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Lipschitz estimates for commutator of fractional integral operators on non-homogeneous metric measure spaces

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Abstract. In this paper, the authors establish the $(L^p(\mu), L^q(\mu))$-type estimate for fractional commutator generated by fractional integral operators T_α with Lipschitz functions ($b \in Lip_\beta(\mu)$), where $1 < p < 1/(\alpha+\beta)$ and $1/q = 1/p - (\alpha+\beta)$, and obtain their weak $(L^1(\mu), L^{1/(1-\alpha-\beta)}(\mu))$-type. Moreover, the authors also consider the boundedness in the case that $1/(\alpha+\beta) < p < 1/\alpha$, $1/\alpha \le p \le \infty$ and the endpoint cases, namely, $p = 1/(\alpha+\beta)$.

§1 Introduction and Notation

It is well known that the doubling condition is a key assumption in the analysis on spaces of homogeneous type. However, some theories have been proved still valid with non-doubling measure (see[6-8]). In 2010, Hytönen [4] introdeced a new class of metric measure spaces satisfying both the so-called geometrically doubling and the upper doubling conditions (see the definition below), which are called non-homogeneous spaces. Recently, many classical results have been proved still valid if the underlying spaces are replaced by the non-homogeneous spaces, for example, the theory of Carlderón-Zygmund operators(see [1,3,6]).

In 2014, J. Zhou and D. Wang [10] established the definition of fractional operator and the definition of Lipschitz space on non-homogeneous metric measure spaces and they also establish some equivalent characterizations for the Lipschitz spaces. Motivated by [10], we consider the endpoint estimates for commutator generated by fractional integral operators with Lipschitz functions.

In this paper, we will prove the $(L^p(\mu), L^q(\mu))$ boundedness of commutator generated by fractional integral operator T_α (see the definition below) with Lipschitz functions $b \in Lip_\beta(\mu)$, where $1 < p < 1/(\alpha+\beta)$ and $1/q = 1/p - (\alpha+\beta)$, and their weak $(L^1(\mu), L^{1/(1-\alpha-\beta)}(\mu))$. We also consider the boundedness in the case that $1/(\alpha+\beta) < p < 1/\alpha$, $1/\alpha \le p \le \infty$ and the endpoint case of $p = 1/(\alpha+\beta)$.

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To state the main results of this paper, we first recall some necessary notions and remarks. Firstly, we make some conventions on notation. Throughout the whole paper, C stands for a positive constant, which is independent of the main parameters, but it may vary from line to line.

Definition 1.1. [4] A metric space $(\mathcal{X}, d, \mu)$ is said to be geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x,r) \subset \mathcal{X}$, there exist a finite ball covering$\{B(x_i, r/2)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most N_0.

Definition 1.2. [4] A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0,\infty) \to (0,\infty)$ and a positive constant c_λ such that, for each $x \in \mathcal{X}, r \to \lambda(x,r)$ is non-decreasing and

$$\mu(B(x,r)) \le \lambda(x,r) \le c_\lambda \lambda(x, r/2) \text{ for all } x \in \mathcal{X}, r > 0. \tag{1.1}$$

A metric measure space $(\mathcal{X}, d, \mu)$ is called a non-homogeneous metric measure space if $(\mathcal{X}, d, \mu)$ is geometrically doubling and $(\mathcal{X}, d, \mu)$ is upper doubling.

Remark 1.1. *Let $(\mathcal{X}, d, \mu)$ be upper doubling with λ being the dominating function on $\mathcal{X} \times (0,\infty)$ as in Definition 1.3. It was proved in [2] that there exists another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \le \lambda$ and, for all $x, y \in \mathcal{X}$ with $d(x,y) \le r$,*

$$\tilde{\lambda}(x,r) \le C_{\tilde{\lambda}} \tilde{\lambda}(y,r). \tag{1.2}$$

Thus, in this paper, we always suppose that λ satisfies (1.2).

Definition 1.3. [4] Let $\alpha, \beta_\alpha \in (1,\infty)$. A ball $B \subset X$ is called (α,β)-doubling if $\mu(\alpha B) \le \beta_\alpha \mu(B)$.

As stated in Lemma of [4], there exist a plenty of doubling balls with small radii and with large radii. In the rest of the paper, unless α and β_α are specified otherwise, by an (α, β_α)-doubling ball we mean a $(6, \beta_6)$-doubling with a fixed number $\beta_6 > \max\{C_\lambda^{3log_2 6}, 6^n\}$, where $n = log_2 N_0$ be viewed as a geometric dimension of the spaces.

Definition 1.4. [3] Let $\epsilon \in (0,\infty)$. A dominating function λ satisfies the ϵ-weak reverse doubling condition if, for all $r \in (0, 2diam(\mathcal{X}))$ and $a \in (1, 2diam(\mathcal{X})/r)$, there exists a number $C(a) \in [1,\infty)$ depending only on a and $\mathcal{X}$, such that, for all $x \in \mathcal{X}$,

$$\lambda(x, ar) \ge C(a)\lambda(x,r) \tag{1.3}$$

and, moreover,

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^\epsilon} < \infty. \tag{1.4}$$

Remark 1.2. (i) It is easy to see that, if $\epsilon_1 < \epsilon_2$ and λ satisfies the ϵ_1-weak reverse doubling condition, then λ also satisfies the ϵ_2-weak reverse doubling condition.

