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On the rate of convergence of two generalized Bernstein type operators

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Abstract. In this paper, we introduce the Bézier variant of two new families of generalized Bernstein type operators. We establish a direct approximation by means of the Ditzian-Totik modulus of smoothness and a global approximation theorem in terms of second order modulus of continuity. By means of construction of suitable functions and the method of Bojanic and Cheng, we give the rate of convergence for absolutely continuous functions having a derivative equivalent to a bounded variation function.

§1 Introduction

In the year 1912, Bernstein [1] introduced a sequence of positive linear operators for $f \in C[0,1]$, as

$$B_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x), \quad x \in [0,1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. Then many scholars have done a lot of relevant research. Lorentz [2] gave an exhaustive exposition of main facts about the Bernstein polynomials and discussed some of their applications in analysis. Cheng [3] obtained an estimate for the rate of convergence of B_n for functions of bounded variation in terms of the arithmetic means of the sequence of total variations and proved that the estimate was essentially the best possible at points of continuity. Bojanic [4] investigated the asymptotic behavior of B_n for some absolutely continuous functions having a derivative equivalent to a bounded variation function. King [5] defined a new type of Bernstein operators which preserve x^2. Quantitative estimates were given and compared with estimates of approximation by the class Bernstein polynomials B_n in [5].

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Very recently, Chen et al. [6] introduced a new family of generalized Bernstein operators based on a non-negative parameter $\alpha(0 \leq \alpha \leq 1)$ as follows:

$$T_{n,\alpha}(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}^{(\alpha)}(x), \quad x \in [0,1], \tag{1}$$

where

$$\begin{aligned} p_{1,0}^{(\alpha)}(x) &= 1-x,\ p_{1,1}^{(\alpha)}(x) = x, \\ p_{n,k}^{(\alpha)}(x) &= \left[\tbinom{n-2}{k}(1-\alpha)x + \tbinom{n-2}{k-2}(1-\alpha)(1-x) + \tbinom{n}{k}\alpha x(1-x)\right] x^{k-1}(1-x)^{n-k-1} \end{aligned}$$

for $n \geq 2$ and $\binom{n}{k} = 0 (k > n)$. When $\alpha = 1$, the operators $T_{n,\alpha}$ reduces to the Bernstein operators B_n.

In [6], the authors studied many approximaiton properties of $T_{n,\alpha}$ such as uniform convergence, rate of convergence in terms of modulus of continuity, voronovskaya-type asymptotic formula, and shape preserving properties.

To approximate Lebesgue integrable functions, Mohiuddine et al. [7] introduced the following integral modification of the operators (1):

$$K_{n,\alpha}(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{2}$$

In [7], the uniform convergence of the operators and rate of convergence in local and global sense in terms of first and second order modulus of continuity are studied. In [8-9], Acar et al. introduced α-Bernstein-Durrmeyer operators and genuine α-Bernstein-Durrmeyer operators. They obtained some approximation results, which include local approximation, error estimation in terms of Ditzian-Totik modulus of smoothness.

As everyone knows, the Bézier curve plays an important role in computer aided design and computer graphics. Zeng and Piriou [10] opened up the work of two Bernstein-Bézier type operators for bounded variation functions. Then many scholars [11-14] have done research work in related fields. Acar et al. [15] introduced the Bézier variant of summation integral type operators based on the parameter α. They studied a direct approximation theorem by means of the first order modulus of smoothness and the rate of convergence for absolutely continuous functions having a derivative equivalent to a function of bounded variation. Also, they obtained the quantitative voronovskaja type theorem.

Base on this, we propose the Bézier variant of the operators (1) and (2) in the following way:

$$T_{n,\alpha}^{(\beta)}(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) Q_{n,k,\alpha}^{(\beta)}(x), \quad x \in [0,1], \tag{3}$$

$$K_{n,\alpha}^{(\beta)}(f,x) = (n+1) \sum_{k=0}^{n} Q_{n,k,\alpha}^{(\beta)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{4}$$

where $\beta \geq 1$, $Q_{n,k,\alpha}^{(\beta)}(x) = \left[J_{n,k,\alpha}(x)\right]^{\beta} - \left[J_{n,k+1,\alpha}(x)\right]^{\beta}, J_{n,k,\alpha}(x) = \sum_{j=k}^{n} p_{n,j}^{(\alpha)}(x)$ and $J_{n,n+1,\alpha}(x) = 0$.

