M_U \frac{\partial A(\Omega)}{\partial \Omega} \right\|_2 \frac{\|\Omega\|_2}{\|U\|_F}.$$

Setting $\mu = \nu = \infty$, and $\beta = \Omega \neq 0$ and $\xi = [\|\text{slvec}(L)\|_\infty, \cdots, \|\text{slvec}(L)\|_\infty]^T \in \mathbb{R}^{\tau_1}$ ($\xi = \text{slvec}(L)$), $\xi = [\|\text{dgvec}(D)\|_\infty, \cdots, \|\text{dgvec}(D)\|_\infty]^T \in \mathbb{R}^{\tau_2}$ ($\xi = \text{dgvec}(D)$) and $\xi = [\|\text{suvec}(U)\|_\infty, \cdots, \|\text{suvec}(U)\|_\infty]^T \in \mathbb{R}^{\tau_1}$ ($\xi = \text{suvec}(U)$), we obtain the structured mixed (componentwise) condition number for the LDU factors L, D and U;

$$\kappa_{mL}(\Omega) = \frac{\left\| |M_L \frac{\partial A(\Omega)}{\partial \Omega}||\Omega| \right\|_\infty}{\|\text{slvec}(L)\|_\infty}, \ \kappa_{cL}(\Omega) = \left\| \frac{|M_L \frac{\partial A(\Omega)}{\partial \Omega}||\Omega|}{\text{slvec}(|L|)} \right\|_\infty,$$
$$\kappa_{mD}(\Omega) = \frac{\left\| |M_D \frac{\partial A(\Omega)}{\partial \Omega}||\Omega| \right\|_\infty}{\|\text{dgvec}(D)\|_\infty}, \ \kappa_{cD}(\Omega) = \left\| \frac{|M_D \frac{\partial A(\Omega)}{\partial \Omega}||\Omega|}{\text{dgvec}(|D|)} \right\|_\infty,$$
$$\kappa_{mU}(\Omega) = \frac{\left\| |M_U \frac{\partial A(\Omega)}{\partial \Omega}||\Omega| \right\|_\infty}{\|\text{suvec}(U)\|_\infty}, \ \kappa_{cU}(\Omega) = \left\| \frac{|M_U \frac{\partial A(\Omega)}{\partial \Omega}||\Omega|}{\text{suvec}(|U|)} \right\|_\infty.$$

Similarly we can obtain the unstructured condition numbers for LDU factorization by setting specific parameters and norms in (3.21), (3.22), and (3.23), respectively. We have the following unstructured normwise condition number for the LDU factors L, D and U ;

$$\kappa_{2L}(A) = \frac{\|M_L\|_2 \|A\|_F}{\|L\|_F}, \ \kappa_{2D}(A) = \frac{\|M_D\|_2 \|A\|_F}{\|D\|_F}, \ \kappa_{2U}(A) = \frac{\|M_U\|_2 \|A\|_F}{\|U\|_F}, \tag{3.24}$$

and the unstructured mixed (componentwise) condition number for the LDU factors L, D and U;

$$\kappa_{mL}(A) = \frac{\||M_L||\text{vec(A)}|\|_\infty}{\|\text{slvec}(L)\|_\infty}, \ \kappa_{cL}(A) = \left\| \frac{|M_L||\text{vec(A)}|}{\text{slvec}(|L|)} \right\|_\infty,$$
$$\kappa_{mD}(A) = \frac{\||M_D||\text{vec(A)}|\|_\infty}{\|\text{dgvec}(D)\|_\infty}, \ \kappa_{cD}(A) = \left\| \frac{|M_D||\text{vec(A)}|}{\text{dgvec}(|D|)} \right\|_\infty,$$
$$\kappa_{mU}(A) = \frac{\||M_U||\text{vec(A)}|\|_\infty}{\|\text{suvec}(U)\|_\infty}, \ \kappa_{cU}(A) = \left\| \frac{|M_U||\text{vec(A)}|}{\text{suvec}(|U|)} \right\|_\infty. \tag{3.25}$$