(ii) Assume that $diam(\mathcal{X}) = \infty$. For any fixed $x \in \mathcal{X}$, we know that

$$\lim_{r\to 0} \lambda(x,r) = 0, \lim_{r\to\infty} \lambda(x,r) = \infty. \tag{1.5}$$

(iii) It is easy to see that the ϵ-weak reverse doubling condition is much weaker than the assumption introduced by Bui and Duong in [2, Subsection 7.3]: there exists $m \in (0,\infty)$ such that, for all $x \in \mathcal{X}$ and $a, r \in (0,\infty)$, $\lambda(x, ar) = a^m \lambda(x,r)$.

Definition 1.5. [4] For any two balls $B \subset S$, define

$$K_{B,S} = 1 + \int_{2S\backslash B} \frac{1}{\lambda(c_B, d(x, c_B))} d\mu(x),$$

where c_B is the center of the ball B.

Definition 1.6. [4] Let $\rho \in (1, \infty)$. A function $f \in L^1_{loc}(\mu)$ is said to be in the space $RBMO(\mu)$ if there exist a positive constant C, and for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| d\mu(x) \le C, \tag{1.6}$$

for any two balls B and B_1 such that $B \subset B_1$,

$$|f_B - f_{B_1}| \le C K_{B,B_1}. \tag{1.7}$$

The infimum of the positive constants C satisfying above two inequalities is defined to be the $RBMO(\mu)$ norm of f and denoted by $\|f\|_{RBMO(\mu)}$.

From Lemma 4.6 of [4], it follows that the space $RBMO(\mu)$ is independent of $\rho \in (1, \infty)$.

In what follows, the definition of fractional integral operators and Lipschitz space are a little different from those in [10], but it can easily be seen that the related results still valid.

Definition 1.7. [6] Let $0 < \alpha < 1$ and $0 < \delta \le 1$. A function $K_\alpha \in L^1_{loc}(\mathcal{X} \times \mathcal{X} \backslash \{(x, y) : x = y\})$ is said to be a fractional kernel of order α and regularity δ if it satisfies the following two conditions:

(i) for all $x, y \in \mathcal{X}$ with $x \ne y$,

$$|K_\alpha(x, y)| \le C \frac{1}{[\lambda(x, d(x, y))]^{1-\alpha}}; \tag{1.8}$$

(ii) for all $x, \tilde{x}, y \in \mathcal{X}$ with $\lambda(x, d(x, y)) \ge 2\lambda(x, d(x, \tilde{x}))$,

$$|K_\alpha(x, y) - K_\alpha(\tilde{x}, y)| + |K_\alpha(y, x) - K_\alpha(y, \tilde{x})| \le C \frac{[\lambda(x, d(x, \tilde{x}))]^\delta}{[\lambda(x, d(x, y))]^{1-\alpha+\delta}}. \tag{1.9}$$

A linear operator T_α is called fractional integral operator with K_α satisfying (1.8) and (1.9), for all $f \in L^\infty_b(\mu)$ and $x \notin supp f$,

$$T_\alpha f(x) := \int_{\mathcal{X}} K_\alpha(x, y) f(y) d\mu(y). \tag{1.10}$$

Definition 1.8. [10] Given $\beta \in (0, 1]$, we say that the function $f : \mathcal{X} \to \mathbb{C}$ satisfies a Lipschitz condition of order β provided

$$|f(x) - f(y)| \le C[\lambda(x, d(x, y))]^\beta \ for\ every\ x, y \in \mathcal{X} \tag{1.11}$$

and the smallest constant in inequality (1.11) will be denoted by $\|f\|_{Lip_\beta(\mu)}$. If we identify two functions whose difference is a constant, it follows that the linear space with the norm $\|\cdot\|_{Lip_\beta(\mu)}$ is a Banach space.

Remark 1.3. The Lipschitz condition can also be defined by

$$|f(x) - f(y)| \le C[\lambda(y, d(x, y))]^\beta \ for\ every\ x, y \in \mathcal{X}, \tag{1.12}$$

by (1.2), it is easy to see that (1.11) and (1.12) are equivalent.

Noting that $b \in Lip_\beta(\mu)$, we discuss the behavior of commutators $T_{\alpha,b}$ generated by fractional integral operator T_α with Lipschitz function b in Lebesgue spaces. For μ-a.e. $x \in \mathrm{supp}(\mu)$,

$$|T_{\alpha,b}(f)(x)| \leq C\|b\|_{Lip(\beta)} I_{\alpha+\beta}(|f|)(x), \tag{1.13}$$

where I_β is defined by

$$I_\beta(f)(x) = \int_{\mathcal{X}} \frac{f(y)}{[\lambda(x, d(x,y))]^{1-\beta}} dy.$$

From now on, we shall assume that $\mu(\mathcal{X}) = \infty$.

Theorem 1.4. Let $1 \leq p < \frac{1}{\beta}$ and $\frac{1}{q} = \frac{1}{p} - \beta$. If λ satisfies the ϵ-weak reverse doubling condition with $\epsilon \in (0, \min\{\beta, (\frac{1}{p} - \beta)p'\})$, then

$$\mu(\{x \in \mathcal{X} : |I_\beta f(x)| > \nu\}) \leq \left(\frac{C\|f\|_{L^p(\mu)}}{\nu}\right)^q,$$

that is, I_β is a bounded operator from $L^p(\mu)$ into the space $L^{q,\infty}(\mu)$.