Obviously for $\beta = 1$, the operators (3) and (4) reduce to the operators (1) and (2) respectively. Our results extend the work of [6] and [7].

Let

$$R^{(1)}_{n,\alpha,\beta}(x,t)=\begin{cases}\sum_{k\le nt}Q^{(\beta)}_{n,k,\alpha}(x), & 0<t\le 1;\\ 0, & t=0.\end{cases}$$

and

$$R^{(2)}_{n,\alpha,\beta}(x,t)=\sum_{k=0}^{n}(n+1)Q^{(\beta)}_{n,k,\alpha}(x)\chi_k(t),$$

where $\chi_k(t)$ is the characteristic function of the interval $[\frac{k}{n+1},\frac{k+1}{n+1}]$ with respect to $I=[0,1]$. By the Lebesgue-Stieltjes integral representations, we have

$$T^{(\beta)}_{n,\alpha}(f,x)=\int_0^1 f(t)d_tR^{(1)}_{n,\alpha,\beta}(x,t) \tag{5}$$

and

$$K^{(\beta)}_{n,\alpha}(f,x)=\int_0^1 f(t)R^{(2)}_{n,\alpha,\beta}(x,t)dt. \tag{6}$$

The aim of this paper is to establish a direct approximaiton by means of the Ditzian-Totik modulus of smoothness and a global approximation theorem in terms of second order modulus of continuity. Furthermore, the rate of convergence for some absolutely continuous functions having a derivative equivalent to a bounded function is obtained. With regard to the research work related to this topic, we can refer to references [16-25].

§2 Some lemmas

The proof of our results are based on the following lemmas.

Lemma 2.1 ([6]) *For $e_i=t^i, i=0,1,2,$ we have*

$$T_{n,\alpha}(e_0,x)=1,\ \ T_{n,\alpha}(e_1,x)=x,$$
$$T_{n,\alpha}(e_2,x)=x^2+\frac{n+2(1-\alpha)}{n^2}x(1-x).$$

By Lemma 2.1 and Cauchy Schwarz inequality, we get

$$T_{n,\alpha}(t-x,x)=0, \tag{7}$$

$$T_{n,\alpha}((t-x)^2,x)=\frac{n+2(1-\alpha)}{n^2}x(1-x)=\gamma^2_{n\alpha}(x). \tag{8}$$

$$T_{n,\alpha}(|t-x|,x)\le\sqrt{T_{n,\alpha}((t-x)^2,x)}\cdot\sqrt{T_{n,\alpha}(1,x)}=\gamma_{n\alpha}(x). \tag{9}$$

According to (8), it is clear that $\gamma_{n\alpha}(x)$ tends to 0 with the help of $(n+2(1-a))x(1-x)/n^2$ tends to 0 as n tends to ∞.

Lemma 2.2 ([7]) *For $e_i=t^i, i=0,1,2,$ we have*

$$K_{n,\alpha}(e_0,x)=1,\ \ K_{n,\alpha}(e_1,x)=\frac{nx}{n+1}+\frac{1}{2(n+1)},$$
$$K_{n,\alpha}(e_2,x)=\frac{n^2}{(n+1)^2}\left(x^2+\frac{n+2(1-\alpha)}{n^2}x(1-x)\right)+\frac{nx}{(n+1)^2}+\frac{1}{3(n+1)^2}.$$

By Lemma 2.2 and Cauchy Schwarz inequality, we get

$$K_{n,\alpha}(t-x,x)=\frac{1-2x}{2(n+1)}, \tag{10}$$

$$K_{n,\alpha}((t-x)^2,x)=\frac{n+2(1-\alpha)-1}{(n+1)^2}x(1-x)+\frac{1}{3(n+1)^2}=\eta^2_{n\alpha}(x). \tag{11}$$

$$K_{n,\alpha}(|t-x|,x) \leq \sqrt{K_{n,\alpha}((t-x)^2,x)} \cdot \sqrt{K_{n,\alpha}(1,x)} = \eta_{n\alpha}(x). \tag{12}$$

Obviously, $\eta_{n\alpha}(x) \to 0 (n \to \infty)$.

