Appl. Math. J. Chinese Univ.
2020, 35(3): 293-307

A constructive method for approximating trigonometric functions and their integrals

CHEN Xiao-diao   WANG Long-quan   WANG Yi-gang*

Abstract. This paper presents an interpolation-based method (IBM) for approximating some trigonometric functions or their integrals as well. It provides two-sided bounds for each function, which also achieves much better approximation effects than those of prevailing methods. In principle, the IBM can be applied for bounding more bounded smooth functions and their integrals as well, and its applications include approximating the integral of $\sin(x)/x$ function and improving the famous square root inequalities.

§1 Introduction

In many applications such as operations research, computer science, mathematics, physical sciences and engineering [1,9,22], computing the integrals of some bounded functions in an interval, is needed such as the trigonometric functions

$$f_1(x) = \frac{\sin(x)}{x} \quad \text{and} \quad f_2(x) = 3\frac{x}{\sin(x)} + \cos(x),\ x \in [0, \pi/2],$$

whose integrals can not be explicitly expressed and must be numerically solved.

Many authors consider estimating the bounds of the integrals, by estimating the bounds of the given bounded functions, which leads to many researches on the corresponding inequalities [2-6, 8, 10-20, 23-25], including some unbounded functions

$$\begin{cases} f_3(x) = (\dfrac{\sin(x)}{x})^2 + \dfrac{\tan(x)}{x}, \\ f_4(x) = 2 \cdot \dfrac{\sin(x)}{x} + \dfrac{\tan(x)}{x}, \qquad x \in [0, \pi/2]. \\ f_5(x) = \dfrac{\sin(x)}{x} + \dfrac{\tan(x)}{x}, \end{cases}$$

Several famous inequalities with $x \in (0, \pi/2)$ are summarized as follows [3]

Received: 2017-07-28. Revised: 2019-11-01.
MR Subject Classification: 26D05 , 26D07, 42A10, 42A15.
Keywords: Padé approximant, trigonometric function, constructive method, interpolation-based method, two-sided bounds.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-020-3562-z.
Supported by the National Natural Science Foundation of China (61672009,61502130).
*Corresponding author.

$$\frac{1+\cos(x)}{2} < L_1(x) < f_1(x) < U_1(x) < \frac{2+\cos(x)}{3}, \tag{1}$$

$$f_2(x) > L_2(x), \tag{2}$$

$$f_3(x) > L_3(x) > 2, \tag{3}$$

$$f_4(x) > L_4(x) > 3, \tag{4}$$

$$f_5(x) > L_5(x) > 2, \tag{5}$$

where $L_1(x) = \frac{60-7x^2}{60+3x^2}$, $U_1(x) = \frac{11x^4-360x^2+2520}{60x^2+2520}$, $L_2(x) = \frac{x^4+8x^2+96}{2x^2+24}$, $L_3(x) = \frac{-303x^6+5550x^4-32400x^2+108000}{-54x^6-2025x^4-16200x^2+54000}$, $L_4(x) = \frac{81x^4-945x^2+2700}{-18x^4-315x^2+900}$, $L_5(x) = \frac{39x^4-480x^2+1800}{-18x^4-315x^2+900}$.

In principle, there are two key issues for an inequality. One is to find the bounds, and the other is to prove it [28]. Malesevic and his coauthors provided several methods for proving some inequalities of some special functions [15,27].

This paper presents an interpolation-based method (IBM) for constructing the bounds of some famous trigonometric functions, including $f_i(x), i = 1, 2, \cdots, 5$. We take the inequalities ($1 \sim 5$) for example, and provide two-sided bounding functions which can achieve much better approximation effects. In principle, the IBM can be applied in approximating any smooth bounded function within some bounded interval, and it provides an improved bounds of the famous square root inequalities. It also proves the bounds of the inequalities in a new way.

§2 The interpolation-based method

For the sake of convenience, let $h = b - a$, and we introduce Theorem 3.5.1 in Page 67, Chapter 3.5 of [7] as follows.

Theorem 1. Let $w_0, w_1, \cdots, w_r$ be $r+1$ distinct points in $[a, b]$, and $n_0, \cdots, n_r$ be $r+1$ integers ≥ 0. Let $N = n_0 + \cdots + n_r + r$. Suppose that $g(t)$ is a polynomial of degree N such that

$$g^{(i)}(w_j) = f^{(i)}(w_j), \quad i = 0, \cdots, n_j, \quad j = 0, \cdots, r.$$

Then there exists $\xi_0(t) \in [a, b]$ such that

$$|f(t) - g(t)| = |\frac{f^{(N+1)}(\xi_0(t))}{(N+1)!}\prod_{i=0}^{r}(t-w_i)^{n_i+1}| = O(\prod_{i=0}^{r}|(t-w_i)^{n_i+1}|). \quad \square$$

2.1 Constructing bounding functions for a bounded function

One can consider bounding functions in the form $g(x) = g_1(x) + g_2(x)\cos(x) + g_3(x)\sin(x)$ (a rational form is also OK) to bound the given bounded smooth function $f(x)$ within $[a, b]$, where $g_i(x) = \sum_{j=0}^{n_i-1} c_{i,j}x^j$ is a polynomial with unknown coefficients $c_{i,j}$ to be determined, $i = 1, 2, 3$. There are altogether $n = n_1 + n_2 + n_3$ unknowns, and it needs n equations. Let $h(x) = f(x) - g(x)$ and $\rho(x) = (x-a)^{n-k}(x-b)^k$, where k is a positive integer number within

$[1, n-1]$. We introduce the interpolation conditions such that

$$h^{(j)}(a) = 0, h^{(l)}(b) = 0,\ j = 0, 1, \cdots, n-k-1,\ l = 0, 1, \cdots, k-1, \tag{6}$$

for determining the unknown $c_{i,j}$. Combining Eq. (6) with Theorem 1, there exists $\psi(x) \in [a, b]$ such that

$$h(x) = \frac{h^{(n)}(\psi(x))}{n!}\rho(x). \tag{7}$$

Note that $\rho(x) = \begin{cases} > 0, & \text{if } k \text{ is even,} \\ < 0, & \text{if } k \text{ is odd,} \end{cases}$ $\forall x \in (a, b)$, if $= \frac{h^{(n)}(\psi(x))}{n!} \geq 0$, we have that $h(x) \geq 0$ for an even k or $h(x) \leq 0$ for a odd k, $\forall x \in [a, b]$. Based on the above observation, one can choose suitable values of k to construct the bounding functions of $f(x)$.

2.2 Constructing bound functions for a unbounded function

Suppose that $f(x)$ is unbounded in $[a, b]$. In this case, we consider a rational form $r(x) = (r_1(x) + r_2(x)\cos(x) + r_3(x)\sin(x))/(r_4(x) + r_5(x)\cos(x) + r_6(x)\sin(x))$ instead, where $r_i(x) = \sum_{j=0}^{n_i-1} d_{i,j}x^j$ is a polynomial with unknown coefficients $d_{i,j}$ to be determined and $d_{4,n_4-1} = 1$, $i = 1, 2, \cdots, 6$.

Firstly, one needs to find a function $\omega(x) > 0, \forall x \in (a, b)$ such that $F(x) = f(x) \cdot \omega(x)$ is bounded in $[a, b]$. So both $f(x) - r(x)$ and $F(x) - r(x) \cdot \omega(x)$ have the same sign, $\forall x \in (a, b)$. To simplify the computation of derivatives, we consider $H(x) = (f(x) - r(x)) \cdot \omega(x) \cdot (r_4(x) + r_5(x)\cos(x) + r_6(x)\sin(x))$ instead, where both $f(x) - r(x)$ and $H(x)$ have the same sign under the assumption that $(r_4(x) + r_5(x)\cos(x) + r_6(x)\sin(x)) > 0$, $\forall x \in (a, b)$. The interpolation conditions become

$$H^{(j)}(a) = 0, H^{(l)}(b) = 0,\ j = 0, 1, \cdots, n-k-1,\ l = 0, 1, \cdots, k-1. \tag{8}$$

2.3 Discussions

In principle, one can utilize more interpolation points $x_i \in (a, b)$ to add the interpolation conditions such that $h(x_i) = 0$ and $h'(x_i) = 0$ to Eq. (7), or $H(x_i) = 0$ and $H'(x_i) = 0$ to Eq. (8), which can also construct two bounding functions with even better approximation effect.

Note that both Eq. (7) and Eq. (8) are linear in the unknowns. Once k is determined, one can compute the interpolation function $g(x)$ or $r(x)$ by solving a system of linear equations. It is trivial to plot the bounded function $h(x)$ or $H(x)$ with software Maple or Mathematics, which can be used to verify whether or not the corresponding interpolation functions bound $f(x)$.