The proof of **Theorem 1.4** is similar to the Theorem 1.13 of [6], we omit the details.

Theorem 1.5. For a measure μ, finite over balls and not having any atoms, condition $\mu(B(c_B, r_B)) \leq C\lambda(c_B, r_B)$ is necessary for the Hardy-Littlewood-Sobolev theorem to hold.

Proof. Suppose that $I_\beta(f)$ is bounded from $L^p(\mu)$ to $L^q(\mu)$, where $1 < p < \frac{1}{\beta}$ and $\frac{1}{q} = \frac{1}{p} - \beta$. Let $B = B(c_B, r_B)$, if $\mu(B) = 0$, then $\mu(B(c_B, r_B)) \leq C\lambda(c_B, r_B)$ is trivially true. Let $\mu(B) \neq 0$. For each $x \in B$, since λ satisfies (1.2), we have $\lambda(x, r_B) \leq C_\lambda \lambda(c_B, r_B)$, then

$$I_\beta \chi_B(x) = \int_B \frac{1}{[\lambda(x, d(x,y))]^{1-\beta}} d\mu(y) \geq \int_B \frac{1}{[\lambda(x, r_B)]^{1-\beta}} d\mu(y) \geq \frac{C}{[\lambda(c_B, r_B)]^{1-\beta}} \mu(B).$$

Assume that $I_\beta f$ is bounded from $L^p(\mu)$ to $L^q(\mu)$, then

$$\begin{aligned} \frac{C}{[\lambda(c_B, r_B)]^{1-\beta}} \mu(B)^{1+\frac{1}{q}} &\leq \left(\int_B |I_\beta \chi_B(x)|^q\right)^{1/q} \\ &\leq C\|\chi_B\|_{L^p(\mu)} = C\mu(B)^{\frac{1}{p}}, \end{aligned}$$

so, we get

$$\mu(B(c_B, r_B))^{1+\frac{1}{q}-\frac{1}{p}} \leq C[\lambda(c_B, r_B)]^{1-\beta},$$

that is, $\mu(B(c_B, r_B)) \leq C\lambda(c_B, r_B)$. A similar proofs if we assume $I_\beta f$ is bounded from $L^p(\mu)$ to $L^{q,\infty}(\mu)$, where $1 \leq p < \frac{1}{\beta}$ and $\frac{1}{q} = \frac{1}{p} - \beta$. □

From (1.13) and **Theorem 1.4**, it is easy to verify the following results.

Theorem 1.6. Let $b \in Lip_\beta(\mu)$. Suppose that $0 < \alpha + \beta < 1$ and λ satisfy the ϵ-weak reverse doubling condition with $\epsilon \in (0, \min\{\alpha + \beta, (\frac{1}{p} - \alpha - \beta)p'\})$, then

(i) for all bounded functions f with compact support,

$$\|T_{\alpha,b}(f)\|_{L^q(\mu)} \leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)},$$

where $1 < p < 1/(\alpha + \beta)$ and $1/q = 1/p - \alpha - \beta$.

(ii) for all bounded functions f with compact support and all $\lambda > 0$,

$$\mu\left(\left\{x \in \mathbb{R}^d : |T_{\alpha,b}(f)| > \nu\right\}\right) \leq C\|b\|_{Lip_\beta(\mu)} \left(\frac{\|f\|_{L^1(\mu)}}{\nu}\right)^{1/(1-\alpha-\beta)}.$$

Using the above Theorem, we consider the case of $p = 1/(\alpha+\beta)$, $1/(\alpha+\beta) < p < 1/\alpha$ and $1/\alpha \le p \le \infty$, respectively. Now our main results can be stated as follows:

Theorem 1.7. Let $b \in Lip_\beta(\mu)$, $1/(\beta+\alpha) < p < 1/\alpha$ with $0 < \alpha+\beta < 1$ and $1/p-\alpha-\beta+\delta > 0$. If λ satisfy the ϵ-weak reverse doubling condition with $\epsilon \in (0, \min\{(1-\alpha-\beta+\delta)p'-1, (1-\alpha)p'-1\})$, then for all bounded functions f with compact support,

$$\|T_{\alpha,b}(f)\|_{Lip_{\alpha+\beta-1/p}(\mu)} \le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}.$$

Theorem 1.8. Let $b \in Lip_\beta(\mu)$, $0 < \alpha+\beta < 1$. If λ satisfy the ϵ-weak reverse doubling condition with $\epsilon \in (0, \min\{\delta/(1-\alpha-\beta), \beta/(1-\alpha-\beta)\})$, then for all bounded functions f with compact support,

$$\|T_{\alpha,b}(f)\|_{RBMO(\mu)} \le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^{1/(\beta+\alpha)}(\mu)}.$$

Theorem 1.9. Let $b \in Lip_\beta(\mu)$, $1/\alpha \le p \le \infty$. If λ satisfies the ϵ-weak reverse doubling condition with $\epsilon \in (0, \min\{(1-\alpha-\beta+\delta)p'-1, \beta/(1-\alpha-\beta)\})$, then the following statements are equivalent.