Corollary 3.5. *Suppose all the assumptions of of Theorem* 3.1 *holds, and* $A(\Omega) \in \mathbb{R}_n^{n\times n}$ *be a symmetric positive definite matrix, then* $A(\Omega)$ *has unique* LDL^T *factorization, we have*

$$\kappa_L(\Omega) = \left\| \text{diag}^\ddagger(\xi) M_{\text{slvec}}(D^{-T} \otimes L) M_{\text{slt}}(L^{-1} \otimes L^{-1}) \frac{\partial A(\Omega)}{\partial \Omega} \text{diag}(\beta) \right\|_{\mu,\nu}$$
$$\kappa_D(\Omega) = \left\| \text{diag}^\ddagger(\xi) M_{\text{dgvec}}(I^T \otimes I) M_{\text{dg}}(L^{-1} \otimes L^{-1}) \frac{\partial A(\Omega)}{\partial \Omega} \text{diag}(\beta) \right\|_{\mu,\nu},$$

where ξ *and* β *are parameter vectors with suitable dimensions and* β *has a requirement like the one in Definition 2.1.*

Proof. The proof is straightforward by considering Lemma 2.2 and Theorem 3.1. □

Remark 3.6. When we set the parameters in Ω to be the entries of A, we can deduce the unstructured unified condition numbers for LDL^T factorization:

$$\kappa_L(A) = \left\|\mathrm{diag}^{\ddagger}(\xi) M_{\mathrm{slvec}}(D^{-T} \otimes L) M_{\mathrm{slt}}(L^{-1} \otimes L^{-1}) \mathrm{diag}(\beta)\right\|_{\mu,\nu},$$
$$\kappa_D(A) = \left\|\mathrm{diag}^{\ddagger}(\xi) M_{\mathrm{dgvec}}(I^T \otimes I) M_{\mathrm{dg}}(L^{-1} \otimes L^{-1}) \mathrm{diag}(\beta)\right\|_{\mu,\nu},$$

because, in this case, it is easy to check that $\frac{\partial A(\Omega)}{\partial \Omega} = I_{n^2}$.

§4 Statistical condition estimates

In this part, we focus on estimating the normwise, mixed and componentwise condition numbers for LDU factorization.

4.1. Estimating normwise condition number

We use two algorithms to estimate the normwise condition number. The first one is from [22] and has been applied to estimate the normwise condition number for matrix equations [24,25], equality constrained linear least squares problem [26], and K-weighted pseudoinverse $L_K^\dagger$ [27]. The second one is based on the SSCE method [23] and has ever been used for some least squares problems [26-28].

Algorithm 1 Probabilistic condition estimator

Input: ϵ, d (d is the dimension of Krylov space and usually determined by the algorithm itself) and matrix M_L, M_D and M_U in (3.2).

Output: Probabilistic spectral norm estimator of the normwise condition numbers (3.24): $\kappa_{2L}(A)$, $\kappa_{2D}(A)$ and $\kappa_{2U}(A)$.

1. Choose a starting random vector v_0 from $\mathcal{U}(S_{t-1})$ with $t = n^2$, the uniform distribution over unit sphere S_{t-1} in R^t.
2. Compute the guaranteed lower bound α_1 and the probabilistic upper bound α_2 of $\|M_L\|_2$, $\|M_D\|_2$ and $\|M_U\|_2$ by the probabilistic spectral norm estimator [22].
3. Estimate the normwise condition numbers (3.24) by

$$\kappa_{p2L}(A) = \frac{(\alpha_1+\alpha_2)\|A\|_F}{2\|L\|_F}, \quad \kappa_{p2D}(A) = \frac{(\alpha_1+\alpha_2)\|A\|_F}{2\|D\|_F} \text{ and } \kappa_{p2U}(A) = \frac{(\alpha_1+\alpha_2)\|A\|_F}{2\|U\|_F}.$$

4.2. Estimating mixed and componentwise condition numbers

To estimate the mixed and componentwise condition numbers, we need the following SSCE method, which is from [23] and has been applied to many problems (see e.g., [25-27]).