By the definition of the operators $T_{n,\alpha}$ and $K_{n,\alpha}$, combined with Lemma 2.1 and 2.2, we have

Lemma 2.3 *For $f \in C[0,1]$, $x \in [0,1]$, the following inequalities hold*

$$\|T_{n,\alpha}(f)\| \leq \|f\|,$$
$$\|K_{n,\alpha}(f)\| \leq \|f\|.$$

We omit the proof of Lemma 2.3.

Lemma 2.4 *For $f \in C[0,1]$, $x \in [0,1]$, we have*

$$\|T^{(\beta)}_{n,\alpha}(f)\| \leq \beta\|f\|,$$
$$\|K^{(\beta)}_{n,\alpha}(f)\| \leq \beta\|f\|.$$

Proof. For $0 \leq x, y \leq 1$ and $\beta \geq 1$, the inequality $|x^\beta - y^\beta| \leq \beta|x-y|$ holds, then we get

$$0 < [J_{n,k,\alpha}(x)]^\beta - [J_{n,k+1,\alpha}(x)]^\beta \leq \beta\left(J_{n,k,\alpha}(x) - J_{n,k+1,\alpha}(x)\right) = \beta p^{(\alpha)}_{n,k}(x).$$

By (3), (4) and Lemma 2.3, we have

$$\|T^{(\beta)}_{n,\alpha}(f)\| \leq \beta\|T_{n,\alpha}(f)\| \leq \beta\|f\|$$

and

$$\|K^{(\beta)}_{n,\alpha}(f)\| \leq \beta\|K_{n,\alpha}(f)\| \leq \beta\|f\|.$$

Lemma 2.5 (i) *For $0 \leq y < x < 1$, there holds*

$$R^{(1)}_{n,\alpha,\beta}(x,y) \leq \frac{\beta}{(x-y)^2}\gamma^2_{n\alpha}(x). \tag{13}$$

(ii) *For $0 < x < z \leq 1$, there holds*

$$1 - R^{(1)}_{n,\alpha,\beta}(x,z) \leq \frac{\beta}{(x-z)^2}\gamma^2_{n\alpha}(x). \tag{14}$$

Proof. (i) By (5) and (8), we get

$$\begin{aligned}
R^{(1)}_{n,\alpha,\beta}(x,y) &\leq \beta R^{(1)}_{n,\alpha,1}(x,y) = \beta\int_0^y d_t R^{(1)}_{n,\alpha,1}(x,t) \\
&\leq \beta\int_0^y \left(\frac{x-t}{x-y}\right)^2 d_t R^{(1)}_{n,\alpha,1}(x,t) \\
&\leq \frac{\beta}{(x-y)^2}\int_0^1 (t-x)^2 d_t R^{(1)}_{n,\alpha,1}(x,t) \\
&= \frac{\beta}{(x-y)^2} T_{n,\alpha}((t-x)^2,x) \\
&= \frac{\beta}{(x-y)^2}\gamma^2_{n\alpha}(x).
\end{aligned}$$

(ii) Using a similar method, we can get (14) easily.

Along the same line of proof, we have

Lemma 2.6 (i) *For $0 \leq y < x < 1$, there holds*

$$\int_0^y R^{(2)}_{n,\alpha,\beta}(x,t)dt \leq \frac{\beta}{(x-y)^2}\eta^2_{n\alpha}(x). \tag{15}$$

(ii) *For* $0 < x < z \le 1$, *there holds*

$$\int_z^1 R_{n,\alpha,\beta}^{(2)}(x,t)dt \le \frac{\beta}{(x-z)^2}\eta_{n\alpha}^2(x). \tag{16}$$