(1) For all bounded functions f with compact support,

$$\|T_{\alpha,b}(f)\|_{Lip_{\alpha+\beta-1/p}(\mu)} \le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}; \tag{1.14}$$

(2) For any ball B and $u \in B$,

$$\frac{1}{\mu(B)}\int_B |b(x)-m_Q(b)|d\mu(x)\left|\int_{\mathcal{X}\setminus 2B} K_\alpha(u,y)f(y)d\mu(y)\right| \le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p}, \tag{1.15}$$

and for any ball U such that $B \subset U$ with $r_U \le 2r_B$, and any $v \in U$,

$$|m_U(b)-m_B(b)|\left|\int_{\mathcal{X}\setminus 2U} K_\alpha(v,y)f(y)d\mu(y)\right| \le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p}. \tag{1.16}$$

§2 Preliminary lemma

To prove our theorems we need the following lemma, which is a little different from those in [10], but it can easily seen that the lemma is still valid.

Lemma 2.1. For a function $f \in L^1_{loc}(\mu)$, the conditions (**A**), (**B**), and (**C**) below, are equivalent.

(**A**) There exist some constant C_1 and a collection of numbers of f_B, one for each B, such that these two properties hold: for any $B = B(c_B, r_B)$

$$\frac{1}{\mu(6B)}\int_B |f(x)-f_B|d\mu(x) \le C_1\lambda(c_B,r_B)^\beta, \tag{2.1}$$

and for any ball U such that $B \subset U$ and $r_U \le 2r_B$,

$$|f_B - f_U| \le C_1\lambda(c_B,r_B)^\beta; \tag{2.2}$$

(**B**) There is a constant C_2 such that

$$|f(x)-f(y)| \le C_2\lambda(x,d(x,y))^\beta, \tag{2.3}$$

for μ-almost every x and y in the support of μ;

(**C**) For any given p, $1 \le p \le \infty$, there is a constant $C(p)$, such that for every ball $B = B(c_B, r_B)$, we have

$$\left(\frac{1}{\mu(B)}\int_B |f(x) - m_B(f)|^p d\mu(x)\right)^{\frac{1}{p}} \le C(p)\lambda(c_B, r_B)^\beta, \tag{2.4}$$

where $m_B(f) = \frac{1}{\mu(B)}\int_B f(y)d\mu(y)$ and also for any ball U such that $B \subset U$ and $r_U \le 2r_B$,

$$|m_B(f) - m_U(f)| \le C(p)\lambda(c_B, r_B)^\beta, \tag{2.5}$$

In addition, the quantities: inf C_1, inf C_2, and inf $C(p)$ with a fixed p are equivalent.

§3 Proofs of Theorem 1.7 and Theorem 1.8

Proof of Theorem 1.7. For any ball $B = B(c_B, r_B)$ and $U = U(c_U, r_U)$ such that $B \subset U$ satisfying $r_U \le 2r_B$. Let

$$a_B = m_B[T_{\alpha,b}(f\chi_{\mathcal{X}\setminus\frac{6}{5}B})],$$

and

$$a_U = m_U[T_{\alpha,b}(f\chi_{\mathcal{X}\setminus\frac{6}{5}U})].$$

By Lemma 2.1, we need to show that there exists a constant $C > 0$ such that

$$\frac{1}{\mu(6B)}\int_B |T_{\alpha,b}(f)(x) - a_B| d\mu(x) \le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}\lambda(c_B, r_B)^{\alpha+\beta-1/p} \tag{3.1}$$

and

$$|a_B - a_U| \le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}\lambda(c_B, r_B)^{\alpha+\beta-1/p}. \tag{3.2}$$

Decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\frac{6}{5}B}$ and $f_2 = f - f_1$. Then

$$\begin{aligned}
&\frac{1}{\mu(6B)}\int_B |T_{\alpha,b}(f)(x) - a_B| d\mu(x)\\
\le\ & \frac{1}{\mu(6B)}\int_B |I_{\alpha,b}(f_1)(x)| d\mu(x) + \frac{1}{\mu(6B)}\int_B |T_{\alpha,b}(f_2)(x) - a_B| d\mu(x)\\
:=\ & \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

Taking $1 < p_1 < 1/(\beta + \alpha) < p$, and q_1 satisfying the condition of $1/q_1 = 1/p_1 - \beta - \alpha$. From **Theorem 1.5** and Hölder inequality, we have

$$\begin{aligned}
\mathrm{I}_1 &\le \frac{1}{\mu(6B)}\left[\int_B |I_{\alpha,b}(f_1)(x)|^{q_1} d\mu(x)\right]^{1/q_1} \mu(B)^{1-1/q_1}\\
&\le C\|b\|_{Lip_\beta(\mu)} \frac{1}{\mu(6B)}\left[\int_{\frac{6}{5}B} |f(x)|^{p_1} d\mu(x)\right]^{1/p_1} \mu(B)^{1-1/q_1}\\
&\le C\|b\|_{Lip_\beta(\mu)} \frac{1}{\mu(6B)}\left[\int_{\frac{6}{5}B} |f(x)|^{p} d\mu(x)\right]^{1/p} \mu(\frac{6}{5}B)^{1/p_1-1/p}\mu(B)^{1-1/q_1}\\
&\le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}\lambda(c_B, \frac{6}{5}r_B)^{\alpha+\beta-1/p}\\
&\le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}\lambda(c_B, r_B)^{\alpha+\beta-1/p}.
\end{aligned}$$

