Appl. Math. J. Chinese Univ.
2020, 35(3): 281-292

Some results on entropy dimension for non-autonomous systems

YANG Yan-juan WANG Lin* WANG Wei

Abstract. In this paper, the preimage branch t-entropy and entropy dimension for nonautonomous systems are studied and some systems with preimage branch t-entropy zero are introduced. Moreover, formulas calculating the s-topological entropy of a sequence of equi-continuous monotone maps on the unit circle are given. Finally, examples to show that the entropy dimension of non-autonomous systems can be attained by any positive number s are constructed.

§1 Introduction

In the study of the autonomous dynamical systems which are induced by the iterations of a single transformation, entropies are important invariants to describe the complexity of dynamical systems. In 1958, with the notion of entropy in information theory, Kolmogorov [16] defined the measure-theoretic entropy for a measure-preserving transformation on a probability space, which describes the maximal information we can get from a system. In 1965, Adler, Konheim and McAndrew [1] introduced the topological entropy for continuous maps on a compact topological space using open covers, which measures the exporential growth rate of the number of orbits of a dynamical system. In 1971, Bowen gave the same definition by spanning sets and separated sets respectively for uniformly continuous maps on a metric space. Since then the notion of entropy including measure-theoretic entropy and topological entropy has been playing an important role in the study of dynamical systems. These two entropies are connected by the variational principle, that is, the topological entropy is equal to the supremum of the measure-theoretic entropies with respect to all invariant probability measures for a given dynamical system [11]. We also refer to [10, 19, 22] for more information about entropies for dynamical systems.

Received: 2017-03-11. Revised: 2019-06-07.
MR Subject Classification:37D30, 37B40, 37C50.
Keywords: entropy dimension, homemorphism, finite graphs, non-autonomous systems.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-020-3537-0.
Lin Wang is supported by the National Natural Science Foundation of China (No.11801336,11771118), the Science and Technology Innovation Project of Shanxi Higher Education (No.2019L0475) and the Applied Basic Research Program of Shanxi Province(No:201901D211417).
*Corresponding author.

In recent years, more and more people keep a watchful eye on the non-autonomous dynamical systems. In order to investigate the complexity of them from different points of view, various of entropy-type invariants are introduced and investigated. Y. Zhu and J. Zhang [25, 29] discuss the topological entropy of non-autonomous systems, for which the lower bound is given. In [15], Kawan and Latushkin obtained some useful results about entropy for non-autonomous systems. In [21, 22], the authors give a description of chaos in term of topological entropy for non-autonomous systems, particularly, some tools are established to give upper and lower bounds of the topological entropy. In [6], the concepts of topological feedback entropy and invariance entropy are given, which gives us a new point to describe the the complexity of a dynamical system. In [24], borrowing the idea from autonomous dynamical systems, Zhang, Zhu and He give the definition of preimage entropy for non-autonomous systems, some similar results being obtained. For more detailed information we refer to [2, 9, 13, 14, 17, 18].

In this paper, we will mainly consider the preimage branch t-entropy of non-autonomous systems. In Section 2, we give the definition of the preimage branch t-entropy, and then we consider two classes of non-autonomous systems, whose preimage branch t-entropies are zero. For more detailed information about the theory of preimage entropies, we refer to [5, 12, 20, 24, 27, 28], and more information about the dynamical systems with zero entropies, we refer to [3, 4, 7, 8]. In Section 3, we will give an interesting result on the entropy dimension of the non-autonomous systems, that is, for every number $s > 0$, we can construct a sequence of equi-continuous monotone maps on the unit circle, satisfying that the entropy dimension is s. For the topological entropy of monotone maps on the circle we refer to [26].

§2 Preliminaries and Preimage Branch t-Entropy

Let (X, d) be a compact metric space and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ a sequence of continuous maps on X. We denote the identity map on X by id. For any $i \in N$, let $f_i^0 = id$ and for any $i, n \in N$, let

$$f_i^n = f_{i+(n-1)} \circ \ldots \circ f_{i+1} \circ f_i,$$
$$f_i^{-n} = (f_i^n)^{-1} = f_i^{-1} \circ f_{i+1}^{-1} \circ \ldots \circ f_{i+(n-1)}^{-1}.$$

$(X, f_{1,\infty})$ is called a non-autonomous topological dynamical system.

For any $x, y \in X$, we define

$$d^n(x, y) = \max_{0 \le i \le n} d(f_1^i(x), f_1^i(y)).$$

For any $\varepsilon > 0$ and a compact subset K, $F \subset K$ is said to be an $(f_{1,\infty}, \varepsilon, d^n)$ spanning set of K, if for any $z \in K$, there exists $z_0 \in F$ such that $d^n(z, z_0) < \varepsilon$. Let $r(f_{1,\infty}, \varepsilon, d^n, K)$ denote the smallest cardinality of any $(f_{1,\infty}, \varepsilon, d^n)$ spanning set of K. $E \subset K$ is said to be an $(f_{1,\infty}, \varepsilon, d^n)$ separated set of K, if $z_1, z_2 \in E, z_1 \neq z_2$, implies $d^n(z_1, z_2) \geq \varepsilon$. Let $s(f_{1,\infty}, \varepsilon, d^n, K)$ denote the largest cardinality of any $(f_{1,\infty}, \varepsilon, d^n)$ separated set of K. It is easy to prove that

$$r(f_{1,\infty}, 3\varepsilon, d^n, K) \leq s(f_{1,\infty}, 2\varepsilon, d^n, K) \leq r(f_{1,\infty}, \varepsilon, d^n, K).$$