§3 Main results

Let $f(x) \in C[0,1]$, $t > 0$ and $W^2[0,1] = \{g \in C[0,1] : g'' \in C[0,1]\}$, the Peetre K-functional $K_2(f,t)$ and the second order modulus of continuity $\omega_2(f,t)$ are defined as follows:

$$K_2(f,t) = \inf_{g \in W^2[0,1]} \{\|f-g\| + t\|g'\| + t^2\|g''\|\},$$

$$\omega_2(f,\sqrt{t}) = \sup_{0<|h|\le\sqrt{t}} \sup_{x,x+h,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

By [26], there exists an absolute constant $C > 0$, such that

$$K_2(f,t) \le C\omega_2(f,\sqrt{t}). \tag{17}$$

Theorem 3.1 *For* $f \in C[0,1]$ *and* $x \in [0,1]$, *we have*

$$|T_{n,\alpha}^{(\beta)}(f,x) - f(x)| \le C\omega_2\left(f, \frac{\sqrt{\beta}\gamma_{n\alpha}(x)}{2}\right), \tag{18}$$

where C *is a positive constant.*

Proof. Let $g \in W^2$. By Taylor's formula, we can write

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Applying the operators $T_{n,\alpha}^{(\beta)}(\cdot,x)$ to the above equation, we have

$$T_{n,\alpha}^{(\beta)}(g,x) = g(x) + g'(x)T_{n,\alpha}^{(\beta)}(t-x,x) + T_{n,\alpha}^{(\beta)}\left(\int_x^t (t-u)g''(u)du, x\right).$$

By Cauchy Schwarz inequality, (8) and Lemma 2.4, we obtain

$$\begin{aligned}
&|T_{n,\alpha}^{(\beta)}(g,x) - g(x)| \\
\le\ & |g'(x)|T_{n,\alpha}^{(\beta)}(|t-x|,x) + \left|T_{n,\alpha}^{(\beta)}\left(\int_x^t (t-u)g''(u)du, x\right)\right| \\
\le\ & \|g'\|T_{n,\alpha}^{(\beta)}(|t-x|,x) + \frac{\|g''\|}{2}T_{n,\alpha}^{(\beta)}((t-x)^2,x) \\
\le\ & \|g'\|T_{n,\alpha}^{(\beta)}\left((t-x)^2,x\right)^{1/2} + \frac{\|g''\|}{2}T_{n,\alpha}^{(\beta)}((t-x)^2,x) \\
\le\ & \sqrt{\beta}\|g'\|T_{n,\alpha}\left((t-x)^2,x\right)^{1/2} + \beta\frac{\|g''\|}{2}T_{n,\alpha}((t-x)^2,x) \\
\le\ & \sqrt{\beta}\|g'\|\gamma_{n\alpha}(x) + \beta\frac{\|g''\|}{2}\gamma_{n\alpha}^2(x).
\end{aligned}$$

Thus

$$\begin{aligned}
|T_{n,\alpha}^{(\beta)}(f,x) - f(x)| &\le |T_{n,\alpha}^{(\beta)}(f-g,x)| + |f-g| + |T_{n,\alpha}^{(\beta)}(g,x) - g(x)| \\
&\le 2\|f-g\| + \sqrt{\beta}\|g'\|\gamma_{n\alpha}(x) + \beta\frac{\|g''\|}{2}\gamma_{n\alpha}^2(x).
\end{aligned}$$

For all $g \in W^2$, taking the infimum on the right hand side, we can get

$$|T_{n,\alpha}^{(\beta)}(f,x) - f(x)| \le 2K_2\left(f, \frac{\beta\gamma_{n\alpha}^2(x)}{4}\right).$$

By (17) and the above inequality, we obtain the desired result of Theorem 3.1.

Using a similar method, we prove

Theorem 3.2 *For $f \in C[0,1]$ and $x \in [0,1]$, we have*

$$|K_{n,\alpha}^{(\beta)}(f,x) - f(x)| \le C\omega_2\left(f, \frac{\sqrt{\beta}\eta_{n\alpha}(x)}{2}\right), \tag{19}$$

where C is a positive constant.