§3 Numerical examples and illustrations

Example 1. Let $l_0(x) = (e_0 + (e_1 + e_2x + e_3x^2)\cos(x) + (e_4 + e_5x + e_6x^2)\sin(x))$, $r_0(x) = (d_0 + (d_1 + d_2x + d_3x^2)\cos(x) + (d_4 + d_5x + d_6x^2)\sin(x))$, $h_{1,1}(x) = f_1(x) - g_{1,1}(x)$ and $h_{1,2}(x) =$

$f_1(x) - g_{1,2}(x)$, $x \in [0, \pi/2]$. By introducing the constraints

$$\begin{cases} h_{1,1}^{(i)}(0) = 0, h_{1,1}^{(j)}(\pi/2) = 0, \ i = 0,1,2,3, \ j = 0,1,2. \\ h_{1,2}^{(k)}(0) = 0, h_{1,2}^{(l)}(\pi/2) = 0, \ k = 0,1,2, \ l = 0,1,2,3. \end{cases}$$

we obtain the values of d_i and e_j as

$$\lambda = \frac{1}{3(\pi^2 - 2\pi - 4)^2\pi^4}, \ \kappa = \frac{1}{3(\pi^3(\pi^4 - 4\pi^3 - 4\pi^2 + 16\pi + 16)},$$
$$d_0 = 2\lambda(\pi^8 - 40\pi^6 + 624\pi^4 - 3456\pi^2 + 2304),$$
$$d_1 = \lambda(\pi^8 - 12\pi^7 + 68\pi^6 + 48\pi^5 - 1200\pi^4 + 6912\pi^2 - 4608),$$
$$d_2 = -\lambda(\pi^8 + 4\pi^7 - 64\pi^6 + 240\pi^5 - 96\pi^4 - 2880\pi^3 + 6336\pi^2 + 2304\pi - 4608),$$
$$d_3 = 2\lambda(1152 + 80\pi^4 - 52\pi^5 + 2\pi^6 + \pi^7 - 1152\pi^2 + 432\pi^3),$$
$$d_4 = \lambda(\pi^8 + 4\pi^7 - 64\pi^6 + 240\pi^5 - 96\pi^4 - 2880\pi^3 + 6336\pi^2 + 2304\pi - 4608),$$
$$d_5 = -2\lambda(3\pi^7 - 16\pi^6 - 60\pi^5 + 384\pi^4 + 432\pi^3 - 2880\pi^2 + 2304),$$
$$d_6 = -8\lambda(7\pi^5 - 30\pi^4 - 60\pi^3 + 312\pi^2 - 48\pi - 288),$$
$$e_0 = -16\kappa(-15\pi^4 + \pi^6 + 144 + 36\pi^2),$$
$$e_1 = \kappa(-192\pi^4 - 12\pi^5 + 48\pi^3 + 3\pi^7 + 4\pi^6 + 2304 + 576\pi^2),$$
$$e_2 = -2(\pi^6 + 4\pi^5 - 32\pi^4 - 24\pi^3 + 192\pi^2 - 192\pi + 384)),$$
$$e_3 = -4\kappa(-4\pi^4 + \pi^5 + 8\pi^3 - 144\pi + 288),$$
$$e_4 = 2(\pi^6 + 4\pi^5 - 32\pi^4 - 24\pi^3 + 192\pi^2 - 192\pi + 384)),$$
$$e_5 = \kappa(2304 + \pi^7 + 4\pi^6 - 120\pi^4 + 288\pi^2 + 48\pi^3 - 576\pi),$$
$$e_6 = -2\kappa(\pi^6 + 4\pi^5 - 32\pi^4 - 24\pi^3 + 192\pi^2 - 192\pi + 384).$$

such that $(2+\cos(x))/3 > r_0(x) > f_1(x) > l_0(x) > (1+\cos(x))/2, \forall x \in (0, \pi/2)$.

Example 2. Let $H_{1,1}(x) = (1+d_{1,3}x^2+d_{1,4}x^4)f_1(x)-(d_{1,0}+d_{1,1}x^2+d_{1,2}x^4)$ and $H_{1,2}(x) = (1+e_{1,3}x^2+e_{1,4}x^4)f_1(x)-(e_{1,0}+e_{1,1}x^2+e_{1,2}x^4)$, $x \in [0, \pi/2]$. It can be verified that $H_{1,i}^{(j)}(0) = 0$, $i = 1, 2$ and $j = 1, 3, 5, 7$. By introducing the constraints

$$\begin{cases} H_{1,1}^{(i)}(0) = 0, H_{1,1}(\pi/2) = 0 \quad \text{and } H'_{1,1}(\pi/2) = 0, \ i = 0,2,4, \\ H_{1,2}^{(j)}(0) = 0 \quad \text{and } H_{1,2}(\pi/2) = 0, \ j = 0,2,4,6, \end{cases}$$

we obtain $r_1(x) = \dfrac{(d_{1,0} + d_{1,1}x^2 + d_{1,2}x^4)}{1 + d_{1,3}x^2 + d_{1,4}x^4}$ and $l_1(x) = \dfrac{e_{1,0} + e_{1,1}x^2 + e_{1,2}x^4}{1 + e_{1,3}x^2 + e_{1,4}x^4}$ such that $U_1(x) > r_1(x) \ge f_1(x) \ge l_1(x) > L_1(x), \forall x \in (0, \pi/2)$, where

$$d_{1,0} = 1, d_{1,1} = \frac{7\pi^5 + 720\pi^3 - 12480\pi^2 + 46080\pi - 46080}{-60\pi^2(\pi^3 - 24\pi^2 + 96\pi - 96)},$$
$$d_{1,2} = \frac{7\pi^5 + 12\pi^4 - 960\pi^3 + 7680\pi^2 - 23040\pi + 23040}{-15\pi^4(\pi^3 - 24\pi^2 + 96\pi - 96)},$$
$$d_{1,3} = \frac{\pi^5 - 80\pi^4 + 80\pi^3 + 3840\pi^2 - 15360\pi + 15360}{20\pi^2(\pi^3 - 24\pi^2 + 96\pi - 96)},$$
$$d_{1,4} = \frac{7\pi^6 + 9\pi^5 - 480\pi^4 + 960\pi^3 + 5760\pi^2 - 23040\pi + 23040}{-15\pi^4(\pi^3 - 24\pi^2 + 96\pi - 96)},$$
$$e_{1,0} = 1, \quad e_{1,1} = \frac{31\pi^5 + 36\pi^4 + 3360\pi^2 - 40320\pi + 80640}{-42\pi^2(7\pi^3 + 6\pi^2 - 240\pi + 480)},$$
$$e_{1,2} = \frac{11\pi^4 + 1240\pi^3 + 1440\pi^2 - 47040\pi + 94080}{420\pi^2(7\pi^3 + 6\pi^2 - 240\pi + 480)},$$
$$e_{1,3} = \frac{3\pi^5 + \pi^4 - 280\pi^3 + 6720\pi - 13440}{7\pi^2(7\pi^3 + 6\pi^2 - 240\pi + 480)}, e_{1,4} = \frac{11\pi^5 - 1440\pi^3 - 480\pi^2 + 40320\pi - 80640}{840\pi^2(7\pi^3 + 6\pi^2 - 240\pi + 480)}.$$

So one obtain the bounds of the integral of $Si(\pi/2) = \int_0^{\pi/2} f_1(x)dx$, i.e., $1.370762103 < Si(\pi/2) < 1.370762206$ by using the integrals of $l_1(x)$ and $r_1(x)$, which is much better than $1.36\cdots < Si(\pi/2) < 1.37\cdots$ in the method in [21].

Example 3. Let $H_{2,1}(x) = (1 + d_{2,3}x^2)f_2(x) - (d_{2,0} + d_{2,1}x^2 + d_{2,2}x^4)$ and $H_{2,2}(x) = (1 + e_{2,3}x^2)f_2(x) - (e_{2,0} + e_{2,1}x^2 + e_{2,2}x^4)$, $x \in [0, \pi/2]$. It can be verified that $H_{2,i}^{(j)}(0) = 0$, $i = 1, 2$ and $j = 1, 3, 5$. By introducing the constraints

$$\begin{cases} H_{2,1}^{(i)}(0) = 0, H_{2,1}(\pi/2) = 0 & \text{and } H'_{2,1}(\pi/2) = 0,\ i = 0, 2, \\ H_{2,2}^{(j)}(0) = 0 & \text{and } H_{2,2}(\pi/2) = 0,\ j = 0, 2, 4, \end{cases}$$

we have that

$$r_2(x) = \frac{4 + \dfrac{640 - 240\pi + \pi^4}{5\pi^2(3\pi - 8)}x^2 + \dfrac{1}{10}x^4}{1 + \dfrac{640 - 240\pi + \pi^4}{20\pi^2(3\pi - 8)}x^2}, l_2(x) = \frac{4 - \dfrac{8(5\pi - 16)}{\pi^2(\pi - 4)}x^2 - \dfrac{4(9\pi^2 - 48\pi + 64))}{\pi^4(\pi - 4)}x^4}{1 - \dfrac{2(5\pi - 16)}{\pi^2(\pi - 4)}x^2}$$

such that $r_2(x) > f_2(x) > l_2(x) > L_2(x), \forall x \in (0, \pi/2)$.