To estimate I_2, using the Hölder's inequality, we obtain

$$\begin{aligned}
&|T_{\alpha,b}(f_2)(x) - T_{\alpha,b}(f_2)(y)| \\
= &\left| \int_{\mathcal{X}\setminus\frac{6}{5}B} \Big[(b(x)-b(z))K_\alpha(x,z) - (b(y)-b(z))K_\alpha(y,z)\Big] f(z)dz \right| \\
\leq &\int_{\mathcal{X}\setminus\frac{6}{5}B} |b(x)-b(z)||K_\alpha(x,z)-K_\alpha(y,z)||f(z)|dz \\
&+ \int_{\mathcal{X}\setminus\frac{6}{5}B} |(b(x)-b(y)||K_\alpha(y,z)||f(z)|dz \\
:= &\ \mathrm{I}_2^{(1)} + \mathrm{I}_2^{(2)}.
\end{aligned}$$

For $\mathrm{I}_2^{(1)}$, by $x,y\in B$, we obtain $d(x,y)\leq r_B$ and $d(x,c_B)\leq r_B$, and $1/p-\alpha-\beta+\delta>0$. Then

$$\begin{aligned}
\mathrm{I}_2^{(1)} &\leq C\|b\|_{Lip_\beta} \int_{\mathcal{X}\setminus\frac{6}{5}B} [\lambda(x,d(x,z))]^\beta \frac{\lambda(x,d(x,y))^\delta}{[\lambda(x,d(x,z))]^{1-\alpha+\delta}} |f(z)|dz \\
&= C\|b\|_{Lip_\beta(\mu)} [\lambda(x,d(x,y))]^\delta \int_{\mathcal{X}\setminus\frac{6}{5}B} \frac{1}{[\lambda(x,d(x,z))]^{1-\alpha-\beta+\delta}} |f(z)|dz \\
&\leq C\|b\|_{Lip_\beta(\mu)} [\lambda(x,r_B)]^\delta \|f\|_{L^p(\mu)} \left(\sum_{k=1}^\infty \int_{2^k\frac{6}{5}B\setminus 2^{k-1}\frac{6}{5}B} \frac{1}{[\lambda(x,d(x,z))]^{(1-\alpha-\beta+\delta)p'}} d\mu(z) \right)^{1/p'} \\
&\leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(x,r_B)]^\delta \left(\sum_{k=1}^\infty \frac{\mu(B(x,2^k\frac{6}{5}r_B))}{[\lambda(x,2^{k-1}\frac{6}{5}r_B)]^{(1-\alpha-\beta+\delta)p'}} \right)^{1/p'} \\
&\leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(x,r_B)]^\delta \left(\sum_{k=1}^\infty [\lambda(x,2^{k-1}r_B)]^{1-(1-\alpha-\beta+\delta)p'} \right)^{1/p'} \\
&\leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(x,r_B)]^\delta \left(\sum_{k=1}^\infty \frac{1}{[C(2^k)]^{(1-\alpha-\beta+\delta)p'-1}} \right)^{1/p'} [\lambda(x,r_B)]^{\alpha+\beta-\delta-1/p} \\
&\leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(x,r_B)]^{\alpha+\beta-1/p} \\
&\leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(c_B,r_B)]^{\alpha+\beta-1/p}.
\end{aligned}$$

For $\mathrm{I}_2^{(2)}$, by $1/p>\alpha$, we obtain

$$\begin{aligned}
\mathrm{I}_2^{(2)} &\leq C\|b\|_{Lip_\beta} \int_{\mathcal{X}\setminus\frac{6}{5}B} [\lambda(x,d(x,y))]^\beta \frac{1}{[\lambda(x,d(x,z))]^{1-\alpha}} |f(z)|dz \\
&\leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(x,r_B)]^\beta \left(\sum_{k=1}^\infty \frac{1}{[C(2^k)]^{(1-\alpha)p'-1}} \right)^{1/p'} [\lambda(x,r_B)]^{\alpha-1/p} \\
&\leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(x,r_B)]^{\alpha+\beta-1/p} \\
&\leq C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(c_B,r_B)]^{\alpha+\beta-1/p}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{I}_2 &= \frac{1}{\mu(6B)} \int_B |I_{\alpha,b}(f_2)(x) - a_Q| d\mu(x) \\
&\leq \frac{1}{\mu(6B)} \int_B \frac{1}{\mu(B)} \int_B |I_{\alpha,b}(f_2)(x) - I_{\alpha,b}(f_2)(y)| d\mu(y) d\mu(x) \\
&\leq C\|b\|_{Lip_\beta} \|f\|_{L^p} [\lambda(c_B,r_B)]^{\alpha+\beta-1/p}.
\end{aligned}$$

The estimates for I_1 and I_2 yield the estimate (3.1).