Let $\phi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$. The first order Ditzian-Totik modulus of smoothness and corresponding K-functional are given by, respectively,

$$\omega_\phi(f,t) = \sup_{0<h\le t}\left|f(x+\frac{h\phi(x)}{2}) - f(x-\frac{h\phi(x)}{2})\right|, x \pm \frac{h\phi(x)}{2} \in [0,1],$$

$$K_\phi(f,t) = \inf_{g\in W_\phi[0,1]}\{\|f-g\| + t\|\phi g'\|\}(t>0),$$

where $W_\phi[0,1] = \{g : g \in AC[0,1], \|\phi g'\| < \infty\}$. By [27], there exists a constant $C > 0$ such that

$$K_\phi(f,t) \le C\omega_\phi(f,t). \tag{20}$$

Theorem 3.3 *For $f \in C[0,1]$, $x \in (0,1)$ and $\phi(x) = \sqrt{x(1-x)}$, we have*

$$|T_{n,\alpha}^{(\beta)}(f,x) - f(x)| \le C\omega_\phi\left(f, \sqrt{\frac{2\beta}{x(1-x)}}\gamma_{n\alpha}(x)\right), \tag{21}$$

where C is a positive constant.

Proof. Applying the operators $T_{n,\alpha}^{(\beta)}(\cdot,x)$ to the representation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we have

$$T_{n,\alpha}^{(\beta)}(g,x) = g(x) + T_{n,\alpha}^{(\beta)}\left(\int_x^t g'(u)du, x\right).$$

For any $x, t \in (0,1)$, we can get

$$\left|\int_x^t g'(u)du\right| \le \|\phi g'\|\left|\int_x^t \frac{1}{\phi(u)}du\right|.$$

On the other hand,

$$\begin{aligned}
\left|\int_x^t \frac{1}{\phi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \\
&\le \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\
&\le 2\left(|\sqrt{t}-\sqrt{x}| + |\sqrt{1-t}-\sqrt{1-x}|\right) \\
&= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right) \\
&\le 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \le \frac{2\sqrt{2}|t-x|}{\phi(x)}.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|T_{n,\alpha}^{(\beta)}(g,x)-g(x)| &\leq 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)T_{n,\alpha}^{(\beta)}(|t-x|,x)\\ &\leq 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)\left(T_{n,\alpha}^{(\beta)}((t-x)^2,x)\right)^{1/2}\\ &\leq 2\sqrt{2}\sqrt{\beta}\|\phi g'\|\phi^{-1}(x)\left(T_{n,\alpha}((t-x)^2,x)\right)^{1/2}\\ &= 2\sqrt{2}\sqrt{\beta}\|\phi g'\|\phi^{-1}(x)\gamma_{n\alpha}(x).\end{aligned}$$

Thus

$$\begin{aligned}|T_{n,\alpha}^{(\beta)}(f,x)-f(x)| &\leq |T_{n,\alpha}^{(\beta)}(f-g,x)|+|f-g|+|T_{n,\alpha}^{(\beta)}(g,x)-g(x)|\\ &\leq 2\|f-g\|+2\sqrt{2}\sqrt{\beta}\|\phi g'\|\phi^{-1}(x)\gamma_{n\alpha}(x).\end{aligned}$$

For all $g\in W_\phi[0,1]$, taking the infimum on the right hand side, we can get

$$|T_{n,\alpha}^{(\beta)}(f,x)-f(x)|\leq 2K_\phi\left(f,\sqrt{\frac{2\beta}{x(1-x)}}\gamma_{n\alpha}(x)\right).$$

By (20) and the above inequality, we get (21) immediately.

Using a similar method, we prove

Theorem 3.4 *For $f\in C[0,1]$, $x\in(0,1)$ and $\phi(x)=\sqrt{x(1-x)}$, we have*

$$|K_{n,\alpha}^{(\beta)}(f,x)-f(x)|\leq C\omega_\phi\left(f,\sqrt{\frac{2\beta}{x(1-x)}}\eta_{n\alpha}(x)\right), \tag{22}$$

where C is a positive constant.

Finally, we study the approximation properties of $T_{n,\alpha}^{(\beta)}(f,x)$ and $K_{n,\alpha}^{(\beta)}(f,x)$ for some absolutely continuous functions $f\in DBV[0,1]$, which is defined by

$$DBV[0,1]=\left\{f|f(x)=f(0)+\int_0^x h(t)dt\right\},$$

where $x\in[0,1], h\in BV[0,1]$, i.e., h is a function of bounded variation on $[0,1]$.