Example 4. Let $H_{3,1}(x) = ((1 + d_{3,4}x^2 + d_{3,5}x^4 + d_{3,6}x^6)f_3(x) - (d_{3,0} + d_{3,1}x^2 + d_{3,2}x^4 + d_{3,3}x^6)) \cdot \cos(x)$ and $H_{3,2}(x) = ((1 + e_{3,4}x^2 + e_{3,5}x^4 + e_{3,6}x^6)f_3(x) - (e_{3,0} + e_{3,1}x^2 + e_{3,2}x^4 + e_{3,3}x^6)) \cdot \cos(x)$, $x \in [0, \pi/2]$. It can be verified that $H_{3,i}^{(j)}(0) = 0$, $i = 1, 2$ and $j = 1, 3, \cdots, 13$. By introducing the constraints

$$\begin{cases} H_{3,1}^{(i)}(0) = 0, H_{3,1}(\pi/2) = 0 & \text{and } H'_{3,1}(\pi/2) = 0,\ i = 0, 2, \cdots, 8, \\ H_{3,2}^{(j)}(0) = 0 & \text{and } H_{3,2}(\pi/2) = 0,\ j = 0, 2, \cdots, 10, \end{cases}$$

it has $r_3(x) = \dfrac{\mu_1x^6 + \mu_2x^4 + \mu_3x^2 + \mu_4}{(\pi^2 - 4x^2)(\mu_5x^4 + \mu_6x^2 + \mu_7)}$ and $l_3(x) = \dfrac{\mu_8x^6 + \mu_9x^4 + \mu_{10}x^2 + \mu_{11}}{(\pi^2 - 4x^2)(\mu_{12}x^4 + \mu_{13}x^2 + \mu_{14})}$ such that $r_3(x) \geq f_3(x) \geq l_3(x) > L_3(x)$, where $\mu_1 = 824\pi^{10} + 2976\pi^8 + 161280\pi^6 - 1814400\pi^4 - 29030400\pi^2 + 203212800$, $\mu_2 = -(2(103\pi^8 + 2520\pi^6 + 10080\pi^4 + 110880\pi^2 - 5443200))\pi^4$, $\mu_3 = (1260(\pi^8 + 14\pi^6 + 78\pi^4 - 30240))\pi^4$, $\mu_4 = -(630(7\pi^6 + 360\pi^2 - 10080))\pi^6$, $\mu_5 = 93\pi^{10} - 196560\pi^4 + 3628800\pi^2 - 25401600$, $\mu_6 = 630(\pi^8 + 78\pi^4 - 720\pi^2 - 10080)\pi^2$, $\mu_7 = -315(7\pi^6 + 360\pi^2 - 10080)\pi^4$, $\mu_8 = 9808\pi^6 + 487368\pi^4 + 8023680\pi^2 - 137047680$, $\mu_9 = -121842\pi^6 - 4656960\pi^4 - 25613280\pi^2 + 838252800$, $\mu_{10} = 662760\pi^6 + 22702680\pi^4 - 2933884800$, $\mu_{11} = -62370(31\pi^4 + 840\pi^2 - 11760)\pi^2$, $\mu_{12} = 25011\pi^4 + 1325520\pi^2 - 15467760$, $\mu_{13} = 331380\pi^4 + 7484400\pi^2 - 104781600$, $\mu_{14} = -966735\pi^4 - 26195400\pi^2 + 366735600$.

Example 5. Let $H_{4,1}(x) = ((1 + d_{4,3}x^2 + d_{4,4}x^4)f_4(x) - (d_{4,0} + d_{4,1}x^2 + d_{4,2}x^4)) \cdot \cos(x)$ and $H_{4,2}(x) = ((1 + e_{4,3}x^2 + e_{4,4}x^4)f_4(x) - (e_{4,0} + e_{4,1}x^2 + e_{4,2}x^4) \cdot \cos(x)$, $x \in [0, \pi/2]$. It can be verified that $H_{4,i}^{(j)}(0) = 0$, $i = 1, 2$ and $j = 1, 3, \cdots, 11$. By introducing the constraints

$$\begin{cases} H_{4,1}^{(i)}(0) = 0, H_{4,1}(\pi/2) = 0 & \text{and } H'_{4,1}(\pi/2) = 0,\ i = 0, 2, 4, \\ H_{4,2}^{(j)}(0) = 0 & \text{and } H_{4,2}(\pi/2) = 0,\ j = 0, 2, 4, 6, \end{cases}$$

we have that $r_4(x) = \dfrac{3(7\pi^4x^4 - 2240x^4 + 200\pi^2x^4 - 50\pi^4x^2 + 140\pi^4)}{10(56x^2 - 5\pi^2x^2 + 14\pi^2)(\pi^2 - 4x^2)}$, and $l_4(x)$ $= \dfrac{3(-12\pi^6x^4 + 32\pi^4x^4 + 10240x^4 + 3\pi^8x^2 - 5120\pi^2x^2 + 640\pi^4)}{(-2560x^2 + 3\pi^6x^2 + 640\pi^2)(\pi^2 - 4x^2)}$ such that $r_4(x) \geq f_4(x) \geq l_4(x) > L_4(x)$.

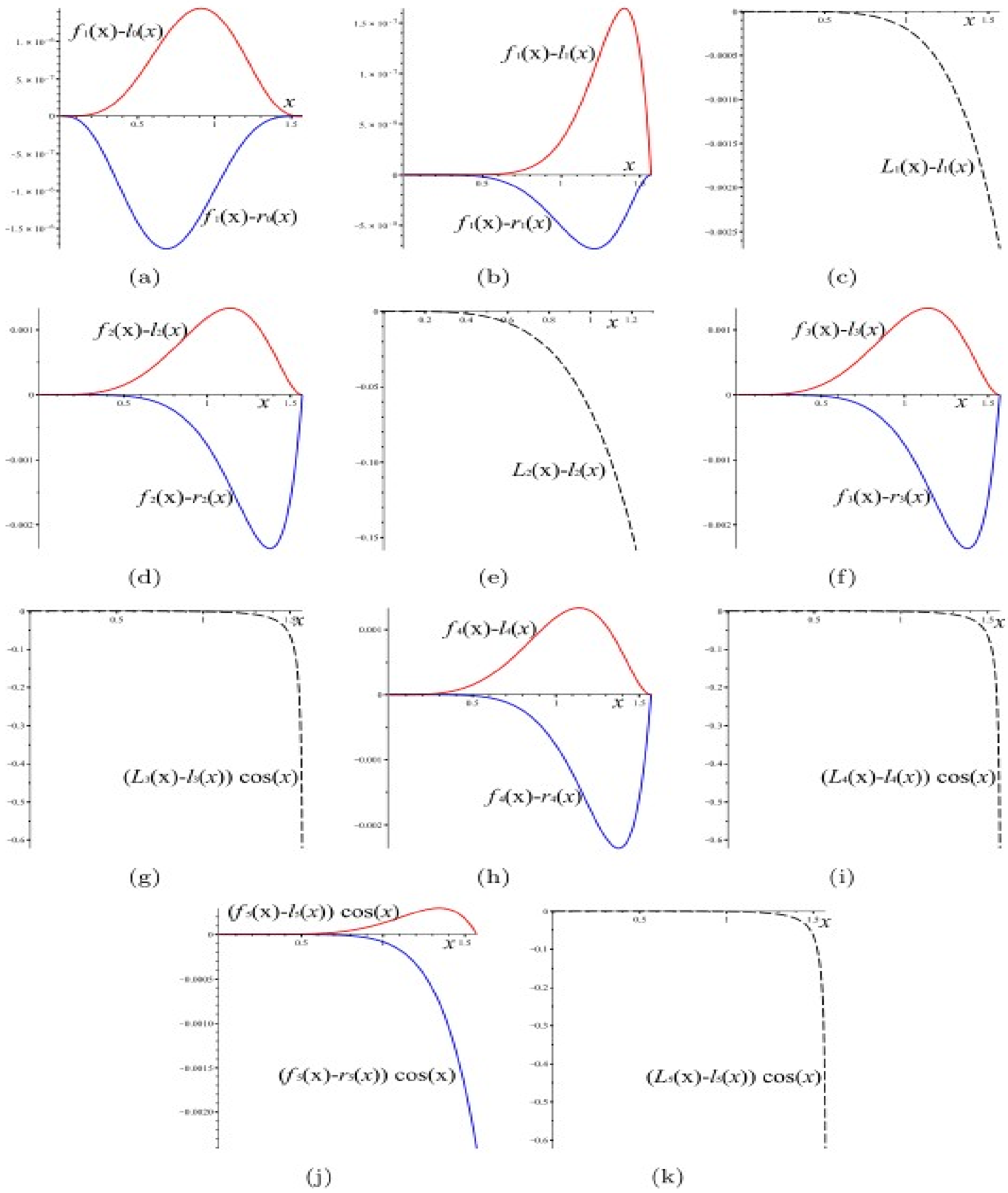

Figure 1: Error plots of (a) $f_1(x)-r_0(x)$ and $f_1(x)-l_0(x)$; (b) $f_1(x)-r_1(x)$ and $f_1(x)-l_1(x)$; (c) $L_1(x)-l_1(x)$; (d) $f_2(x)-r_2(x)$ and $f_2(x)-l_2(x)$; (e) $L_2(x)-l_2(x)$; (f) $f_3(x)-r_3(x)$ and $f_3(x)-l_3(x)$; (g) $(L_3(x)-l_3(x))\cdot\cos(x)$; (h) $f_4(x)-r_4(x)$ and $f_4(x)-l_4(x)$; (i) $(L_4(x)-l_4(x))\cdot\cos(x)$; (j) $(f_5(x)-r_5(x))\cdot\cos(x)$ and $(f_5(x)-l_5(x))\cdot\cos(x)$; and (k) $(L_5(x)-l_5(x))\cdot\cos(x)$.