Definition 2.1. *Let $f_{1,\infty}$ be a sequence of continuous maps of compact metric space (X, d), then the pointwise preimage t-entropy of $f_{1,\infty}$ is defined as*

$$\begin{aligned} h_{p,t}(f_{1,\infty}) &= \sup_{x\in X}\{\lim_{\varepsilon\to 0}(\limsup_{n\to\infty}\frac{1}{n^t}\log r(f_{1,\infty},\varepsilon,d^n,f_1^{-n}(x)))\} \\ &= \sup_{x\in X}\{\lim_{\varepsilon\to 0}(\limsup_{n\to\infty}\frac{1}{n^t}\log s(f_{1,\infty},\varepsilon,d^n,f_1^{-n}(x)))\}. \\ h_{m,t}(f_{1,\infty}) &= \lim_{\varepsilon\to 0}\{\limsup_{n\to\infty}\frac{1}{n^t}\log\sup_{x\in X} r(f_{1,\infty},\varepsilon,d^n,f_1^{-n}(x))\} \\ &= \lim_{\varepsilon\to 0}\{\limsup_{n\to\infty}\frac{1}{n^t}\log\sup_{x\in X} s(f_{1,\infty},\varepsilon,d^n,f_1^{-n}(x))\}. \end{aligned}$$

It is easy to check the pointwise preimage t-entropy is an equi-topologically conjugate invariant, and $h_{p,t}(f_{1,\infty}) = h_{m,t}(f_{1,\infty})$ if $f_{1,\infty}$ is a sequence of equi-continuous homeomorphisms.

For any $x \in X$, $k = 1, 2\ldots$, let

$$f_1^{-k}(x) = \{z \in X \mid f_1^k(z) = x\}$$

be the kth preimage set of x under $f_{1,\infty}$. Let

$$T_n(x, f_{1,\infty}) = \{[z_k, z_{k-1}, \ldots, z_0] \mid z_0 = x, 0 \le k \le n, f_{k-j+1}(z_j) = z_{j-1}, 1 \le j \le k\}$$

be the nth order preimage tree of x under $f_{1,\infty}$. Each element $[z_k, z_{k-1}, \ldots, z_0]$ in $T_n(x, f_{1,\infty})$ is called a branch, of which the length is k. If the length of a branch is n, we call it a full branch.

For any $n > 0, x, x' \in X$, let

$$\beta = [z_k,\ z_{k-1},\ \ldots,\ z_1,\ z_0 = x] \in T_n(x,\ f_{1,\infty}),$$
$$\beta' = [z'_l,\ z'_{l-1},\ \ldots,\ z'_1,\ z'_0 = x'] \in T_n(x',\ f_{1,\infty}).$$

Define their branch distance as follows

$$d^b(\beta,\ \beta') = \begin{cases} d^k(z_k,\ z'_l) & l = k \\ \operatorname{diam}(X) & l \ne k. \end{cases}$$

Then we define a branch Hausdorff metric d_n^b on X:

$d_n^b(x, x') < \varepsilon$, if for any branch $\beta \in T_n(x, f_{1,\infty})$ there exists a branch $\beta' \in T_n(x', f_{1,\infty})$ such that $d^b(\beta, \beta') < \varepsilon$, and vice-versa (for branches of $T_n(x',\ f_{1,\infty})$).

Let $\mathcal{T}_n = \bigcup_{x\in X} T_n(x, f_{1,\infty})$, $s(f_{1,\infty}, \varepsilon, d_n^b, X)$ denote the largest cardinality of any $(f_{1,\infty}, \varepsilon, d_n^b)$ separated set of X, $r(f_{1,\infty}, \varepsilon, d_n^b, X)$ denote the smallest cardinality of any $(f_{1,\infty}, \varepsilon, d_n^b)$ spanning set of X. Then, we have

$$r(f_{1,\infty}, 2\varepsilon, d_n^b, X) \le s(f_{1,\infty}, 2\varepsilon, d_n^b, X) \le r(f_{1,\infty}, \varepsilon, d_n^b, X).$$

Definition 2.2. *Let $f_{1,\infty}$ be a a sequence of continuous maps of compact metric space (X, d), then the preimage branch t-entropy of $f_{1,\infty}$ is defined by*

$$h_{i,t}(f_{1,\infty}) = \lim_{\varepsilon\to 0}\{\limsup_{n\to\infty}\frac{1}{n^t}\log s(f_{1,\infty}, \varepsilon, d_n^b, X)\}.$$

Clearly, the preimage branch t-entropy is an equi-topologically conjugate invariant.

Next, we establish the relationship among $h_{m,t}(f_{1,\infty})$, $h_t(f_{1,\infty})$, and $h_{i,t}(f_{1,\infty})$ as follows.

Proposition 2.3. *Let X be a compact metric space and $f : X \to X$ a sequence of continuous maps, for $t > 0$, we have*

$$h_{m,t}(f_{1,\infty}) \le h_t(f_{1,\infty}) \le h_{m,t}(f_{1,\infty}) + h_{i,t}(f_{1,\infty}).$$

In particular, if $h_{i,t}(f_{1,\infty})=0$, then $h_{m,t}(f_{1,\infty})=h_t(f_{1,\infty})$.

Proof. By definition 2.1, we have $h_{m,t}(f_{1,\infty})\le h_t(f_{1,\infty})$. Now we only need to prove $h_t(f_{1,\infty})\le h_{m,t}(f_{1,\infty})+h_{i,t}(f_{1,\infty})$.