Algorithm 2 SSCE method for the normwise condition number

Input: Sample size k and matrix M_L, M_D and M_U in (3.2).
Output: SSCE estimates of the normwise condition number of LDU factorization: $\kappa_{s2L}(A)$, $\kappa_{s2D}(A)$ and $\kappa_{s2U}(A)$

1. Let $t = n^2$. Generate q random vectors $[z_1, \cdots, z_k] \to Z$ from $\mathcal{U}(S_{t-1})$.
2. Orthonormalize these vectors using the QR factorization $[Z, \sim] = QR(Z)$.
3. For $i = 1, \cdots, k$, compute $\kappa_{i2L}(A)$, $\kappa_{i2D}(A)$ and $\kappa_{i2U}(A)$ by:
$$\kappa_{i2L}(A) = \frac{M_{iL}\|A\|_F}{\|L\|_F},\ \kappa_{i2D}(A) = \frac{M_{iD}\|A\|_F}{\|D\|_F} \text{ and } \kappa_{i2U}(A) = \frac{M_{iU}\|A\|_F}{\|U\|_F}.$$
where
$$\begin{aligned} M_{iL} &= z_i^T M_{\text{slvec}}(D^{-T} \otimes L) M_{\text{slt}}(U^{-T} \otimes L^{-1}) z_i, \\ M_{iD} &= z_i^T M_{\text{dgvec}}(I^T \otimes I) M_{\text{dg}}(U^{-T} \otimes L^{-1}) z_i, \\ M_{iU} &= z_i^T M_{\text{suvec}}(U^{-T} \otimes D) M_{\text{sut}}(U^{-T} \otimes L^{-1}) z_i. \end{aligned}$$
4. Approximate ω_k and ω_n by:
$$\omega_k \approx \sqrt{\frac{2}{\pi(k - \frac{1}{2})}}.$$
5. Estimate the normwise condition numbers (3.24) by:
$$\kappa_{s2L}(A) = \frac{\omega_k}{\omega_n}\sqrt{\sum_{i=1}^{k} \kappa_{i2L}^2(A)},\ \ \kappa_{s2D}(A) = \frac{\omega_k}{\omega_n}\sqrt{\sum_{i=1}^{k} \kappa_{i2D}^2(A)},\ \ \kappa_{s2U}(A) = \frac{\omega_k}{\omega_n}\sqrt{\sum_{i=1}^{k} \kappa_{i2U}^2(A)}.$$

Algorithm 3 SSCE method for the mixed and componentwise condition numbers

Input: Sample size k and matrix M_L, M_D and M_U in (3.2).

Output: SSCE estimates of mixed and componentwise condition numbers of LDU factorization: $\kappa_{mL}(A)$, $\kappa_{cL}(A)$, $\kappa_{mD}(A)$, $\kappa_{cD}(A)$, $\kappa_{mU}(A)$ and $\kappa_{cU}(A)$.

1. Let $t = n^2$. Generate q random vectors $[z_1, \cdots, z_k] \to Z$ from $\mathcal{U}(S_{t-1})$.

2. Orthonormalize these vectors using the QR factorization $[Z, \sim] = QR(Z)$.

3. Compute $u_iL = M_L z_i$, $u_iD = M_D z_i$, $u_iU = M_U z_i$, and estimate the mixed and componentwise condition numbers in (3.25) by

$$\kappa_{smL}(A) = \frac{\|\kappa_{imL}(A)\|_\infty}{\|\mathrm{slvec}(L)\|_\infty}, \quad \kappa_{scL}(A) = \left\|\frac{\kappa_{imL}(A)}{\mathrm{slvec}(L)}\right\|_\infty,$$
$$\kappa_{smD}(A) = \frac{\|\kappa_{imD}(A)\|_\infty}{\|\mathrm{dgvec}(D)\|_\infty}, \quad \kappa_{scD}(A) = \left\|\frac{\kappa_{imD}(A)}{\mathrm{dgvec}(D)}\right\|_\infty,$$
$$\kappa_{smU}(A) = \frac{\|\kappa_{imU}(A)\|_\infty}{\|\mathrm{suvec}(U)\|_\infty}, \quad \kappa_{scU}(A) = \left\|\frac{\kappa_{imU}(A)}{\mathrm{suvec}(U)}\right\|_\infty,$$

where

$$\kappa_{imL}(A) = \frac{\omega_k}{\omega_t}\left|\sum_{i=1}^{k}|u_iL|^2\right|^{\frac{1}{2}}, \quad \kappa_{imD}(A) = \frac{\omega_k}{\omega_t}\left|\sum_{i=1}^{k}|u_iD|^2\right|^{\frac{1}{2}}, \quad \kappa_{imU}(A) = \frac{\omega_k}{\omega_t}\left|\sum_{i=1}^{k}|u_iU|^2\right|^{\frac{1}{2}},$$

and the power and square root operation are performed on each entry of u_i, $i = 1, \cdots, k$.