Example 6. Let $H_{5,1}(x) = ((1 + d_{5,3}x^2 + d_{5,4}x^4)f_5(x) - (d_{5,0} + d_{5,1}x^2 + d_{5,2}x^4)) \cdot \cos(x)$ and $H_{5,2}(x) = ((1 + e_{5,3}x^2 + e_{5,4}x^4)f_5(x) - (e_{5,0} + e_{5,1}x^2 + e_{5,2}x^4) \cdot \cos(x)$, $x \in [0, \pi/2]$. It can be verified that $H^{(j)}_{5,i}(0) = 0$, $i = 1, 2$ and $j = 1, 3, \cdots, 11$. By introducing the constraints

$$\begin{cases} H^{(i)}_{5,1}(0) = 0, H_{5,1}(\pi/2) = 0 & \text{and } H'_{5,1}(\pi/2) = 0,\ i = 0, 2, 4, \\ H^{(j)}_{5,2}(0) = 0 & \text{and } H_{5,2}(\pi/2) = 0,\ j = 0, 2, 4, 6, \end{cases}$$

we have that

$$\begin{aligned} r_5(x) = {} & \frac{(-1236480 + 101520\pi^2 + 3359\pi^4)x^4 + (1612800 - 33600\pi^2 - 25380\pi^4)x^2}{-60(271\pi^2x^2 - 2856x^2 - 714\pi^2 + 3360)(\pi^2 - 4x^2)} \\ & + \frac{-403200\pi^2 + 85680\pi^4}{-60(271\pi^2x^2 - 2856x^2 - 714\pi^2 + 3360)(\pi^2 - 4x^2)}, \\ l_5(x) = {} & \frac{4((-92\pi^6 + 408\pi^4 - 3840\pi^2 + 92160)x^4 + (-46080\pi^2 + 480\pi^4 + 120\pi^6 + 23\pi^8)x^2)}{3(-15360x^2 + 17\pi^6x^2 - 20\pi^6 + 3840\pi^2)(\pi^2 - 4x^2)} \\ & + \frac{4(5760\pi^4 - 30\pi^8)}{3(-15360x^2 + 17\pi^6x^2 - 20\pi^6 + 3840\pi^2)(\pi^2 - 4x^2)}, \end{aligned}$$

such that $r_5(x) \geq f_5(x) \geq l_5(x) > L_5(x)$.

The results are summarized as follows, which are simply illustrated by using Fig.1. In principle, the method in [15] has provided a way for proving the following inequalities of mixed trigonometric polynomial functions, this paper also shows proofs in another way, see also proofs in Appendix.

Theorem 2. (Main results) We have that

$$r_0(x) > f_1(x) > l_0(x),\ \forall x \in (0, \pi/2), \tag{9}$$

$$U_1(x) > r_1(x) > f_1(x) > l_1(x) > L_1(x),\ \forall x \in (0, \pi/2), \tag{10}$$

$$r_2(x) > f_2(x) > l_2(x) > L_2(x),\ \forall x \in (0, \pi/2), \tag{11}$$

$$r_3(x) > f_3(x) > l_3(x) > L_3(x),\ \forall x \in (0, \pi/2), \tag{12}$$

$$r_4(x) > f_4(x) > l_4(x) > L_4(x),\ \forall x \in (0, \pi/2), \tag{13}$$

$$r_5(x) > f_5(x) > l_5(x) > L_5(x),\ \forall x \in (0, \pi/2). \tag{14}$$

§4 More applications

Firstly, we consider to approximate the integral of some trigonometric functions, such as $f_1(x), x \in [0, \pi/2]$, by using the bounding functions in form $D_i(x) = d_{i,0} + (\sum_{j=0}^{4} d_{i,j+1}x^j)\cos(x) + (\sum_{j=0}^{4} d_{i,j+6}x^j)\sin(x)$, $i = 1, 2$, where $d_{i,j}$ can be determined by the constraints that

$$\begin{aligned} f^{(j)}_1(0) = G^{(j)}_1(0),\ f^{(k)}_1(\pi/2) = G^{(k)}_1(\pi/2),\ j = 0, 1, \cdots, 5,\ k = 0, 1, \cdots, 4, \\ f^{(j)}_1(0) = G^{(j)}_2(0),\ f^{(k)}_1(\pi/2) = G^{(k)}_2(\pi/2),\ j = 0, 1, \cdots, 4,\ k = 0, 1, \cdots, 5. \end{aligned}$$

Then, $Si(\pi/2) = \int_0^{\pi/2} f_1(x)dx$ is bounded by [1.370762168133,1.370762168179] with length $4.6 \cdot 10^{-11}$. The integral of any subinterval of $[0, \pi/2]$ can be well-approximated whose approximation error is less than $4.6 \cdot 10^{-11}$ of the whole interval $[0, \pi/2]$. In principle, the more unknowns and the more interpolation conditions, the better the approximation effect.

Secondly, we consider the famous square root inequalities

$$2\sqrt{n+1}-2\sqrt{n}<1/\sqrt{n}<2\sqrt{n}-2\sqrt{n-1}, n\in\mathbb{N}.$$

In this case, from the new method, we can search the bounding function in the form $a_0\sqrt{n-1}+a_1\sqrt{n}+a_2\sqrt{n+1}$ and obtain that

$$(\sqrt{2}/2-1)\sqrt{n-1}-\sqrt{2}\sqrt{n}+(1+\sqrt{2}/2)\sqrt{n+1}\le 1/\sqrt{n}\le\sqrt{n+1}-\sqrt{n-1},$$

which has a much tighter bound.

Appendix: Proofs of Eq. (10 ∼ 14)

This section will prove Eq. (10 ∼ 14), where all of the figures are plotted by using the Maple software. The method of proof is similar to that of the method developed in [15,29]. Note that it is trivial to compare two rational polynomials of degrees less than 6 within $[0,\pi/2]$ by using Maple software,we omit the proofs of $l_i(x)-L_i(x)>0, \forall x\in(0,\pi/2)$, $i=1,2,\cdots,5$.

1 Lemmas

Let $E_1(x)=\sin(x)-s_1(x)$, $E_2(x)=\sin(x)-s_2(x)$, $E_3(x)=\sin(x)\cdot\cos(x)-c_1(x)$, $E_4(x)=\sin(x)\cdot\cos(x)-c_2(x)$, $E_5(x)=\cos(x)-s_3(x)$ and $E_6(x)=\cos(x)-s_4(x)$, where

$$\alpha_0=\frac{-382588157952000+178541140377600\pi-6376469299200\pi^3+66421555200\pi^5}{1556755200\pi^{14}}+\frac{-316293120\pi^7+823680\pi^9-1248\pi^{11}+\pi^{13}}{1556755200\pi^{14}},$$

$$\alpha_1=\frac{-714164561510400+331576403558400\pi-11690193715200\pi^3+119558799360\pi^5}{-1556755200\pi^{15}}+\frac{-553512960\pi^7+1372800\pi^9-1872\pi^{11}+\pi^{13}}{-1556755200\pi^{15}},$$

$$\alpha_2=\frac{1144494489600-531372441600\pi+18734284800\pi^3-191600640\pi^5+887040\pi^7}{9979200\pi^{13}}+\frac{-2200\pi^9+3\pi^{11}}{9979200\pi^{13}}$$

$$\alpha_3=-\frac{531372441600-245248819200\pi+8515584000\pi^3-85155840\pi^5+380160\pi^7}{2494800\pi^{14}}-\frac{-880\pi^9+\pi^{11}}{2494800\pi^{14}},$$

$$s_1(x)=\sum_{i=0}^{6}(-1)^i\frac{x^{2i+1}}{(2i+1)!}+\alpha_0x^{14}+\alpha_1x^{15},\quad s_2(x)=\sum_{i=0}^{5}(-1)^i\frac{x^{2i+1}}{(2i+1)!}+\alpha_2x^{13}+\alpha_3x^{14},$$

$$\alpha_4=\frac{\pi^{12}-1056\pi^{10}+570240\pi^8-170311680\pi^6+25546752000\pi^4}{-119750400\pi^{13}}+\frac{-1471492915200\pi^2-490497638400\pi+13733933875200}{-119750400\pi^{13}},$$

$$\alpha_5=\frac{\pi^{12}-1584\pi^{10}+950400\pi^8-298045440\pi^6+45984153600\pi^4}{119750400\pi^{14}}+\frac{-2697737011200\pi^2-980995276800\pi+25505877196800}{119750400\pi^{14}},$$

$$\alpha_6=\frac{\pi^{10}-600\pi^8+188160\pi^6-29030400\pi^4+1703116800\pi^2+619315200\pi-16102195200}{302400\pi^{12}},$$

$$\alpha_7 = \frac{\pi^{10} - 720\pi^8 + 241920\pi^6 - 38707200\pi^4 + 2322432000\pi^2 + 928972800\pi - 22295347200}{-226800\pi^{13}},$$
$$s_3(x) = \sum_{i=0}^{6}(-1)^i\frac{x^{2i}}{(2i)!} + \alpha_4x^{13} + \alpha_5x^{14}, \quad s_4(x) = \sum_{i=0}^{5}(-1)^i\frac{x^{2i}}{(2i)!} + \alpha_6x^{12} + \alpha_7x^{13},$$
$$\beta_0 = -\frac{40475635200 - 6227020800\pi^2 + 259459200\pi^4 - 4942080\pi^6 + 51480\pi^8 - 312\pi^{10} + \pi^{12}}{6081075\pi^{14}},$$
$$\beta_1 = \frac{74724249600 - 11416204800\pi^2 + 467026560\pi^4 - 8648640\pi^6 + 85800\pi^8 - 468\pi^{10} + \pi^{12}}{6081075\pi^{15}},$$
$$\beta_2 = \frac{16(-219542400 + 33264000\pi^2 - 1330560\pi^4 + 23760\pi^6 - 220\pi^8 + \pi^{10})}{155925\pi^{11}}$$
$$\beta_3 = -\frac{16(-399168000 + 59875200\pi^2 - 2328480\pi^4 + 39600\pi^6 - 330\pi^8 + \pi^{10})}{155925\pi^{12}},$$
$$c_1(x) = \sum_{i=0}^{6}(-1)^i\frac{2^{2i}x^{2i+1}}{(2i+1)!} + \beta_0x^{14} + \beta_1x^{15}, \quad c_2(x) = \sum_{i=0}^{5}(-1)^i\frac{2^{2i}x^{2i+1}}{(2i+1)!} + \beta_2x^{12} + \beta_3x^{13}.$$

We have the following lemmas.