For any $0<\varepsilon<3\cdot\mathrm{diam}(X), n\ge 1$, let Y be the maximal $(f_{1,\infty},\frac{\varepsilon}{3},d_n^b,X)$ separated set of X. For any $x\in X$, let $S(x)$ be the maximal $(\frac{\varepsilon}{3},d^n)$ separated set of $f_1^{-n}(x)$. Set $S=\bigcup_{y\in Y}S(y)$, we claim that S is an $(f_{1,\infty},\varepsilon,d^n)$ spanning set of X. Then we have

$$\begin{aligned} r(n,\ \varepsilon) &\le |S| \\ &\le |Y|\cdot\sup_{y\in Y}|S(y)| \\ &\le |Y|\cdot\sup_{x\in X}s(\frac{\varepsilon}{3},\ d^n,\ f_1^{-n}(x)) \\ &= s(\frac{\varepsilon}{3},\ d_n^b,\ X)\cdot\sup_{x\in X}s(\frac{\varepsilon}{3},\ d^n,\ f_1^{-n}(x)). \end{aligned}$$

Therefore,

$$\begin{aligned} h_t(f_{1,\infty}) &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^t}\log r(f_{1,\infty},\varepsilon,d^n) \\ &\le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^t}\log s(f_{1,\infty},\frac{\varepsilon}{3},\ d_n^b,\ X)\cdot\sup_{x\in X}s(f_{1,\infty},\ \frac{\varepsilon}{3},\ d^n,\ f^{-n}(x)) \\ &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^t}\log s(f_{1,\infty},\frac{\varepsilon}{3},\ d_n^b,\ X)+ \\ &\quad \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^t}\log\sup_{x\in X}s(f_{1,\infty},\ \frac{\varepsilon}{3},\ d^n,\ f^{-n}(x)) \\ &= h_{i,\ t}(f_{1,\infty})+h_{m,\ t}(f_{1,\infty}). \end{aligned}$$

Now we turn to prove the claim.

For any $x\in X$, let $\omega=f_1^n(x)$, since $\mathcal{M}$ is the maximal $(f_{1,\infty},\frac{\varepsilon}{3},d_n^b,X)$ separated set of $\mathcal{T}_n$, then there exists $T_n(y,f_{1,\infty})\in\mathcal{M}$, such that $d_n^b(\omega,\ y)<\frac{\varepsilon}{3}$.

For the branch $\beta=[x,f_1(x),f_1^2(x),\dots,f_1^n(x)=\omega]\in T_n(\omega,f_{1,\infty})$, there exists $\beta'\in T_n(y,f_{1,\infty})$ such that $d^b(\beta,\ \beta')<\frac{\varepsilon}{3}$, by the condition $\frac{\varepsilon}{3}<\mathrm{diam}(X)$, we know β and β' have the same order, assume $\beta'=[y_n,y_{n-1},\dots,y_1,y_0=y]$, we get $d^n(x,y_n)<\frac{\varepsilon}{3}$.

Because $S(y)$ is the maximal $(\frac{\varepsilon}{3},d^n)$ separated set of $f_1^{-n}(y)$, $y_n\in f_1^{-n}(y)$, there exists a point $p\in S(y)$ such that $d^n(y_n,p)\le\frac{\varepsilon}{3}$. Therefore, we have

$$d^n(x,p)\le d^n(x,y_n)+d^n(y_n,p)\le\frac{2\varepsilon}{3}<\varepsilon.$$

Thus,

$$h_{m,\ t}(f_{1,\infty})\le h_t(f_{1,\infty})\le h_{m,\ t}(f_{1,\infty})+h_{i,\ t}(f_{1,\infty}).$$

□

Recall that $f:X\to X$ is expanding if there exists $C>0,\lambda>0$, such that for any $v\in T_xM$ satisfies $\|Tf^n(v)\|\ge C\lambda^n\|v\|, n\in\mathbb{Z}^+$.

A sequence of continuous maps $f_{1,\infty}:X\to X$ are called covering maps if for every $x\in X$, there is an open neighborhood U_x such that $f_1^{-1}(U_x)=\bigcup_i V_i(x)$, where $V_i(x)$ are disjoint, and $f_1|_{V_i(x)}: V_i(x)\to U_x$ is a homeomorphism.

Next, we introduce a non-autonomous system, whose preimage branch t-entropy is zero.

Theorem 2.4. *Let M be a closed Riemannian Manifold, $f_{1,\infty}: M \to M$ a sequence of maps generated by C^1 perturbation of an expanding map f, then $h_{i,t}(f_{1,\infty}) = 0$. In particular $h_t(f_{1,\infty}) = h_{m,t}(f_{1,\infty})$.*

Proof. By [23], an expanding map f is also a covering map. By the compactness of M, if the sequence $f_{1,\infty}$ is generated by small enough C^1 perturbation of the map f then we can take sufficiently small constant $\varepsilon > 0$ such that for any $x \in X$, we have

$$f_i^{-1}(B(x, \varepsilon/2)) = V_{i,1}(x) \cup \ldots \cup V_{i,k(x)}(x).$$

Where $f_i \mid_{V_{i,j}(x)}: V_{i,j}(x) \to B(x, \varepsilon/2)(j = 1, \ldots, k(x))$ is a homeomorphism, $V_i(x)$ are disjoint and $\text{diam}(V_i(x)) < b$. In addition, by [23], the expansion map f is structurally stable. So as long as the above C^1 perturbation is sufficiently small, we can take the constant $0 < \lambda_0 < \lambda(\lambda$ is the expanding constant of $f)$ such that for any $x', x'' \in V_{i,j}(x)(j = 1, \ldots, k(x))$, $d(x', x'') \leq \frac{1}{\lambda_0} d(f_i(x'), f_i(x''))$. Note that $B(x, \varepsilon/2) = f_i(V_{i,j}(x))$ and by the expansivity of f_i on $V_{i,j}(x)$, we can get $diam V_{i,j}(x) \leq \varepsilon$.