Now we turn to estimate (3.2). For $x \in B$, $y \in U$ and $B \subset U$, we write

$$\begin{aligned}|a_B - a_U| &= \bigg| \int_{\mathcal{X} \setminus \frac{6}{5}U} [b(x) - b(z)] K_\alpha(x,z) f(z) d\mu(z) \\ &\quad + \int_{\frac{6}{5}U \setminus \frac{6}{5}B} [b(x) - b(z)] K_\alpha(x,z) f(z) d\mu(z) \\ &\quad - \int_{\mathcal{X} \setminus \frac{6}{5}U} [b(y) - b(z)] K_\alpha(y,z) f(z) d\mu(z) \bigg| \\ &\le \int_{\mathcal{X} \setminus \frac{6}{5}U} \bigg| [b(x) - b(z)] K_\alpha(x,z) - [b(y) - b(z)] K_\alpha(y,z) \bigg| |f(z)| d\mu(z) \\ &\quad + \int_{\frac{6}{5}R \setminus \frac{6}{5}B} |b(x) - b(z)| |K_\alpha(x,z)| |f(z)| d\mu(z) \\ &:= \mathrm{II}_1 + \mathrm{II}_2.\end{aligned}$$

Arguing similarly to that in the estimate of I_2, we obtain that

$$\mathrm{II}_1 \le C \|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(c_B, r_B)]^{\alpha+\beta-1/p}.$$

Now we give the estimate for II_2. Noting that $Q \subset U$ and $r_U \le 2r_B$, we obtain

$$\begin{aligned}\mathrm{II}_2 &\le C\|b\|_{Lip_\beta} \int_{\frac{6}{5}U \setminus \frac{6}{5}B} \frac{[\lambda(x, d(x,z))]^\beta}{[\lambda(x, d(x,z))]^{1-\alpha}} |f(z)| d\mu(z) \\ &\le C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} \left(\int_{\frac{6}{5}R \setminus \frac{6}{5}Q} \frac{1}{[\lambda(x, d(x,z))]^{(1-\alpha-\beta)p'}} d\mu(z) \right)^{1/p'} \\ &\le C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)} [\lambda(c_B, r_B)]^{\alpha+\beta-1/p}.\end{aligned}$$

Combining the estimates for II_1 and II_2 yield the estimate (3.2). This completes the proof of **Theorem1.7**.

The aim of the following is to prove **Theorem 1.8**, it should be pointed out that the proof of **Theorem 1.7** and **Theorem 1.8** have little different.

Proof of Theorem 1.8. By *Lemma* 2.1 and **Theorem 1.7**, it suffices to show

$$|a_B - a_U| \le C K_{B,U} \|b\|_{Lip_\beta(\mu)} \|f\|_{L^{1/(\alpha+\beta)}(\mu)}, \tag{3.3}$$

where $B \subset U$,

$$a_B = m_B[T_{\alpha,b}(f\chi_{\mathcal{X} \setminus \frac{6}{5}B})],$$

and

$$a_U = m_U[T_{\alpha,b}(f\chi_{\mathcal{X} \setminus \frac{6}{5}U})].$$

Now we verify (3.3). Let N be the first integer k such that $U \subset 6^k B$. We denote $\bar{B} = 6^{N+1}B$.

$$\begin{aligned}|a_B - a_U| &= |m_B[T_{\alpha,b}(f\chi_{\mathcal{X} \setminus \frac{6}{5}B})] - m_U[T_{\alpha,b}(f\chi_{\mathcal{X} \setminus \frac{6}{5}U})]| \\ &\le |m_B[T_{\alpha,b}(f\chi_{\mathcal{X} \setminus \bar{B}})] - m_U[T_{\alpha,b}(f\chi_{\mathcal{X} \setminus \bar{B}})]| \\ &\quad + |m_B[T_{\alpha,b}(f\chi_{\bar{B} \setminus \frac{6}{5}B})]| + |m_U[T_{\alpha,b}(f\chi_{\bar{B} \setminus \frac{6}{5}U})]| \\ &:= \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3.\end{aligned}$$

Arguing similarly to that in the estimate of I_2, we obtain that

$$\mathrm{III}_1 \le C\|b\|_{Lip_\beta(\mu)} \|f\|_{L^p(\mu)}.$$

Now we deal with the term III_2. For any $x \in B$,

$$\begin{aligned}|T_{\alpha,b}(f\chi_{\bar{B}\setminus\frac{6}{5}B})(x)| &\leq \int_{\bar{B}\setminus\frac{6}{5}B}\frac{|f(y)(b(x)-b(y))|}{[\lambda(x,d(x,y))]^{1-\alpha}}d\mu(y)\\ &\leq \|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}\left(\int_{\bar{B}\setminus\frac{6}{5}B}\frac{1}{[\lambda(x,d(x,y))]^{(1-\alpha-\beta)p'}}d\mu(y)\right)^{1/p'}\\ &\leq CK_{\frac{6}{5}B,\bar{B}}\|b\|_{Lip_\beta(\mu)}\|f\|_{L^{1/(\alpha+\beta)}(\mu)}.\end{aligned}$$

Therefore, $\mathrm{III}_2 \leq CK_{B,U}\|b\|_{Lip_\beta(\mu)}\|f\|_{L^{1/(\alpha+\beta)}(\mu)}$.

Similarly, since $r_{\bar{B}} \approx r_U$, we have

$$|T_{\alpha,b}(f\chi_{\bar{B}\setminus\frac{6}{5}U})(x)| \leq C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^{1/(\alpha+\beta)}}.$$

To sum up, we have

$$|a_B - a_U| \leq CK_{B,U}\|b\|_{Lip_\beta(\mu)}\|f\|_{L^{1/(\alpha+\beta)}(\mu)}, \tag{3.3}$$

this complete the proof of **Theorem 1.8**.