Lemma 1. We have that $E_1(x) > 0$ and $E_2(x) < 0$, $E_3(x) > 0$ and $E_4(x) < 0$, $E_5(x) > 0$ and $E_6(x) < 0$, $\forall x \in (0, \pi/2)$.

Proof. It can be verified that $\forall x \in (0, \pi/2)$, we have that

$$\begin{aligned}
&(E_1(x))^{(16)} = \sin(x) > 0, \quad E_1^{(i)}(0) = 0 = E_1^{(j)}(\pi/2), \ i = 0, 1, \cdots, 13, \ j = 0, 1,\\
&(E_2(x))^{(15)} = -\cos(x) < 0, \quad E_2^{(i)}(0) = 0 = E_2^{(j)}(\pi/2), \ i = 0, 1, \cdots, 12, \ j = 0, 1,\\
&(E_3(x))^{(16)} = (\cos(x)\cdot\sin(x))^{(16)} = 32768\sin(2x) > 0,\\
&E_3^{(i)}(0) = 0 = E_3^{(j)}(\pi/2), \ i = 0, 1, \cdots, 13, \ j = 0, 1,\\
&(E_4(x))^{(14)} = (\cos(x)\cdot\sin(x))^{(14)} = -8192\sin(2x) < 0,\\
&E_4^{(i)}(0) = 0 = E_4^{(j)}(\pi/2), \ i = 0, 1, \cdots, 11, \ j = 0, 1,\\
&(E_5(x))^{(15)} = \sin(x) > 0, \quad E_5^{(i)}(0) = 0 = E_5^{(j)}(\pi/2), \ i = 0, 1, \cdots, 12, \ j = 0, 1,\\
&(E_6(x))^{(14)} = -\cos(x) < 0, \quad E_6^{(i)}(0) = 0 = E_6^{(j)}(\pi/2), \ i = 0, 1, \cdots, 11, \ j = 0, 1,
\end{aligned} \tag{15}$$

Combining Eq. (15) with Theorem 1, there exists $\xi_i \in (0, \pi/2)$, $i = 1, 2, \cdots, 6$, such that

$$\begin{aligned}
E_1(x) &= \frac{E_1^{(16)}(\xi_1(x))}{16!}(x-0)^{14}(x-\pi/2)^2 = \frac{\sin(\xi_1(x))}{16!}(x-0)^{14}(x-\pi/2)^2 > 0,\\
E_2(x) &= \frac{E_2^{(15)}(\xi_2(x))}{15!}(x-0)^{13}(x-\pi/2)^2 = \frac{-\cos(\xi_2(x))}{15!}(x-0)^{13}(x-\pi/2)^2 < 0,\\
E_3(x) &= \frac{E_3^{(16)}(\xi_3(x))}{16!}(x-0)^{14}(x-\pi/2)^2 = \frac{32768\sin(2\xi_3(x))}{16!}(x-0)^{14}(x-\pi/2)^2 > 0,\\
E_4(x) &= \frac{E_4^{(14)}(\xi_4(x))}{14!}(x-0)^{12}(x-\pi/2)^2 = \frac{-8192\sin(2\xi_4(x))}{14!}(x-0)^{12}(x-\pi/2)^2 < 0,\\
E_5(x) &= \frac{E_5^{(15)}(\xi_5(x))}{15!}(x-0)^{13}(x-\pi/2)^2 = \frac{\sin(\xi_5(x))}{15!}(x-0)^{13}(x-\pi/2)^2 > 0,\\
E_6(x) &= \frac{E_2^{(14)}(\xi_6(x))}{14!}(x-0)^{12}(x-\pi/2)^2 = \frac{-\cos(\xi_6(x))}{14!}(x-0)^{12}(x-\pi/2)^2 < 0,
\end{aligned}$$

and we have completed the proof. □ From Lemma 1, we have that **Lemma 2.** $s_1(x) < \sin(x) < s_2(x)$, $s_3(x) < \cos(x) < s_4(x)$ and $c_1(x) < \sin(x)\cdot\cos(x) < c_2(x)$, $\forall x \in (0, \pi/2)$.

2 Proof of Eq. (10)

Firstly, we prove that $r_1(x) > f_1(x), \forall x \in (0, \pi/2)$. It is equivalent to prove that $G_1(x) = (f_1(x) - r_1(x))\cdot x\cdot(1 + d_{1,3}x^2 + d_{1,4}x^4) = \sin(x)\cdot(1 + d_{1,3}x^2 + d_{1,4}x^4) - x\cdot(d_{1,0} + d_{1,1}x^2 + d_{1,2}x^4) <$

$0, \forall x \in (0, \pi/2)$. It can be verified that $\forall x \in (0, \pi/2)$,

$$\begin{cases} 1 + d_{1,3}x^2 + d_{1,4}x^4 > 0, \\ G_1(x) < s_2(x) \cdot (1 + d_{1,3}x^2 + d_{1,4}x^4) - x \cdot (d_{1,0} + d_{1,1}x^2 + d_{1,2}x^4) \\ = x^7(x - \pi/2)^2(\sum_{i=0}^{9} \gamma_{9,i} B_{9,i}(x)), \end{cases} \tag{16}$$

where $B_{9,i}(x) = C_9^i(x-0)^{9-i}(\pi/2 - x)^i/(\pi/2)^9 > 0, \forall x \in (0, \pi/2)$, $\gamma_{9,0} \approx -1.5e-8 < 0$, $\gamma_{9,1} \approx -1.9e-8 < 0$, $\gamma_{9,2} \approx -2.3e-8 < 0$, $\gamma_{9,3} \approx -2.3e-8 < 0$, $\gamma_{9,4} \approx -3.3e-8 < 0$, $\gamma_{9,5} \approx -4.0e-8 < 0$, $\gamma_{9,6} \approx -4.6e-8 < 0$, $\gamma_{9,7} \approx -5.4e-8 < 0$, $\gamma_{9,8} \approx -3.6e-8 < 0$ and $\gamma_{9,9} \approx -7.3e-8 < 0$, which leads to $G_1(x) < 0$, and the proof is completed.

Secondly, we prove that $l_1(x) \le f_1(x), \forall x \in [0, \pi/2]$. It is equivalent to prove that $G_2(x) = (f_1(x) - l_1(x)) \cdot x \cdot (1 + e_{1,3}x^2 + e_{1,4}x^4) = \sin(x) \cdot (1 + e_{1,3}x^2 + e_{1,4}x^4) - x \cdot (e_{1,0} + e_{1,1}x^2 + e_{1,2}x^4) > 0, \forall x \in (0, \pi/2)$. It can be verified that $\forall x \in (0, \pi/2)$,

$$\begin{cases} 1 + e_{1,3}x^2 + e_{1,4}x^4 > 0, \\ G_2(x) > s_1(x) \cdot (1 + e_{1,3}x^2 + e_{1,4}x^4) - x \cdot (e_{1,0} + e_{1,1}x^2 + e_{1,2}x^4) \\ = x^9(\pi/2 - x)(\sum_{i=0}^{9} \bar{\gamma}_{9,i} B_{9,i}(x)), \end{cases} \tag{17}$$

where $B_{9,i}(x) = C_9^i(x-0)^{9-i}(\pi/2 - x)^i/(\pi/2)^9 > 0, \forall x \in (0, \pi/2)$, $\bar{\gamma}_{9,0} \approx 1.9e-8 > 0$, $\bar{\gamma}_{9,1} \approx 2.1e-8 > 0$, $\bar{\gamma}_{9,2} \approx 2.4e-8 > 0$, $\bar{\gamma}_{9,3} \approx 2.6e-8 > 0$, $\bar{\gamma}_{9,4} \approx 2.8e-8 > 0$, $\bar{\gamma}_{9,5} \approx 2.9e-8 > 0$, $\bar{\gamma}_{9,6} \approx 3.2e-8 > 0$, $\bar{\gamma}_{9,7} \approx 3.2e-8 > 0$, $\bar{\gamma}_{9,8} \approx 3.0e-8 > 0$ and $\bar{\gamma}_{9,9} \approx 3.6e-8 > 0$, which leads to $G_2(x) > 0$, $\forall x \in (0, \pi/2)$. And the proof is completed.