Given $\widetilde{x} \in B(x, \varepsilon)$ and a branch $\beta = [z_l, z_{l-1}, \ldots, z_1, z_0 = x] \in T_n(x, f_{1,\infty})$. We can find a branch $\widetilde{\beta} = [\widetilde{z}_l, \widetilde{z}_{l-1}, \ldots, \widetilde{z}_1, \widetilde{z}_0 = \widetilde{x}] \in T_n(\widetilde{x}, f_{1,\infty})$ with $d^b(\beta, \widetilde{\beta}) = d(x, \widetilde{x})$, by induction. Assume $d(\widetilde{z}_j, \ z_j) < \varepsilon$, pick $\widetilde{z}_{j+1}$ to be the unique point of $f_{n-j+1}^{-1}(\widetilde{z}_j)$ that belongs to the "piece" $V_i(z_j)$ of $f_{n-j+1}^{-1}(B(z_j, \ \varepsilon))$ that contains z_{j+1}, then we have

$$d(z_{j+1}, \widetilde{z}_{j+1}) \leq \frac{1}{\lambda_0} d(z_j, \widetilde{z}_j) \leq d(x, \widetilde{x}) < \varepsilon.$$

In particular, if $d(x, \widetilde{x}) < \varepsilon$, then for all $n \geq 0$, $d_n^b(x, \widetilde{x}) = d(x, \widetilde{x})$, so that for $0 < \varepsilon' < \varepsilon$, $r(f_{1,\infty}, \varepsilon', d_n^b, X)$ is independent of n. Hence

$$\begin{aligned} h_{i,\,t}(f_{1,\infty}) &= \lim_{\varepsilon' \to 0} \limsup_{n \to \infty} \frac{1}{n^t} \log r(f_{1,\infty}, \varepsilon', \ d_n^b, \ X) \\ &= 0. \end{aligned}$$

□

Now we introduce another non-autonomous system. Suppose X is a finite graph, obviously, it can be characterized as a compact metric space with a distinguished finite set of points (vertices), whose complement has finitely many components (edges). We use the geodesic metric, which means that each edge has length 1, and the distance between two points is the length of the shortest path joining them. Any homeomorphic image of $[0, 1]$ in X is a closed interval in X, and the interior of a closed interval is an open interval in X.

Suppose $\mathcal{P}$ is a finite set of points of X such that each component of $X \backslash \mathcal{P}$ is an open interval, Call the closures of these intervals the atoms of $\mathcal{P}$. If any two distinct atoms of $\mathcal{P}$ share at most one common endpoint, we call $\mathcal{P}$ a division of X.

Now, give a division $\mathcal{P}(f_{1,\infty})$ of X and a sequence of local homeomorphisms $f_{1,\infty}: X \to X$ (that is, for any $x \in X$, there exists an open neighborhood U_x of x, such that $f_{1,\infty}(U_x)$ is an open set and $f_{1,\infty}|_{U_x}: U_x \to f_{1,\infty}(U_x)$ is a homeomorphism), define a sequence of divisions:

$$\begin{aligned}\mathcal{P}_0(f_{1,\infty}) &= \mathcal{P}(f_{1,\infty}),\\ \mathcal{P}_1(f_{1,\infty}) &= \mathcal{P}_0(f_{1,\infty}) \cup f_1(\mathcal{P}_0(f_{1,\infty})),\\ \mathcal{P}_2(f_{1,\infty}) &= \mathcal{P}_1(f_{1,\infty}) \cup f_2(\mathcal{P}_1(f_{1,\infty})),\\ &\vdots\ .\end{aligned}$$

By the compactness of X, there exists $\varepsilon_0 > 0$ such that for any $x \in X$, if $\operatorname{diam}(I_x) < \varepsilon_0$, I_x and $f_i(I_x)$ $(i \in \mathbb{N}^+)$ are homeomorphic. For any given $0 < \varepsilon < \varepsilon_0$, $\mathcal{P}(f_{1,\infty})$ is a division with any atom having the length at most ε, then for the divisions $\mathcal{P}_n(f_{1,\infty})$ defined as above, it is not difficult to draw the following conclusion.

Lemma 2.5. *Suppose X is a finite graph and $f_{1,\infty} : X \to X$ is a sequence of local homeomorphisms. $\mathcal{P}_n(f_{1,\infty}), i = 0, 1, 2, \ldots$, are divisions defined as above. If x and x' are both interior to the same atom of $\mathcal{P}_n(f_{1,\infty})$, $\beta = [z_l, z_{l-1}, \ldots, z_1, z_0 = x]$ is a branch of $T_n(x, f_{1,\infty})$, then there exists a branch $\beta' = [z'_l, z'_{l-1}, \ldots, z'_1, z'_0 = x']$ of $T_n(x', f_{1,\infty})$, such that z_i, $z'_i(i = 0, 1, \ldots, l)$ are both interior to the same atom of $\mathcal{P}_{n-i}(f_{1,\infty})$.*

Theorem 2.6. *Suppose X is a finite graph and $f_{1,\infty} : X \to X$ is a sequence of equi-continuous local homeomorphisms. Then $h_{i,t}(f_{1,\infty}) = 0$, in particular $h_t(f_{1,\infty}) = h_{m,t}(f_{1,\infty})$.*

Proof. Define $\mathcal{P}_i(f_{1,\infty})$ as above, for any $x \in X$, $n > 0$, by Lemma 2.5, there exists a point $x' \in \mathcal{P}_n(f_{1,\infty})$ satisfying $d^b_n(x, x') < \varepsilon$. So $\mathcal{P}_n(f_{1,\infty})$ is an $(f_{1,\infty}, \varepsilon, d^b_n, X)$ spanning set of $\mathcal{T}_n$, then we have

$$r(f_{1,\infty}, \varepsilon, d^b_n, X) \le |\mathcal{P}_n| \le n|\mathcal{P}_0|.$$

Therefore, for $t > 0$,we have

$$\begin{aligned} h_{i,\,t}(f_{1,\infty}) &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^t}\log r(f_{1,\infty}, \varepsilon,\ d^b_n,\ X)\\ &\le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^t}[\log n + \log|\mathcal{P}_0|]\\ &= 0.\end{aligned}$$

□

Theorem 2.7. *Suppose X is a finite graph and $f_{1,\infty} : X \to X$ is a sequence of equi-continuous homeomorphisms. Then $h_t(f_{1,\infty}) = 0$, and the entropy dimension $D_{f_{1,\infty}}(X) = 0$.*