§4 Proof of Theorem 1.9

Now we consider the endpoint case that $1/\alpha \leq p \leq \infty$. The basic idea is form [6].

Proof of Theorem 1.9. The proof of the above Theorem is divided into the following two steps.

(i) Let us first prove

$$\frac{1}{\mu(B)}\int_B |T_{\alpha,b}(f)(x) - m_B(T_{\alpha,b}(f))|d\mu(x) \leq C\|f\|_{L^p(\mu)}[\lambda(c_B,r_B)]^{\alpha-1/p} \tag{4.1}$$

is equivalent to (1.15), for any bounded function f with compact support and any ball B.

Decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. Then for $x \in B$,

$$\begin{aligned}&T_{\alpha,b}(f)(x) - m_B(T_{\alpha,b}(f))\\ =\ &T_{\alpha,b}(f_1)(x) - m_B(T_{\alpha,b}(f_1)) + T_{\alpha,b}(f_2)(x) - \frac{1}{\mu(B)}\int_B T_{\alpha,b}(f_2)(z)d\mu(z)\\ =\ &T_{\alpha,b}(f_1)(x) - m_B(T_{\alpha,b}(f_1)) + [b(x)-m_B(b)]T_\alpha(f_2)(x) - T_\alpha([b-m_B(b)]f_2)(x)\\ &-\frac{1}{\mu(B)}\int_B[b(z)-m_B(b)]T_\alpha(f_2)(z)d\mu(z) + \frac{1}{\mu(B)}\int_B T_\alpha([b-m_B(b)]f_2)(z)d\mu(z).\end{aligned}$$

Let $u \in Q$,

$$\begin{aligned}&T_{\alpha,b}(f)(x) - m_B(T_{\alpha,b}(f))\\ =\ &T_{\alpha,b}(f_1)(x) - m_B(T_{\alpha,b}(f_1)) + [b(x)-m_B(b)]\Big[T_\alpha(f_2)(x) - T_\alpha(f_2)(u)\Big]\\ &+[b(x)-m_B(b)]T_\alpha(f_2)(u) - \frac{1}{\mu(B)}\int_B[b(z)-m_Q(b)]\Big[T_\alpha(f_2)(z) - T_\alpha(f_2)(u)\Big]d\mu(z)\\ &+\frac{1}{\mu(B)}\int_B\Big[T_\alpha([b-m_B(b)]f_2)(z) - T_\alpha([b-m_B(b)]f_2)(x)\Big]d\mu(z).\end{aligned}$$

Now, if we let

$$\eta_1(x) = T_{\alpha,b}(f_1)(x)$$
$$\eta_2(x,u) = [b(x) - m_B(b)]\Big[T_\alpha(f_2)(x) - T_\alpha(f_2)(u)\Big]$$
$$\eta_3(x,u) = T_\alpha([b - m_B(b)]f_2)(u) - T_\alpha([b - m_B(b)]f_2)(x).$$

and

$$\eta_4(x,u) = [b(x) - m_B(b)]T_\alpha(f_2)(u),$$

we have

$T_{\alpha,b}(f)(x) - m_B(T_{\alpha,b}(f)) = \eta_1(x) - m_B(\eta_1) + \eta_2(x,u) + \eta_4(x,u) - m_B[\eta_2(\cdot,u)] + m_B[\eta_3(\cdot,u)]$.

We claim that the following estimates hold:

$$\frac{1}{\mu(B)}\int_B |\eta_1(x) - m_B(\eta_1)|d\mu(x) \le C\|b\|_{Lip_\beta}\|f\|_{L^p(\mu)}[\lambda(c_B, r_B)]^\beta, \tag{4.2}$$

$$\frac{1}{\mu(B)}\int_B |\eta_2(x,u)|d\mu(x) \le C\|b\|_{Lip_\beta}\|f\|_{L^p(\mu)}[\lambda(c_B, r_B)]^\beta, \tag{4.3}$$

$$\frac{1}{\mu(B)}\int_B |\eta_3(x,u)|d\mu(x) \le C\|b\|_{Lip_\beta}\|f\|_{L^p(\mu)}[\lambda(c_B, r_B)]^\beta. \tag{4.4}$$

By **Theorem 1.7** with $1/(\alpha+\beta) < p_1 < 1/\alpha$, then $f_1 \in L^{p_2}$, it follows that

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |\eta_1(x) - m_B(\eta_1)|d\mu(x) &\le C\|b\|_{Lip_\beta}\|f_1\|_{L^{p_1}(\mu)}[\lambda(c_B, r_B)]^{\alpha+\beta-1/p_1} \\ &\le C\|b\|_{Lip_\beta}\|f\|_{L^p(\mu)}[\lambda(c_B, r_B)]^{\alpha+\beta-1/p}.\end{aligned}$$