§5 Numerical experiments

In this section, we first, illustrate the reliability of Algorithms 1, 2, 3 and then compare the structured condition numbers and the unstructured ones. All computations are carried out in MATLAB 2016a.

Example 5.1. The matrices have the form $A = D_1 B D_2$, where $D_1 = \mathrm{diag}(1, d_1, \cdots, d_1^{m-1})$, $D_2 = \mathrm{diag}(1, d_2, \cdots, d_2^{m-1})$ and B is an $n \times n$ random matrix produced by MATLAB function randn. The result for $n = 10$, $d_1, d_2 = 1$ and the same matrix B. For Algorithm 1, we choose the parameters to be $\delta = 0.01$ and $\epsilon = 0.001$. For Algorithms 2 and 3, we set $k = 2$. We define the ratios between exact condition numbers and the corresponding estimated ones as follows:

$$r_{p2L} = \frac{\kappa_{p2L}(A)}{\kappa_{2L}(A)}, \quad r_{p2D} = \frac{\kappa_{p2D}(A)}{\kappa_{2D}(A)}, \quad r_{p2U} = \frac{\kappa_{p2U}(A)}{\kappa_{2U}(A)};$$
$$r_{s2L} = \frac{\kappa_{s2L}(A)}{\kappa_{2L}(A)}, \quad r_{s2D} = \frac{\kappa_{s2D}(A)}{\kappa_{2D}(A)}, \quad r_{s2U} = \frac{\kappa_{s2U}(A)}{\kappa_{2U}(A)};$$
$$r_{mL} = \frac{\kappa_{smL}(A)}{\kappa_{mL}(A)}, \quad r_{mD} = \frac{\kappa_{smD}(A)}{\kappa_{mD}(A)}, \quad r_{mU} = \frac{\kappa_{smU}(A)}{\kappa_{mU}(A)};$$
$$r_{cL} = \frac{\kappa_{scL}(A)}{\kappa_{cL}(A)}, \quad r_{cD} = \frac{\kappa_{scD}(A)}{\kappa_{cD}(A)}, \quad r_{cU} = \frac{\kappa_{scU}(A)}{\kappa_{cU}(A)}.$$

The ratios are displayed in Figures 1 and 2. Among 2000 tests, the ratios in most cases are of order 1, except a few exceptional cases. The average values of $r_{p2L}, r_{p2D}, r_{p2U}, r_{s2L}, r_{s2D}, r_{s2U}$, r_{mL}, r_{mD}, r_{mU}, r_{cL}, r_{cD} and r_{cU} are 1.0002, 1.0001, 1.0003, 1.0952, 1.8665, 1.1761, 1.5270, 1.7779, 1.3427, 1.2680, 1.5661 and 1.6355, respectively. We see that the Probabilistic condition estimator and the small sample statistical method are quite effective for condition numbers estimation.