Proof. As we all know, $f_{1,\infty}$ preserves points and edges. Here we take the geodesic metric, pick $\varepsilon' > 0$ satisfying

$$d(x, y) < \varepsilon' \Rightarrow d(f_1^{-1}(x), f_1^{-1}(y)) \le \frac{1}{2} \quad \forall x,\ y \in X.$$

Given $0 < \varepsilon < \varepsilon'$, consider the (n, ε) spanning set of X. Besides the vertices, we pick some other points such that all the points become a division $\mathcal{P}(f_{1,\infty})$ of X, with any atom of it having the length no more than ε, then $\mathcal{P}(f_{1,\infty})$ is a $(1, \varepsilon)$ spanning set of X. Now define a sequence of divisions:

$$\begin{aligned}\mathcal{P}_0(f_{1,\infty}) &= \mathcal{P}(f_{1,\infty}),\\ \mathcal{P}_1(f_{1,\infty}) &= \mathcal{P}_0(f_{1,\infty}) \cup f_1^{-1}(\mathcal{P}_0(f_{1,\infty})),\\ \mathcal{P}_2(f_{1,\infty}) &= \mathcal{P}_1(f_{1,\infty}) \cup f_2^{-1}(\mathcal{P}_1(f_{1,\infty})),\\ &\vdots \ .\end{aligned}$$

We know that any atom of $\mathcal{P}_n(f_{1,\infty})(n=1,2,\ldots)$ has the length at most ε. We claim that $\mathcal{P}_{n-1}(f_{1,\infty})(n=1,2,\ldots)$ is an (n,ε) spanning set of X. In fact, for any $x\in X$, there exists an atom $A_{n-1}\in\mathcal{P}_{n-1}(f_{1,\infty})$, such that $x\in A_{n-1}$, since any atom is a closed interval, suppose $A_{n-1}=[x_n,\widetilde{x}_n]$, and x is interior to A_{n-1}, so we have $d(x_n,x)<\varepsilon$.

We also know that there exists an atom $A_{n-2}\in\mathcal{P}_{n-2}(f_{1,\infty})$ such that $f_i^{-1}([x_n,x])\subset A_{n-2}$ $(i\in\mathbb{N}^+)$. Otherwise, there is at least one point in $\mathcal{P}_{n-2}(f_{1,\infty})$ belonging to $f_i^{-1}([x_n,x])$, since $f_{1,\infty}$ is homeomorphic, so x and x_n can not in the same atom, it is contradictory.

By the structure of $\mathcal{P}_n(f_{1,\infty})$, $f_i^{-1}(x)$ is interior to A_{n-2}, so we have

$$d(f_i^{-1}(x_n),f_i^{-1}(x))<\mathrm{diam}(A_{n-2})\le\varepsilon.$$

Then, we get

$$d(f_i^{-1}\circ f_{i-1}^{-1}\circ\ldots\circ f_1^{-1}(x_n),f_i^{-1}\circ f_{i-1}^{-1}\circ\ldots\circ f_1^{-1}(x))<\varepsilon \quad i=0,1,2,\ldots,$$

hence $\mathcal{P}_n(f_{1,\infty})$ is an (n,ε) spanning set of X, and then we have,

$$r(n,\varepsilon)\le|\mathcal{P}_n|\le n\cdot|\mathcal{P}_0|.$$

Therefore, for $t>0$, we have

$$\begin{aligned}h_t(f_{1,\infty}) &= \lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n^t}\log r(n,\ \varepsilon)\\ &\le \lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n^t}\log n\cdot|\mathcal{P}_0|\\ &= \lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n^t}\log n\\ &= 0,\end{aligned}$$

and then it is obvious that $D_{f_{1,\infty}}(X)=0$. □

§3 A Result on Entropy Dimension

In this section, we consider the entropy dimension of non-autonomous dynamical systems. Let $(X,\mathcal{A},\mu)$ be a probability space, $T_{1,\infty}=\{T_i\}_{i=1}^{\infty}$ a sequence of maps that preserve the same measure μ. $(X,T_{1,\infty})$ is called an non-autonomous measure-preserving dynamical system. For any $i,n\in N$, let $T_i^0=T_i$ and

$$T_i^n=T_{i+(n-1)}\circ\ldots\circ T_{i+1}\circ T_i,$$
$$T_i^{-n}=(T_i^n)^{-1}=T_i^{-1}\circ T_{i+1}^{-1}\circ\ldots\circ T_{i+(n-1)}^{-1}.$$

Definition 3.1. *Let $(X,f_{1,\infty})$ be an non-autonomous dynamical system, $\mathcal{U}$ an any open cover* *of X, for $s>0$, the s-topological up-entropy is defined as*

$$\overline{D}(s,f_{1,\infty})=\sup_{\mathcal{U}}\limsup_{n\to\infty}\frac{1}{n^s}\log N(\bigvee_{i=0}^{n-1}f_1^{-i}\mathcal{U}),$$

and the topological entropy up-dimension is defined as

$$\overline{D}(f_{1,\infty})=\inf\{s>0,\overline{D}(s,f_{1,\infty})=0\}=\sup\{s>0,\overline{D}(s,f_{1,\infty})=\infty\}.$$

We can obtained the s-topological down-entropy $\underline{D}(s,f_{1,\infty})$ and topological entropy down-dimension $\underline{D}(f_{1,\infty})$ if we replace " lim sup" by " lim inf ".