Theorem 3.5 *Let $f\in DBV[0,1]$. If $h(x+)$ and $h(x-)$ exist at a fixed point $x\in(0,1)$, then we have*

$$\begin{aligned}\left|T_{n,\alpha}^{(\beta)}(f,x)-f(x)\right| &\leq \beta\big(|h(x+)|+|h(x-)|\big)\gamma_{n\alpha}(x)\\ &\quad+\frac{2\beta\gamma_{n\alpha}^2(x)}{x(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}}(\varphi_x)+\frac{1}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}}(\varphi_x),\end{aligned}$$

where

$$\varphi_x(t)=\begin{cases}h(t)-h(x+), & x<t\leq 1;\\ 0, & t=x;\\ h(t)-h(x-), & 0\leq t<x.\end{cases}$$

Proof. Let f satisfy the conditions of Theorem 3.5, by using Bojanic-Cheng's method [4], we have

$$f(t)-f(x)=\int_x^t h(u)du \tag{23}$$

and $h(u)$ can be expressed as

$$\begin{aligned} h(u) &= \frac{h(x+)+h(x-)}{2} + \varphi_x(u) + \frac{h(x+)-h(x-)}{2} sign(u-x) \\ &\quad + \delta_x(u) \left[h(x) - \frac{h(x+)+h(x-)}{2} \right], \end{aligned} \tag{24}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x; \\ 0, & u \neq x. \end{cases}$$

$$sign(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

From (23), (24), and noting $\int_x^t sign(u-x)du = |t-x|$, $\int_x^t \delta_x(u)du = 0$, we find that

$$\begin{aligned} & \left| T_{n,\alpha}^{(\beta)}(f,x) - f(x) \right| \\ = & \left| T_{n,\alpha}^{(\beta)}(f(t)-f(x),x) \right| = \left| T_{n,\alpha}^{(\beta)}(\int_x^t h(u)du, x) \right| \\ = & \Big| \frac{h(x+)+h(x-)}{2} T_{n,\alpha}^{(\beta)}(t-x,x) \\ & + \frac{h(x+)-h(x-)}{2} T_{n,\alpha}^{(\beta)}(|t-x|,x) + T_{n,\alpha}^{(\beta)}(\int_x^t \varphi_x(u)du, x) \Big| \\ \leq & \left(|h(x+)| + |h(x-)| \right) T_{n,\alpha}^{(\beta)}(|t-x|,x) + \left| T_{n,\alpha}^{(\beta)}(\int_x^t \varphi_x(u)du, x) \right|. \end{aligned}$$

By the inequality $T_{n,\alpha}^{(\beta)}(|t-x|,x) \leq \beta T_{n,\alpha}(|t-x|,x)$ and (9), we have

$$\begin{aligned} & \left| T_{n,\alpha}^{(\beta)}(f,x) - f(x) \right| \\ \leq & \beta \left(|h(x+)| + |h(x-)| \right) \gamma_{n\alpha}(x) + \left| T_{n,\alpha}^{(\beta)}(\int_x^t \varphi_x(u)du, x) \right|. \end{aligned} \tag{25}$$

Next, we estimate another item $T_{n,\alpha}^{(\beta)}(\int_x^t \varphi_x(u)du, x)$.

By the Lebesgue-Stieltjes integral representations of (5), the term $T_{n,\alpha}^{(\beta)}(\int_x^t \varphi_x(u)du, x)$ can be expressed as

$$\begin{aligned} & T_{n,\alpha}^{(\beta)}\left(\int_x^t \varphi_x(u)du, x \right) \\ = & \int_0^1 (\int_x^t \varphi_x(u)du) d_t R_{n,\alpha,\beta}^{(1)}(x,t) \\ = & \int_0^x (\int_x^t \varphi_x(u)du) d_t R_{n,\alpha,\beta}^{(1)}(x,t) + \int_x^1 (\int_x^t \varphi_x(u)du) d_t R_{n,\alpha,\beta}^{(1)}(x,t). \end{aligned}$$