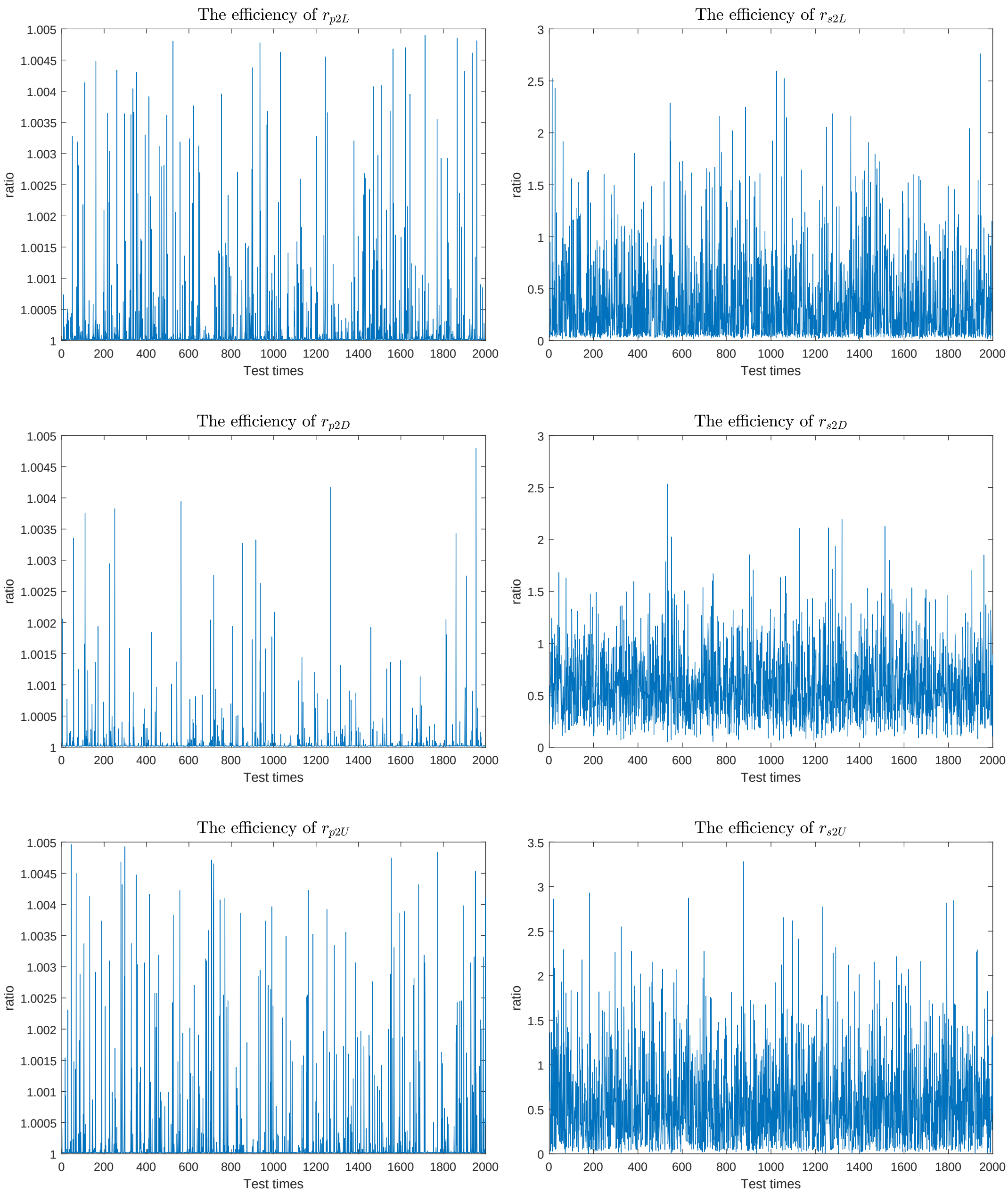

Figure 1: Efficiency of condition estimators of Algorithm 1 and 2.