For any $\varepsilon>0$, $E\subset X$ is said to be an (n,ε)-spanning set of $(X,f_{1,\infty})$, if for any $x\in X$, there exists $y\in E$ such that $d(f_1^ix,f_1^iy)<\varepsilon$, $i=0,1,\dots,n-1$. Let $r(n,f_{1,\infty},\varepsilon)$ denote the smallest cardinality of any (n,ε) spanning set of $(X,f_{1,\infty})$. $F\subset X$ is said to be an (n,ε) separated set of $(X,f_{1,\infty})$, if for any $x,y\in F$, $x\neq y$, there exists $i\in\{0,1,\dots,n-1\}$ such that $d(f_1^ix,f_1^iy)\geq\varepsilon$. Let $s(n,f_{1,\infty},\varepsilon)$ denote the largest cardinality of any (n,ε) separated set of $(X,f_{1,\infty})$.

Definition 3.2. *Let $(X,f_{1,\infty})$ be an non-autonomous dynamical system, then the s-topological up-entropy is defined by*

$$\overline{D}_r(s,f_{1,\infty})=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n^s}\log r(n,f_{1,\infty},\varepsilon),$$

$$\overline{D}_s(s,f_{1,\infty})=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n^s}\log s(n,f_{1,\infty},\varepsilon).$$

By the classical topological entropy theory, we can easily get $\overline{D}(s,f_{1,\infty})=\overline{D}_r(s,f_{1,\infty})=\overline{D}_s(s,f_{1,\infty})$.

Definition 3.3. *Let $(X,T_{1,\infty})$ be an non-autonomous measure-preserving dynamical system, η an arbitrary partition of X, then the s-measure-theoretic up-entropy is defined as*

$$\overline{D}(s,\mu,T_{1,\infty})=\sup_{\eta}\limsup_{n\to\infty}\frac{1}{n^s}H_\mu(\bigvee_{i=0}^{n-1}T_1^{-i}\eta),$$

and the measure-theoretic entropy up-dimension is defined as

$$\overline{D}(\mu,T_{1,\infty})=\inf\{s>0,\overline{D}(s,\mu,T_{1,\infty})=0\}=\sup\{s>0,\overline{D}(s,\mu,T_{1,\infty})=\infty\}.$$

Now we will calculate the s-topological up-entropy of a sequence of equi-continuous monotone maps on the circle. For any $s>0$, we construct a sequence of monotone maps on the circle such that both the measure-theoretic entropy dimension and the topological entropy dimension are equal to s.

Definition 3.4. *Let S^1 be the unit circle, $f:S^1\to S^1$ a continuous monotone map, and the degree is $\mid degf\mid=q$, for any $x\in S^1$, $f^{-1}(x)=\{x_1,\dots,x_q\}$ is a set that consists q points. Let $\eta_{f,1}=(x_1,x_2),\dots,\eta_{f,q-1}=(x_{q-1},x_q),\eta_{f,q}=(x_q,x_1)$, then we can get a finite partition $\mathcal{A}_f=\{\eta_{f,1},\dots,\eta_{f,q}\}$ of S^1, where $\overline{f(\eta_{f,i})}=S^1$, and $\eta_{f,i}\cap\eta_{f,j}=\emptyset$ for $1\leq i\leq q$.*

Lemma 3.5. *[26] Let $f_{1,\infty}=\{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous monotone maps of S^1, then there exists a constant $c>0$, such that for any $f_i(i\geq1)$ and any partition $\mathcal{A}_{f_i}=\{\alpha_{f_i,1},\dots,\alpha_{f_i,q_i}\}$ of S^1, we have diam $(\alpha_{f_i,j})\geq c$, $1\leq j\leq q_i$, where $q_i=\mid degf_i\mid$.*

Proposition 3.6. *Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous monotone maps of S^1, $s > 1$, then*

$$\overline{D}(s, f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n^s} \log \prod_{i=1}^{n} \mid deg f_i \mid .$$

Proof. Let $\{\mathcal{A}_{f_i} = \{\alpha_{f_i,1}, \ldots, \alpha_{f_i,q_i}\}\}_{i=1}^{\infty}$ be a sequence of partitions of S^1, we construct a new sequence of partitions

$$\mathcal{A}_{f_1^n} = \{\bigcap_{i=1}^{n} f_1^{-(i-1)}(\alpha_{f_{i,k}}) \mid \alpha_{f_{i,k}} \in \mathcal{A}_{f_i}, 1 \le k \le q_i\}.$$

We will prove $\overline{D}(s, f_{1,\infty}) = \limsup\limits_{n\to\infty} \frac{1}{n^s} \log card\mathcal{A}_{f_1^n}$.

For any $x \in S^1$, let $0 \le \varepsilon \le \frac{c}{2}$, here c is the one in lemma 3.5. Let $B_d(x, \varepsilon) = \{y \in S^1 \mid d(x, y) < \varepsilon\}$, E be the minimal $(n, f_{1,\infty}, \varepsilon)$ spanning set, so for any $x \in E, 0 \le j \le n-1$, the ε-neighborhood $\overline{B_{d_n}(f_1^j, \varepsilon)}$ of f_1^j intersects at most two elements of $\mathcal{A}_{f_j}$. Therefore $\overline{B_{d_n}(x, \varepsilon)}$ intersects at most 2^n elements of $\mathcal{A}_{f_j}$. By $\overline{\bigcup_{x\in E} B_{d_n}(x, \varepsilon)} = S^1$, then $card\mathcal{A}_{f_1^n} \le 2^n card E$, hence

$$\frac{1}{n^s} \log card\mathcal{A}_{f_1^n} \le \frac{1}{n^s} \log r(n, f_{1,\infty}, \varepsilon) + \frac{n}{n^s} \log 2.$$

If $s > 1$, we have

$$\overline{D}(s, f_{1,\infty}) \ge \limsup_{n\to\infty} \frac{1}{n^s} \log card\mathcal{A}_{f_1^n}.$$