Let

$$\Delta_{1n}(f,x) = \int_0^x (\int_x^t \varphi_x(u)du) d_t R_{n,\alpha,\beta}^{(1)}(x,t),$$
$$\Delta_{2n}(f,x) = \int_x^1 (\int_x^t \varphi_x(u)du) d_t R_{n,\alpha,\beta}^{(1)}(x,t).$$

Then we have

$$T_{n,\alpha}^{(\beta)}\left(\int_x^t \varphi_x(u)du, x\right) = \Delta_{1n}(f,x) + \Delta_{2n}(f,x). \tag{26}$$

Applying the integration by parts and noticing $R_{n,\alpha,\beta}^{(1)}(x,0)=0$, $\int_x^x \varphi_x(u)du = 0$, we get

$$\begin{aligned}
&\Delta_{1n}(f,x)\\
=\ & R_{n,\alpha,\beta}^{(1)}(x,t)\int_x^t \varphi_x(u)du\Big|_0^x - \int_0^x R_{n,\alpha,\beta}^{(1)}(x,t)\varphi_x(t)dt\\
=\ & -\int_0^x R_{n,\alpha,\beta}^{(1)}(x,t)\varphi_x(t)dt = -(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^x) R_{n,\alpha,\beta}^{(1)}(x,t)\varphi_x(t)dt.
\end{aligned}$$

Thus, it follows that

$$\left|\Delta_{1n}(f,x)\right| \le \int_0^{x-\frac{x}{\sqrt{n}}} R_{n,\alpha,\beta}^{(1)}(x,t)\bigvee_t^x(\varphi_x)dt + \int_{x-\frac{x}{\sqrt{n}}}^x R_{n,\alpha,\beta}^{(1)}(x,t)\bigvee_t^x(\varphi_x)dt.$$

From Lemma 2.5 (i) and $0 \le R_{n,\alpha,\beta}^{(1)}(x,t) \le 1$, we get

$$\left|\Delta_{1n}(f,x)\right| \le \beta\gamma_{n\alpha}^2(x)\int_0^{x-\frac{x}{\sqrt{n}}} \frac{\bigvee_t^x(\varphi_x)}{(x-t)^2}dt + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\varphi_x). \tag{27}$$

Putting $t = x - \frac{x}{u}$ for the integral of (27), we get

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{\bigvee_t^x(\varphi_x)}{(x-t)^2}dt = \frac{1}{x}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x(\varphi_x)du \le \frac{2}{x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(\varphi_x). \tag{28}$$

From (27),(28), it follows that

$$\left|\Delta_{1n}(f,x)\right| \le \frac{2\beta\gamma_{n\alpha}^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(\varphi_x) + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\varphi_x). \tag{29}$$

From Lemma 2.5 (ii), using the same method, we also get

$$\left|\Delta_{2n}(f,x)\right| \le \frac{2\beta\gamma_{n\alpha}^2(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{1-x}{k}}(\varphi_x) + \frac{1-x}{\sqrt{n}}\bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(\varphi_x). \tag{30}$$

Theorem 3.5 now follows from (25),(26),(29) and (30).

From Lemma 2.2, Lemma 2.4 and Lemma 2.6, using a similar method, we prove

Theorem 3.6 *Let $f \in DBV[0,1]$. If $h(x+)$ and $h(x-)$ exist at a fixed point $x \in (0,1)$, then we have*

$$\begin{aligned}
\left|K_{n,\alpha}^{(\beta)}(f,x) - f(x)\right| \le\ & \beta\big(|h(x+)| + |h(x-)|\big)\eta_{n\alpha}(x)\\
& + \frac{2\beta\eta_{n\alpha}^2(x)}{x(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}}(\varphi_x) + \frac{1}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}}(\varphi_x).
\end{aligned}$$

§4 Conclusion

The Bézier variant of two new families of generalized Bernstein operators has been introduced. A direct approximaiton by means of the Ditzian-Totik modulus of smoothness and a global approximation theorem in terms of second order modulus of continuity have been established. The approximation of functions with derivatives of bounded variation has been studied.

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