3 Proof of Eq. (11)

Firstly, we prove that $r_2(x) > f_2(x), \forall x \in (0, \pi/2)$. It is equivalent to prove that $G_3(x) = (f_2(x) - r_2(x)) \cdot \sin(x) \cdot (1 + e_{2,3}x^2) = 3x \cdot (1 + e_{2,3}x^2) + \sin(x) \cdot \cos(x) \cdot (1 + e_{2,3}x^2) - \sin(x) \cdot (e_{2,0} + e_{2,1}x^2 + e_{2,2}x^4) < 0, \forall x \in (0, \pi/2)$. It can be verified that

$$\begin{aligned} &(1 + e_{2,3}x^2) > 0,\ (e_{2,0} + e_{2,1}x^2 + e_{2,2}x^4) > 0, \\ &G_3(x) = 3x \cdot (1 + e_{2,3}x^2) + \sin(x) \cdot \cos(x) \cdot (1 + e_{2,3}x^2) - \sin(x) \cdot (e_{2,0} + e_{2,1}x^2 + e_{2,2}x^4) \\ &< 3x \cdot (1 + e_{2,3}x^2) + c_2(x) \cdot (1 + e_{2,3}x^2) - s_1(x) \cdot (e_{2,0} + e_{2,1}x^2 + e_{2,2}x^4) \\ &= (\pi - 2x)x^7 \cdot (\sum_{i=0}^{11} \gamma_{11,i} B_{11,i}(x)), \end{aligned} \tag{18}$$

where $B_{11,i}(x) = C_{11}^i(x-0)^{11-i}(\pi/2 - x)^i/(\pi/2)^{11} > 0, \forall x \in (0, \pi/2)$, $\gamma_{11,0} \approx -3.6e-4 < 0$, $\gamma_{11,1} \approx -3.9e-4 < 0$, $\gamma_{11,2} \approx -4.2e-4 < 0$, $\gamma_{11,3} \approx -4.5e-4 < 0$, $\gamma_{11,4} \approx -4.8e-4 < 0$, $\gamma_{11,5} \approx -5.0e-4 < 0$, $\gamma_{11,6} \approx -5.2e-4 < 0$, $\gamma_{11,7} \approx -5.4e-4 < 0$, $\gamma_{11,8} \approx -5.5e-4 < 0$, $\gamma_{11,9} \approx -5.6e-4 < 0$, $\gamma_{11,10} \approx -5.7e-4 < 0$ and $\gamma_{11,11} \approx -5.7e-4 < 0$, which leads to $G_3(x) < 0$, $\forall x \in (0, \pi/2)$, and the proof is completed. Secondly, we prove that $l_2(x) \le f_2(x), \forall x \in [0, \pi/2]$. It is equivalent to prove that $G_4(x) = (f_2(x) - l_2(x)) \cdot \sin(x) \cdot (1 + d_{2,3}x^2) > 0, \forall x \in (0, \pi/2)$. It can be verified that

$$(1 + d_{2,3}x^2) > 0,\ (d_{2,0} + d_{2,1}x^2 + d_{2,2}x^4) > 0,$$

$$\begin{aligned} G_4(x) &= 3x \cdot (1 + d_{2,3}x^2) + \sin(x) \cdot \cos(x) \cdot (1 + d_{2,3}x^2) - \sin(x) \cdot (d_{2,0} + d_{2,1}x^2 + d_{2,2}x^4) \\ &> 3x \cdot (1 + d_{2,3}x^2) + c_1(x) \cdot (1 + d_{2,3}x^2) - s_2(x) \cdot (d_{2,0} + d_{2,1}x^2 + d_{2,2}x^4) \qquad (19) \\ &= (\pi - 2x)^2 x^5 \cdot (\sum_{i=0}^{11} \bar{\gamma}_{11,i} B_{11,i}(x)), \end{aligned}$$

where $B_{11,i}(x) = C_{11}^i (x-0)^{11-i}(\pi/2 - x)^i/(\pi/2)^{11} > 0, \forall x \in (0, \pi/2)$, $\bar{\gamma}_{11,0} \approx 2.9e-4 > 0$, $\bar{\gamma}_{11,1} \approx 3.5e-4 > 0$, $\bar{\gamma}_{11,2} \approx 4.0e-4 > 0$, $\bar{\gamma}_{11,3} \approx 4.6e-4 > 0$, $\bar{\gamma}_{11,4} \approx 5.3e-4 > 0$, $\bar{\gamma}_{11,5} \approx 5.9e-4 > 0$, $\bar{\gamma}_{11,6} \approx 6.5e-4 > 0$, $\bar{\gamma}_{11,7} \approx 7.2e-4 > 0$, $\bar{\gamma}_{11,8} \approx 7.8e-4 > 0$, $\bar{\gamma}_{11,9} \approx 8.3e-4 > 0$, $\bar{\gamma}_{11,10} \approx 8.8e-4 > 0$ and $\bar{\gamma}_{11,11} \approx 9.3e-4 > 0$, which leads to $G_4(x) > 0$, $\forall x \in (0, \pi/2)$. The proof is completed.

4 Proof of Eq. (12)

Firstly, we prove that $r_3(x) \geq f_3(x), \forall x \in [0, \pi/2]$. It is equivalent to prove that $G_5(x) = (f_3(x) - r_3(x)) \cdot \cos(x) \cdot x^2 \cdot (1 + d_{3,4}x^2 + d_{3,5}x^4 + d_{3,6}x^6) = (\sin(x)^2\cos(x) + x\sin(x)) \cdot (1 + d_{3,4}x^2 + d_{3,5}x^4 + d_{3,6}x^6) - \cos(x) \cdot x^2 \cdot (d_{3,0} + d_{3,1}x^2 + d_{3,2}x^4 + d_{3,3}x^6) < 0, \forall x \in (0, \pi/2)$. It can be verified that

$$\begin{aligned} &(1 + d_{3,4}x^2 + d_{3,5}x^4 + d_{3,6}x^6) > 0,\ (d_{3,0} + d_{3,1}x^2 + d_{3,2}x^4 + d_{3,3}x^6) > 0, \\ &G_5(x) = (\sin(x)^2\cos(x) + x\sin(x)) \cdot (1 + d_{3,4}x^2 + d_{3,5}x^4 + d_{3,6}x^6) \\ &- \cos(x) \cdot x^2 \cdot (d_{3,0} + d_{3,1}x^2 + d_{3,2}x^4 + d_{3,3}x^6) \\ &< (c_2(x) \cdot s_2(x) + x \cdot s_2(x)) \cdot (1 + d_{3,4}x^2 + d_{3,5}x^4 + d_{3,6}x^6) \qquad (20) \\ &- s_3(x) \cdot x^2 \cdot (d_{3,0} + d_{3,1}x^2 + d_{3,2}x^4 + d_{3,3}x^6) \\ &= (\pi - 2x)^2 x^{12} \cdot (\sum_{i=0}^{19} \gamma_{19,i} B_{19,i}(x)), \end{aligned}$$

where $B_{19,i}(x) = C_{19}^i (x-0)^{19-i}(\pi/2 - x)^i/(\pi/2)^{19} > 0, \forall x \in (0, \pi/2)$, $\gamma_{19,0} \approx -2.5e-6 < 0$, $\gamma_{19,1} \approx -2.8e-6 < 0$, $\gamma_{19,2} \approx -3.0e-6 < 0$, $\gamma_{19,3} \approx -3.3e-6 < 0$, $\gamma_{19,4} \approx -3.6e-6 < 0$, $\gamma_{19,5} \approx -3.8e-6 < 0$, $\gamma_{19,6} \approx -4.1e-6 < 0$, $\gamma_{19,7} \approx -4.4e-6 < 0$, $\gamma_{19,8} \approx -4.6e-6 < 0$, $\gamma_{19,9} \approx -4.9e-6 < 0$, $\gamma_{19,10} \approx -5.1e-6 < 0$, $\gamma_{19,11} \approx -5.3e-6 < 0$, $\gamma_{19,12} \approx -5.5e-6 < 0$, $\gamma_{19,13} \approx -5.7e-6 < 0$, $\gamma_{19,14} \approx -5.9e-6 < 0$, $\gamma_{19,15} \approx -6.0e-6 < 0$, $\gamma_{19,16} \approx -6.1e-6 < 0$, $\gamma_{19,17} \approx -6.2e-6 < 0$, $\gamma_{19,18} \approx -6.3e-6 < 0$ and $\gamma_{19,19} \approx -6.3e-6 < 0$, which leads to $G_5(x) < 0$, $\forall x \in (0, \pi/2)$, and the proof is completed.