For any $0 < \varepsilon \le \frac{1}{2}$, we choose a subset $F \subset S^1$ such that it is the maximal $(n, f_{1,\infty}, \varepsilon)$ separated set of S^1. By the definition of separated set, for any $\alpha \in \mathcal{A}_{f_1^n}$ and any two adjacent points $x, y \in \alpha \cap F$, there exists i_0 such that $d(f_1^{i_0}(x), f_1^{i_0}(y)) > \varepsilon$, where $0 \le i_0 \le n-1$. Since $f_1^{i_0}$ is monotone on α, then $f_1^{i_0}(x)$ and $f_1^{i_0}(y)$ are also two adjacent points, because S^1 is a unit circle, there are at most $M = [\frac{1}{\varepsilon}] + 1$ pairs adjacent points in $f_1^{i_0}(\alpha \cap F)$, and the distance between any two adjacent points is greater than ε. For every $0 \le i_0 \le n-1$, we know $card(\alpha \cap F) \le nM + 1$, and then $card F \le (nM + 1)$, therefore, we have

$$\frac{1}{n^s} \log s(n, f_{1,\infty}, \varepsilon) \le \frac{1}{n^s} \log card\mathcal{A}_{f_1^n} + \frac{1}{n^s} \log(nM + 1),$$

and

$$\overline{D}(s, f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n^s} \log card\mathcal{A}_{f_1^n}.$$

Hence

$$\begin{aligned} \overline{D}(s, f_{1,\infty}) &= \limsup_{n\to\infty} \frac{1}{n^s} \log card\mathcal{A}_{f_1^n} \\ &= \limsup_{n\to\infty} \frac{1}{n^s} \log \prod_{i=1}^{n} \mid deg f_i \mid . \end{aligned}$$

□

As we all know, for autonomous systems, the entropy dimension is in the interval $(0, 1]$, however in non-autonomous systems, we can construct a sequence of maps such that the entropy dimension can be attained by any positive real number. Now we introduce the example.

Example 3.7. (1) *For any $0 < s \le 1$, define a sequence of monotone maps $\{f_i\}_{i=1}^{\infty}$ on S^1:*

$$f_1(x) = kx(mod 1)$$

$$f_{i+1}(x)=\begin{cases} kx(mod1)[(i+1)^s]>[(i)^s],\\ Id[(i+1)^s]=[(i)^s],\end{cases}$$

where $i=1,\dots,n-1,k\in Z_+$*, then* $D(f_{1,\infty})=s$.

(2) *For* $s>1$*, define a sequence of monotone maps* $\{f_i\}_{i=1}^{\infty}$ *on* S^1*:*

$$\begin{cases} f_1(x)=k^{[2^s]}x(mod1),\\ f_2(x)=k^{[3^s]-[2^s]}x(mod1),\\ \dots,\\ f_{n-1}(x)=k^{[n^s]-[(n-1)^s]}x(mod1),\end{cases}$$

then $D(f_{1,\infty})=s$.

Proof. (1) For $q\geq 1$, let $E_q=\{\frac{c}{k^q}:c=0,1,2,\dots,k^q-1\}$ (i.e the cardinality is k^q). Then for any $\varepsilon>0$, there exists $q\geq 1$ such that $\frac{1}{k^q}\leq\varepsilon<\frac{1}{k^{q-1}}$. Next, we prove that for $n\geq 1$, $E_{[(n-1)^s]+q-1}$ is an (n,ε) separated set.

For any $\frac{c_1}{k^{[(n-1)^s]+q-1}},\frac{c_2}{k^{[(n-1)^s]+q-1}}\in E_{[(n-1)^s]+q-1}$, we have

$$\begin{aligned} & d(f_1^{n-1}(\frac{c_1}{k^{[(n-1)^s]+q-1}}),f_1^{n-1}(\frac{c_2}{k^{[(n-1)^s]+q-1}}))\\ = & d(k^{[(n-1)^s]}(\frac{c_1}{k^{[(n-1)^s]+q-1}}),k^{[(n-1)^s]}(\frac{c_2}{k^{[(n-1)^s]+q-1}}))\\ = & d(\frac{c_1}{k^{q-1}},\frac{c_2}{k^{q-1}})\\ \geq & \frac{1}{k^{q-1}}\\ > & \varepsilon.\end{aligned}$$

Therefore $s(n,f_{1,\infty},\varepsilon)\geq k^{[(n-1)^s]+q-1}$.

For any $x\in S^1$, we take $\frac{c}{k^{[(n-1)^s]+q}}\in E_{[(n-1)^s]+q}$, such that $d(x,\frac{c}{k^{[(n-1)^s]+q}})\leq\frac{1}{\frac{c}{k^{[(n-1)^s]+q}}}$, then for $i=0,1,\dots,n-1$, we have

$$d(f_1^i(\frac{c}{k^{[(n-1)^s]+q}}),f_1^ix)\leq\frac{1}{k^q}\leq\varepsilon.$$

Therefore, $E_{[(n-1)^s]+q}$ is an (n,ε) spanning set and $r(n,f_{1,\infty},\varepsilon)\leq k^{[(n-1)^s]+q}$.

By the definition of the entropy dimension, we have

$$\begin{aligned} \overline{D}(\alpha,f_{1,\infty}) &= \limsup_{n\to\infty}\frac{1}{n^\alpha}\log s(n,f_{1,\infty},\varepsilon)\\ &\geq \limsup_{n\to\infty}\frac{1}{n^\alpha}\log k^{[(n-1)^s]+q-1}\\ &= \limsup_{n\to\infty}\frac{[(n-1)^s]}{n^\alpha}\log k.\\ \overline{D}(\alpha,f_{1,\infty}) &= \limsup_{n\to\infty}\frac{1}{n^\alpha}\log r(n,f_{1,\infty},\varepsilon)\\ &\leq \limsup_{n\to\infty}\frac{1}{n^\alpha}\log k^{[(n-1)^s]+q}\\ &= \limsup_{n\to\infty}\frac{[(n-1)^s]}{n^\alpha}\log k.\end{aligned}$$

Therefore, $\overline{D}(\alpha,f_{1,\infty})=\limsup\limits_{n\to\infty}\frac{[(n-1)^s]}{n^\alpha}\log k$, and then we know $\overline{D}(\alpha,f_{1,\infty})=0$ for $\alpha>s$,

$\overline{D}(\alpha, f_{1,\infty}) = \infty$ for $\alpha < s$. Consequently $\overline{D}(f_{1,\infty}) = s$.