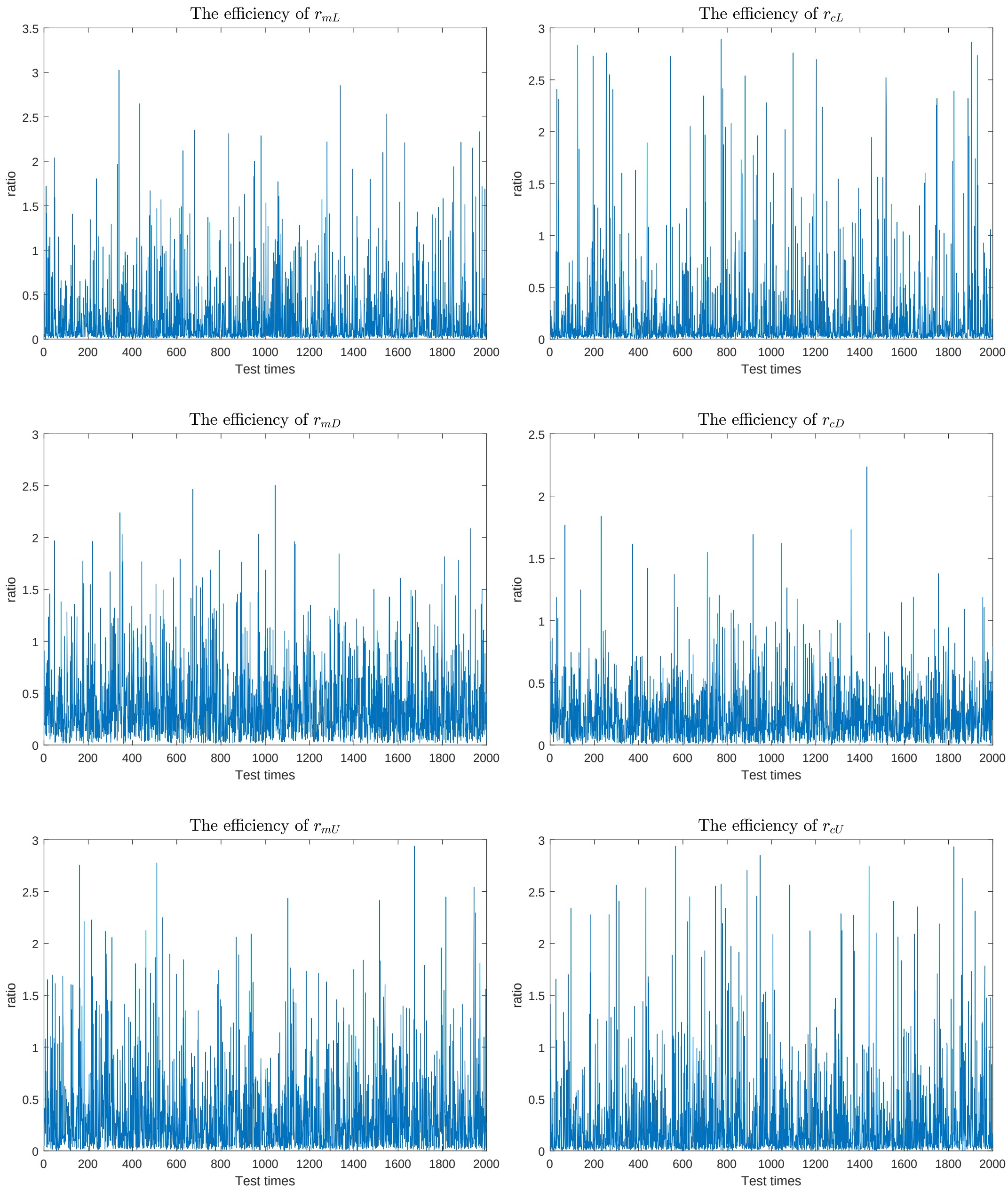

Figure 2: Efficiency of condition estimators of Algorithm 3.

Example 5.2. Consider Toeplitz matrix, we will compare the structured normwise, mixed, and componentwise condition numbers with the corresponding unstructured ones for LDU factorization of linear structured unsymmetric Toeplitz matrix. In the numerical experiments, we generate the test matrices, i.e., the Toeplitz matrices, by the Matlab function $\mathbf{toeplitz}(c, r)$ with $c = \mathbf{randn}(m, 1)$ and $r = \mathbf{randn}(n, 1)$.

In the practical experiments for LDU factorization, we set $c = \mathbf{randn}(n, 1)$ and $r = \mathbf{randn}(n, 1)$, and to make sure that the generated test matrix has the unique LDU factorization, we will check its leading principal sub-matrices. In the specific experiments, we set $n = 10, 20, 30$ and generate 2000 unsymmetric Toeplitz matrices. The numerical results on the ratios defined by