Secondly, we prove that $l_3(x) < f_3(x), \forall x \in (0, \pi/2)$. It is equivalent to prove that $G_6(x) = (f_3(x) - l_3(x)) \cdot \cos(x) \cdot x^2 \cdot (1 + e_{3,4}x^2 + e_{3,5}x^4 + e_{3,6}x^6) = (\sin(x)^2\cos(x) + x\sin(x)) \cdot (1 + e_{3,4}x^2 + e_{3,5}x^4 + e_{3,6}x^6) - \cos(x) \cdot x^2 \cdot (e_{3,0} + e_{3,1}x^2 + e_{3,2}x^4 + e_{3,3}x^6) > 0, \forall x \in (0, \pi/2)$. It can be verified that

$$\begin{aligned} &(1 + e_{3,4}x^2 + e_{3,5}x^4 + e_{3,6}x^6) > 0,\ (e_{3,0} + e_{3,1}x^2 + e_{3,2}x^4 + e_{3,3}x^6) > 0, \\ &G_6(x) = (\sin(x)^2\cos(x) + x\sin(x)) \cdot (1 + e_{3,4}x^2 + e_{3,5}x^4 + e_{3,6}x^6) \\ &- \cos(x) \cdot x^2 \cdot (e_{3,0} + e_{3,1}x^2 + e_{3,2}x^4 + e_{3,3}x^6) \\ &> (c_1(x) \cdot s_2(x) + x \cdot s_1(x)) \cdot (1 + e_{3,4}x^2 + e_{3,5}x^4 + e_{3,6}x^6) \qquad (21) \\ &- s_4(x) \cdot x^2 \cdot (e_{3,0} + e_{3,1}x^2 + e_{3,2}x^4 + e_{3,3}x^6) \\ &= (\pi - 2x) x^{13} \cdot (\sum_{i=0}^{20} \gamma_{20,i} B_{20,i}(x)), \end{aligned}$$

where $B_{20,i}(x) = C_{20}^{i}(x-0)^{20-i}(\pi/2-x)^{i}/(\pi/2)^{20} > 0, \forall x \in (0,\pi/2)$, $\gamma_{20,0} \approx 2.8e-7 > 0$, $\gamma_{20,1} \approx 5.2e-7 > 0$, $\gamma_{20,2} \approx 7.8e-7 > 0$, $\gamma_{20,3} \approx 1.1e-6 > 0$, $\gamma_{20,4} \approx 1.4e-6 > 0$, $\gamma_{20,5} \approx 1.7e-6 > 0$, $\gamma_{20,6} \approx 2.0e-6 > 0$, $\gamma_{20,7} \approx 2.3e-6 > 0$, $\gamma_{20,8} \approx 2.7e-6 > 0$, $\gamma_{20,9} \approx 3.1e-6 > 0$, $\gamma_{20,10} \approx 3.4e-6 > 0$, $\gamma_{20,11} \approx 3.8e-6 > 0$, $\gamma_{20,12} \approx 4.1e-6 > 0$, $\gamma_{20,13} \approx 4.4e-6 > 0$, $\gamma_{20,14} \approx 4.8e-6 > 0$, $\gamma_{20,15} \approx 5.1e-6 > 0$, $\gamma_{20,16} \approx 5.3e-6 > 0$, $\gamma_{20,17} \approx 5.6e-6 > 0$, $\gamma_{20,18} \approx 5.8e-6 > 0$, $\gamma_{20,19} \approx 6.0e-6 > 0$ and $\gamma_{20,20} \approx 6.2e-6 > 0$, which leads to $G_6(x) > 0$, $\forall x \in (0,\pi/2)$. The proof is completed.

5 Proof of Eq. (13)

Firstly, we prove that $r_4(x) > f_4(x), \forall x \in (0,\pi/2)$. It is equivalent to prove that $G_7(x) = (f_4(x)-r_4(x))\cdot\cos(x)\cdot x\cdot(1+e_{4,3}x^2+e_{4,4}x^4) = (2\sin(x)\cdot\cos(x)+\sin(x))\cdot(1+e_{4,3}x^2+e_{4,4}x^4) - \cos(x)\cdot x\cdot(e_{4,0}+e_{4,1}x^2+e_{4,2}x^4) < 0, \forall x \in (0,\pi/2)$. It can be verified that $\forall x \in (0,\pi/2)$,

$$\begin{cases} (1+e_{4,3}x^2+e_{4,4}x^4) > 0,\ (e_{4,0}+e_{4,1}x^2+e_{4,2}x^4) > 0, \\ G_7(x) = (2\sin(x)\cdot\cos(x)+\sin(x))\cdot(1+e_{4,3}x^2+e_{4,4}x^4) \\ -\cos(x)\cdot x\cdot(e_{4,0}+e_{4,1}x^2+e_{4,2}x^4) \\ < (2c_2(x)+s_2(x))\cdot(1+e_{4,3}x^2+e_{4,4}x^4) - s_3(x)\cdot x\cdot(e_{4,0}+e_{4,1}x^2+e_{4,2}x^4) \\ = x^9(\pi/2-x)(\sum_{i=0}^{9}\hat{\gamma}_{9,i}B_{9,i}(x)), \end{cases} \tag{22}$$

where $B_{9,i}(x) = C_9^i(x-0)^{9-i}(\pi/2-x)^i/(\pi/2)^9 > 0, \forall x \in (0,\pi/2)$, $\hat{\gamma}_{9,0} \approx -5.9e-5 < 0$, $\hat{\gamma}_{9,1} \approx -6.5e-5 < 0$, $\hat{\gamma}_{9,2} \approx -7.1e-5 < 0$, $\hat{\gamma}_{9,3} \approx -7.6e-5 < 0$, $\hat{\gamma}_{9,4} \approx -8.1e-5 < 0$, $\hat{\gamma}_{9,5} \approx -8.4e-5 < 0$, $\hat{\gamma}_{9,6} \approx -8.6e-5 < 0$, $\hat{\gamma}_{9,7} \approx -8.7e-5 < 0$, $\hat{\gamma}_{9,8} \approx -8.7e-5 < 0$ and $\hat{\gamma}_{9,9} \approx -8.6e-5 < 0$, which leads to $G_7(x) < 0$, and the proof is completed. Secondly, we prove that $l_4(x) < f_4(x), \forall x \in (0,\pi/2)$. It is equivalent to prove that $G_8(x) = (f_4(x)-l_4(x))\cdot\cos(x)\cdot x\cdot(1+d_{4,3}x^2+d_{4,4}x^4) = (2\sin(x)\cdot\cos(x)+\sin(x))\cdot(1+d_{4,3}x^2+d_{4,4}x^4) - \cos(x)\cdot x\cdot(d_{4,0}+d_{4,1}x^2+d_{4,2}x^4) > 0, \forall x \in (0,\pi/2)$. It can be verified that $\forall x \in (0,\pi/2)$,

$$\begin{cases} (1+d_{4,3}x^2+d_{4,4}x^4) > 0,\ (d_{4,0}+d_{4,1}x^2+d_{4,2}x^4) > 0, \\ G_8(x) = (2\sin(x)\cdot\cos(x)+\sin(x))\cdot(1+d_{4,3}x^2+d_{4,4}x^4) \\ -\cos(x)\cdot x\cdot(d_{4,0}+d_{4,1}x^2+d_{4,2}x^4) \\ < (2c_2(x)+s_2(x))\cdot(1+d_{4,3}x^2+d_{4,4}x^4) - s_3(x)\cdot x\cdot(d_{4,0}+d_{4,1}x^2+d_{4,2}x^4) \\ = x^7(\pi/2-x)^2(\sum_{i=0}^{10}\hat{\gamma}_{10,i}B_{10,i}(x)), \end{cases} \tag{23}$$

where $B_{10,i}(x) = C_{10}^i(x-0)^{10-i}(\pi/2-x)^i/(\pi/2)^{10} > 0, \forall x \in (0,\pi/2)$, $\hat{\gamma}_{10,0} \approx 4.8e-5 > 0$, $\hat{\gamma}_{10,1} \approx 5.8e-5 > 0$, $\hat{\gamma}_{10,2} \approx 6.8e-5 > 0$, $\hat{\gamma}_{10,3} \approx 7.9e-5 > 0$, $\hat{\gamma}_{10,4} \approx 9.0e-5 > 0$, $\hat{\gamma}_{10,5} \approx 1.0e-4 > 0$, $\hat{\gamma}_{10,6} \approx 1.1e-4 > 0$, $\hat{\gamma}_{10,7} \approx 1.2e-4 > 0$, $\hat{\gamma}_{10,8} \approx 1.3e-4 > 0$, $\hat{\gamma}_{10,9} \approx 1.4e-4 > 0$ and $\hat{\gamma}_{10,10} \approx 1.4e-4 > 0$, which leads to $G_8(x) > 0$. The proof is completed.

6 Proof of Eq. (14)

Firstly, we prove that $r_5(x) > f_5(x), \forall x \in (0,\pi/2)$. It is equivalent to prove that $G_9(x) = (f_5(x)-r_5(x))\cdot\cos(x)\cdot x\cdot(1+e_{5,3}x^2+e_{5,4}x^4) = (2\sin(x)\cdot\cos(x)+\sin(x))\cdot(1+e_{5,3}x^2+e_{5,4}x^4) -$