For (4.3), using the Hölder inequality, we have

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B |\eta_2(x,u)|d\mu(x) \\ \le\ &\frac{1}{\mu(B)}\int_B |b(x) - m_B(b)|\int_{\mathcal{X}\setminus 2B} |K_\alpha(x,z) - K_\alpha(u,z)||f(z)|d\mu(z)d\mu(x) \\ \le\ &C\|f\|_{L^p(\mu)}\frac{1}{\mu(B)}\int_B |b(x) - m_B(b)| \\ &\times\Bigg\{\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} \frac{[\lambda(x, d(x,u))]^{\delta p'}}{[\lambda(x, d(x,z))]^{(1-\alpha+\delta)p'}}d\mu(z)\Bigg\}^{1/p'} d\mu(x) \\ \le\ &C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B, r_B)]^{\alpha+\beta-1/p}.\end{aligned}$$

Finally we give the proof of (4.4).

$$\begin{aligned}&\Big|T_\alpha([b - m_B(b)]f_2)(u) - T_\alpha([b - m_B(b)]f_2)(x)\Big| \\ \le\ &\int_{\mathcal{X}\setminus 2B} |K_\alpha(u,z) - K_\alpha(x,z)||b(z) - m_B(b)||f(z)|d\mu(z) \\ \le\ &C\|f\|_{L^p(\mu)}\Bigg\{\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} \frac{[\lambda(x, d(x,u))]^{\delta p'}}{[\lambda(x, d(x,z))]^{(1-\alpha+\delta)p'}}|b(z) - m_B(b)|^{p'}d\mu(z)\Bigg\}^{1/p'} \\ \le\ &C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B, r_B)]^{\alpha+\beta-1/p}.\end{aligned}$$

From this, it follows that

$$\frac{1}{\mu(B)}\int_B |\eta_3(x,u)|d\mu_x \le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p}.$$

Assume (4.1) holds and the estimates (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned}
&\frac{1}{\mu(B)}\int_B |\eta_4(x,u)|d\mu(x)\\
= &\frac{1}{\mu(B)}\int_B \Big|T_{\alpha,b}(f)(x)-m_B[T_{\alpha,b}f]-(\eta_1(x)-m_B(\eta_1))\\
&\qquad -\eta_2(x,u)+m_B(\eta_2(\cdot,u))-m_B(\eta_3(\cdot,u))\Big|d\mu(x)\\
\le &\frac{1}{\mu(B)}\int_B |T_{\alpha,b}(f)(x)-m_B[T_{\alpha,b}f]|d\mu(x)+\frac{1}{\mu(B)}\int_B|\eta_1(x)-m_B(\eta_1)|d\mu(x)\\
&+\frac{2}{\mu(B)}\int_B|\eta_2(x,u)|d\mu(x)+\frac{1}{\mu(B)}\int_B|\eta_3(x,u)|d\mu(x)\\
\le &C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p},
\end{aligned}$$

that is, for any B and $u\in B$, (1.15) holds. Conversely, (1.15) implies that (4.1) holds.

(ii)Now let us verify that

$$|m_U[T_{\alpha,b}f]-m_B[T_{\alpha,b}f]|\le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p} \tag{4.5}$$

is equivalent to (1.16). We write $f_1=f\chi_{2U}$ and $f_2=f-f_1$, then

$$\begin{aligned}
&m_B(T_{\alpha,b}(f))-m_U(T_{\alpha,b}(f))\\
= &\Big[m_B[T_{\alpha,b}(f_1)]-m_U[T_{\alpha,b}(f_1)]\Big]+\Big[m_B[T_{\alpha,b}(f_2)]-m_U[T_{\alpha,b}(f_2)]\Big]\\
:= &\mathrm{IV}_1+\mathrm{IV}_2.
\end{aligned}$$

From **Theorem 1.7** with $1/(\alpha+\beta)<p_1<1/\alpha$, it follows that

$$|\mathrm{IV}_1|\le C\|f_1\|_{L^{p_1}}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p_1}\le C\|f\|_{L^p}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p}.$$

To estimates IV_2, for any $x\in B$ and $v\in U$, we write

$$T_{\alpha,b}(f)(x)-m_U(T_{\alpha,b}(f))=\eta_2'(x,v)+\eta_4'(x,v)-m_U[\eta_2'(\cdot,v)]+m_U[\eta_3'(\cdot,v)],$$

where

$$\begin{aligned}
\eta_2'(x,v)&=[b(x)-m_U(b)]\Big[T_\alpha(f_2)(x)-T_\alpha(f_2)(v)\Big]\\
\eta_3'(x,v)&=T_\alpha([b-m_U(b)]f_2)(v)-T_\alpha([b-m_U(b)]f_2)(x).
\end{aligned}$$

and

$$\eta_4'(x,v)=[b(x)-m_U(b)]T_\alpha(f_2)(v),$$

Some computations similar above, implies that

$$\begin{aligned}
m_B[\eta_2'(x,v)] &\le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p},\\
m_U[\eta_2'(x,v)] &\le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_U,r_U)]^{\alpha+\beta-1/p}\\
&\le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,2r_B)]^{\alpha+\beta-1/p}\\
&\le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p},\\
m_U[\eta_3'(x,v)] &\le C\|f\|_{L^p(\mu)}\|b\|_{Lip_\beta(\mu)}[\lambda(c_B,r_B)]^{\alpha+\beta-1/p}.
\end{aligned}$$

An argument similar to the proof of the equivalence between (4.1) and (1.15), we obtain (4.5) is equivalent to (1.16). By Lemma 2.1, we complete the proof of theorem.

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