We can obtain $\underline{D}(f_{1,\infty}) = s$ if we change " lim sup " to " lim inf ", hence $D(f_{1,\infty}) = s$.

(2) Construct spanning sets and separated sets just as in (1), we have

$$s(n, f_{1,\infty}, \varepsilon) \geq k^{[n]^s+q-1}, r(n, f_{1,\infty}, \varepsilon) \leq k^{[n]^s+q}.$$

Then

$$\overline{D}(\alpha, f_{1,\infty}) = \limsup_{n\to\infty} \frac{[n]^s}{n^\alpha} \log k,$$

and $\overline{D}(\alpha, f_{1,\infty}) = 0$ for $\alpha > s$, $\overline{D}(\alpha, f_{1,\infty}) = \infty$ for $\alpha < s$. Therefore $D(f_{1,\infty}) = s$. □

Acknowledgement. We would like to show our great thanks to Professor Yujun Zhu for his constructive comments and useful suggestions.

References

[1] R Adler, A Konheim. *McAndrew M Topological entropy*, Trans Amer Math Soc, 1965, 114: 309-319.

[2] J S Cánovas. *On entropy of nonautonomous discrete systems*, Progress and Challenges in Dynamical Systems, Springer, 2013, 143-159.

[3] M D Carvalho. *Entropy dimension of dynamical systems*, Portugal Math, 1997, 54(1): 19-40.

[4] W C Cheng, B Li. *Zero entropy systems*, J Stat Phys, 2010, 140(5): 1006-1021.

[5] W C Cheng, S Newhouse. *Preimage entropy*, Ergodic Theory Dynam Systems, 2005, 25: 1091-1113.

[6] F Colonius, C Kawana, G Nair. *A note on topological feedback entropy and invariance entropy*, Systems Control Lett, 2013, 62: 377-381.

[7] D Dou, W Huang, K K Park. *Entropy dimension of topological dynamics*, Trans Amer Math Soc, 2011, 363(2): 659-680.

[8] S Ferenczi, K K Park. *Entropy dimensions and a class of constructive examples*, Discrete Contin Dyn Syst, 2007, 17(1): 133-141.

[9] G Fuhrmann, M Gröger, T Jäger. *Amorphic complexity*, Nonlinearity, 2016, 29(2).

[10] T N Goodman. *Maximal measures for expansive homeomorphisms*, Bull London Math Soc, 1972, 5: 439-444.

[11] T N Goodman. *Relating topological entropy and measure entropy*, Bull London Math Soc, 1971, 3: 176-180.

[12] M Hurley. *On topological entropy of maps*, Ergodic Theory Dynam Systems, 1995, 15(3): 557–568.

[13] C Kawan. *Metric entropy of nonautonomous dynamical systems*, Nonauton Stoch Dyn Syst, 2013, 1: 26-52.

[14] C Kawan. *Expanding and expansive time-dependent dynamics*, Nonlinearity, 2015, 28(3): 669-695.

[15] C Kawan, Y Latushkin. *Some results on the entropy of non-autonomous dynamical systems*, Dyn Syst, 2016, 31(3): 251-279.

[16] A N Kolmogorov. *New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces*, Dokl Akad Nauk SSSR, 1958, 119: 861-864.

[17] S Kolyada, M Misiurewicz, L Snoha. *Topological entropy of non-autonomous piecewise monotone dynamical systems on the interval*, Fund Math, 1999, 160(2): 161-181.

[18] S Kolyada, L Snoha. *Topological entropy of non-autonomous dynamical systems*, Random Comput Dynamics, 1996, 4(2-3): 205-233.

[19] M Misiurevicz. *Topological conditional entropy*, Studia Math, 1976, 55(2): 175-200.

[20] Z Niitecki, F Preimage. *Preimage entropy of mappings*, Internat J Bifur Chaos Appl Sci Engrg, 1999, 9: 1815-1843.

[21] P Oprocha, P Wilczynski. *Chaos in non-autonomous dynamical systems*, An Stiint Univ Ovidius Constanta Ser Mat, 2009, 17(3): 209-221.

[22] P Oprocha, P Wilczynski. *Topological entropy of local processes*, J Differential Equations, 2010, 249(8): 1929-1967.

[23] M Shub. *Expanding maps*, Proc Sympos Pure Math, 1969, 14: 273-276.

[24] J L Zhang, Y J Zhu, L F He. *Preimage entropy for nonautonomous dynamical systems*, Acta Math Sinica(Chin Ser), 2005, 48(4): 693-702.

[25] J Zhang, L Chen. *Lower bounds of the topological entropy for non-autonomous dynamical systems*, Appl Math J Chinese Univ Ser B, 2009, 24(1): 76-82.

[26] J L Zhang, Y J Zhu, L F He. *Topological entropy of a sequence of monotone maps on circles*, J Korean Math Soc, 2006, 43(2): 373-382.

[27] Y J Zhu. *Preimage entropy for random dynamical systems*, Discrete Contin Dyn Syst, 2007, 18(4): 529-551.

[28] Y J Zhu, Z M Li, X H Li. *Preimage pressure for random transformations*, Ergodic Theory Dynam Systems, 2009, 29: 1669-1687.

[29] Y J Zhu, L X Xu, D W Zhang. *Entropy of non-autonomous dynamical systems*, J Korean Math Soc, 2012, 49(1): 165-185.

School of Mathematics and Statistics, Shandong Normal University, Ji'nan 250014, China.
Email: yjyang90@163.com

College of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030006, China.
Email: wanglin198762@163.com

College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, China.
Email: wangweiufo012@163.com