$\cos(x) \cdot x \cdot (e_{5,0} + e_{5,1}x^2 + e_{5,2}x^4) < 0, \forall x \in (0, \pi/2)$. It can be verified that $\forall x \in (0, \pi/2)$,

$$\begin{cases} (1 + e_{5,3}x^2 + e_{5,4}x^4) > 0, \ (e_{5,0} + e_{5,1}x^2 + e_{5,2}x^4) > 0, \\ G_9(x) = (2\sin(x) \cdot \cos(x) + \sin(x)) \cdot (1 + e_{5,3}x^2 + e_{5,4}x^4) \\ - \cos(x) \cdot x \cdot (e_{5,0} + e_{5,1}x^2 + e_{5,2}x^4) \\ < (2c_2(x) + s_2(x)) \cdot (1 + e_{5,3}x^2 + e_{5,4}x^4) - s_3(x) \cdot x \cdot (e_{5,0} + e_{5,1}x^2 + e_{5,2}x^4) \\ = x^9(\pi/2 - x)(\sum_{i=0}^{9} \tilde{\gamma}_{9,i} B_{9,i}(x)), \end{cases} \tag{24}$$

where $B_{9,i}(x) = C_9^i (x - 0)^{9-i}(\pi/2 - x)^i/(\pi/2)^9 > 0, \forall x \in (0, \pi/2)$, $\tilde{\gamma}_{9,0} \approx -3.2e - 5 < 0$, $\tilde{\gamma}_{9,1} \approx -3.5e - 5 < 0$, $\tilde{\gamma}_{9,2} \approx -3.8e - 5 < 0$, $\tilde{\gamma}_{9,3} \approx -4.1e - 5 < 0$, $\tilde{\gamma}_{9,4} \approx -4.4e - 5 < 0$, $\tilde{\gamma}_{9,5} \approx -4.5e - 5 < 0$, $\tilde{\gamma}_{9,6} \approx -4.7e - 5 < 0$, $\tilde{\gamma}_{9,7} \approx -4.7e - 5 < 0$, $\tilde{\gamma}_{9,8} \approx -4.7e - 5 < 0$ and $\tilde{\gamma}_{9,9} \approx -4.6e - 5 < 0$, which leads to $G_9(x) < 0$, and the proof is completed. Secondly, we prove that $l_5(x) < f_5(x), \forall x \in (0, \pi/2)$. It is equivalent to prove that $G_{10}(x) = (f_5(x) - l_5(x)) \cdot \cos(x) \cdot x \cdot (1 + d_{5,3}x^2 + d_{5,4}x^4) = (2\sin(x) \cdot \cos(x) + \sin(x)) \cdot (1 + d_{5,3}x^2 + d_{5,4}x^4) - \cos(x) \cdot x \cdot (d_{5,0} + d_{5,1}x^2 + d_{5,2}x^4) > 0, \forall x \in (0, \pi/2)$. It can be verified that $\forall x \in (0, \pi/2)$,

$$\begin{cases} (1 + d_{5,3}x^2 + d_{5,4}x^4) > 0, \ (d_{5,0} + d_{5,1}x^2 + d_{5,2}x^4) > 0, \\ G_{10}(x) = (2\sin(x) \cdot \cos(x) + \sin(x)) \cdot (1 + d_{5,3}x^2 + d_{5,4}x^4) \\ - \cos(x) \cdot x \cdot (d_{5,0} + d_{5,1}x^2 + d_{5,2}x^4) \\ < (2c_2(x) + s_2(x)) \cdot (1 + d_{5,3}x^2 + d_{5,4}x^4) - s_3(x) \cdot x \cdot (d_{5,0} + d_{5,1}x^2 + d_{5,2}x^4) \\ = x^7(\pi/2 - x)^2(\sum_{i=0}^{10} \tilde{\gamma}_{10,i} B_{10,i}(x)), \end{cases} \tag{25}$$

where $B_{10,i}(x) = C_{10}^i (x - 0)^{10-i}(\pi/2 - x)^i/(\pi/2)^{10} > 0, \forall x \in (0, \pi/2)$, $\tilde{\gamma}_{10,0} \approx 2.6e - 5 > 0$, $\tilde{\gamma}_{10,1} \approx 3.1e - 5 > 0$, $\tilde{\gamma}_{10,2} \approx 3.7e - 5 > 0$, $\tilde{\gamma}_{10,3} \approx 4.3e - 5 > 0$, $\tilde{\gamma}_{10,4} \approx 4.9e - 5 > 0$, $\tilde{\gamma}_{10,5} \approx 5.5e - 5 > 0$, $\tilde{\gamma}_{10,6} \approx 6.1e - 5 > 0$, $\tilde{\gamma}_{10,7} \approx 6.6e - 5 > 0$, $\tilde{\gamma}_{10,8} \approx 7.1e - 5 > 0$, $\tilde{\gamma}_{10,9} \approx 7.4e - 5 > 0$ and $\tilde{\gamma}_{10,10} \approx 7.7e - 5 > 0$, which leads to $G_{10}(x) > 0$. The proof is completed.

References

[1] Abdel-Raouf O, Abdel-Baset M, El-Henawy I. *Chaotic firefly algorithm for solving definite integrals*, International Journal of Information Technology & Computer Science, 2014, 6(6): 19-24.

[2] Alzer H, Liu X, Shi X. *Inequalities for alternating trigonometric sums*, Results in Mathematics, 2013, 63(2013): 1215-1223.

[3] Bercu G. *Padé approximant related to remarkable inequalities involving trigonometric functions*, Journal of Inequalities and Applications, 2016, 2016(99): 1-11.

[4] Bhayo B A, Sándor J. *On Jordan's, Redheffer's and Wilker's inequality*, Mathematical Inequalities & Applications, 2016, 19(3): 823-839.

[5] Chen, C P, Cheung, W S. *Sharpness of Wilker and Huygens type inequalities*, J Inequal Appl, 2012, 72(2012), https://doi.org/10.1186/1029-242x-2012-72.

[6] Chen C P, Cheung W S. *Wilker-and Huygens-type inequalities and solution to Oppenheim's problem*, Integral Transforms & Special Functions, 2012, 23(5): 325-336.

[7] Davis P J. *Interpolation and approximation*, Dover Publications, New York, 1975.

[8] Debnath L, Mortici C, Zhu L. *Refinements of Jordan-Steckin and Becker-Stark inequalities*, Results in Mathematics, 2015, 67(1-2): 207-215.

[9] González-Santander J L. *Calculation of some integrals arising in the Samara-Valencia solution for dry flat grinding*, Mathematical Problems in Engineering, 2015, 2015, https://doi.org/10.1155/2015/428461.

[10] Hua Y. *Sharp Wilker and Huygens type inequalities for trigonometric and hyperbolic functions*, Hacettepe Journal of Mathematics and Statistics, 2016, 45(3): 731-741.

[11] Krnić M, Vuković P. *A class of Hilbert-type inequalities obtained via the improved young inequality*, Results in Mathematics, 2015, 71(2017): 185-196.

[12] Klén R, Visuri M, Vuorinen M. *On Jordan type inequalities for hyperbolic functions*, Journal of Inequalities and Applications, 2010, 2010, https://doi.org/10.1155/2010/362548.

[13] Kalmykov S I, Nagy B. *Polynomial and rational inequalities on analytic Jordan arcs and domains*, Journal of Mathematical Analysis & Applications, 2015, 430(2): 874-894.

[14] Liu J, Chen C P. *Padé approximant related to inequalities for Gauss lemniscate functions*, Journal of Inequalities & Applications, 2016, 320(2016), https://doi.org/10.1186/s13660-016-1262-2.

[15] Malesevic B, Makragic M. *A method for proving some inequalities on mixed trigonometric polynomial functions*, Journal of Mathematical Inequalities, 2016, 10(3): 849-876.

[16] Mitrinovic D S. *Analytic Inequalities*, Springer, Berlin, 1970.

[17] Mortici, C. *The natural approach of Wilker-Cusa-Huygens inequalities*, Math Inequal Appl, 2011, 14(3): 535-541.

[18] Mortici C. *A subtly analysis of Wilker inequality*, Applied Mathematics & Computation, 2014, 231(1): 516-520.

[19] Nenezić M, Malešević B, Mortici C. *New approximations of some expressions involving trigonometric functions*, Applied Mathematics & Computation, 2016, 283: 299-315.

[20] Nursultanov E, Tikhonov S. *A sharp Remez inequality for trigonometric polynomials*, Constructive Approximation, 2013, 38(1): 101-132.

[21] Olbryś A, Szostok T. *Inequalities of the HermiteCHadamard type involving numerical differentiation formulas*, Results in Mathematics, 2015, 67(3): 403-416.

[22] Qi F, Luo Q M, Guo B N. *A simple proof of Oppenheim's double inequality relating to the cosine and sine functions*, Journal of Mathematical Inequalities, 2012, 6(4): 645-654.

[23] J L G Santander. *Calculation of some integrals involving the Macdonald function by using Fourier transform*, Journal of Mathematical Analysis & Applications, 2016, 441(1): 349-363.

[24] Sun J L, Chen C P. *Shafer-type inequalities for inverse trigonometric functions and Gauss lemniscate functions*, Journal of Inequalities and Applications, 2016, 212(1): 1-9.

[25] Yang Z H. *Refinements of a two-sided inequality for trigonometric functions*, Journal of Mathematical Inequalities, 2013, 7(4): 601-615.

[26] Zhu L. *On Wilker-type inequality*, Math Inequal Appl, 2007, 10(4): 727-731.

[27] Lutovac T, Malesevic B, Rasajski M. *A new method for proving some inequalities related to several special functions*, Results in Mathematics, 2018, 73(100), https://doi.org/10.1007/s00025-018-0862-1.

[28] Zhu L, Malesevic B. *Inequalities between the inverse hyperbolic tangent and the inverse sine and the analogue for corresponding functions*, Journal of Inequalities and Applications, 2019, 93, https://doi.org/10.1186/s13660-019-2046-2.

[29] Banjac B. *System for automatic proving of some classes of analytic inequalities*, Doctoral dissertation (in Serbian), School of Electrical Engineering, Belgrade, 2019.

Key Laboratory of Complex Systems Modeling and Simulation, Hangzhou Dianzi University, Hangzhou 310027, China.

Email: xiaodiao@hdu.edu.cn, yigang.wang@hdu.edu.cn