$$\sigma_{2L} = \frac{\kappa_{2L}(A)}{\kappa_{2L}(\Omega)},\ \sigma_{2D} = \frac{\kappa_{2D}(A)}{\kappa_{2D}(\Omega)},\ \sigma_{2U} = \frac{\kappa_{2U}(A)}{\kappa_{2U}(\Omega)};$$
$$\sigma_{mL} = \frac{\kappa_{mL}(A)}{\kappa_{mL}(\Omega)},\ \sigma_{mD} = \frac{\kappa_{mD}(A)}{\kappa_{mD}(\Omega)},\ \sigma_{mU} = \frac{\kappa_{mU}(A)}{\kappa_{mU}(\Omega)};$$
$$\sigma_{cL} = \frac{\kappa_{cL}(A)}{\kappa_{cL}(\Omega)},\ \sigma_{cD} = \frac{\kappa_{cD}(A)}{\kappa_{cD}(\Omega)},\ \sigma_{cU} = \frac{\kappa_{cU}(A)}{\kappa_{cU}(\Omega)}$$

are presented in Table 1, we find that the structured condition numbers are always smaller than the unstructured ones, however, the former is not much smaller than the latter.

Table 1: Comparisons of structured condition numbers and unstructured ones for LDU factorizations of unsymmetric Toeplitz matrices.

n	10		20		30	
	mean	max	mean	max	mean	max
σ_{2L}	1.7610	2.6975	2.1584	3.3916	2.5600	4.4721
σ_{2D}	1.7867	3.2464	2.2773	4.3897	2.6894	6.4786
σ_{2U}	1.7618	2.7497	2.1661	3.3774	2.5538	4.4717
σ_{mL}	1.2253	2.5830	1.4203	2.8633	1.5644	3.0865
σ_{mD}	1.3787	3.3006	1.6223	3.1454	1.8017	4.1569
σ_{mU}	1.2276	2.2803	1.4219	2.9426	1.5655	3.0989
σ_{cL}	1.2945	2.9710	1.5478	3.5392	1.7362	4.0040
σ_{cD}	1.4188	3.3016	1.7376	3.7993	1.9896	6.4452
σ_{cU}	1.2972	3.2643	1.5901	3.2439	1.7423	3.6081

Example 5.3. Now, we investigate the comparisons of condition numbers of non-linear structured matrices. We first consider Vandermonde matrix. In numerical experiments, we set $c = \mathbf{randn}(n, 1)$ and $v_{ij} = c(j)^i$ with $i = 0, 1, \cdots, m-1; j = 0, 1, \cdots, n-1$ to generate the $m \times n$ Vandermonde matrix $V = (v_{ij})$. In the specific experiments for LDU factorization, we set $m = n = 5, 8, 10$. We generate 2000 nonsingular Vandermonde matrices, and report the numerical results on the ratios in Table 2. These results show that, for Vandermonde matrices, the structured condition numbers for LDU factorization can be much smaller than the corresponding unstructured ones, which is very unlike the case for Toeplitz matrices. In a word, for linear structured Toeplitz matrices, these are little differences between the structured condition numbers and the corresponding unstructured ones for LDU factorization. Whereas, the results for non-linear structured Vandermonde are very encouraging.

Table 2: Comparisons of structured condition numbers and unstructured ones for LDU factorizations of Vandermonde matrices.

m, n	$5, 5$		$8, 8$		$10, 10$	
	mean	max	mean	max	mean	max
σ_{2L}	2.5870e+03	3.8770e+06	6.9073e+04	4.9048e+07	8.2419e+06	1.3959e+09
σ_{2D}	2.1502e+01	1.3438e+03	2.6357e+03	1.8443e+06	6.9354e+04	7.3116e+06
σ_{2U}	2.4597e+01	3.2133e+03	6.9219e+03	2.8603e+06	4.2975e+05	4.4017e+07
σ_{mL}	2.1021e+01	1.2118e+05	1.3092e+03	6.5363e+05	5.4926e+04	6.3929e+06
σ_{mD}	9.6708e+00	5.2805e+02	5.4654e+02	5.5244e+05	1.3198e+03	2.1332e+05
σ_{mU}	5.3874e+00	5.9792e+02	1.8341e+02	1.1070e+05	1.7871e+03	1.9244e+05
σ_{cL}	1.1218e+02	4.9423e+04	2.0462e+03	1.2126e+06	5.0450e+04	5.7688e+06
σ_{cD}	1.7428e+01	3.4426e+03	3.5746e+02	2.4314e+05	1.4742e+03	3.1332e+05
σ_{cU}	7.5854e+00	5.4458e+02	2.2005e+02	7.9495e+04	2.5060e+03	2.3320e+05

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