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Boundedness in a fully parabolic quasilinear repulsion chemotaxis model of higher dimension

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Abstract. We deal with the boundedness of solutions to a class of fully parabolic quasilinear repulsion chemotaxis systems

$$\begin{cases} u_t = \nabla\cdot(\phi(u)\nabla u) + \nabla\cdot(\psi(u)\nabla v), & (x,t)\in\Omega\times(0,T),\\ v_t = \Delta v - v + u, & (x,t)\in\Omega\times(0,T),\end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset\mathbb{R}^N (N\geq 3)$, where $0<\psi(u)\leq K(u+1)^\alpha$, $K_1(s+1)^m\leq\phi(s)\leq K_2(s+1)^m$ with $\alpha, K, K_1, K_2>0$ and $m\in\mathbb{R}$. It is shown that if $\alpha - m < \frac{4}{N+2}$, then for any sufficiently smooth initial data, the classical solutions to the system are uniformly-in-time bounded. This extends the known result for the corresponding model with linear diffusion.

§1 Introduction

In general chemotaxis models, such as those described by the classical Keller-Segel system [8], cells move toward to the increasing signal concentration. The attraction mechanism in these chemotaxis models results in possible blow-up of solutions, see [2, 3, 4, 7, 11] and references therein. On the other hand, contrary phenomena can be observed in biology that cells move away from the increasing signal concentration to resist the chemical signals, the so-called chemorepulsion. In this paper, we consider the following quasilinear repulsion chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(\phi(u)\nabla u) + \nabla\cdot(\psi(u)\nabla v), & (x,t)\in\Omega\times(0,T),\\ v_t = \Delta v - v + u, & (x,t)\in\Omega\times(0,T),\\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), v(x,0)=v_0(x), & x\in\overline{\Omega},\end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $u = u(x,t)$ is the cell density, and $v = v(x,t)$ denotes the concentration of a repulsion signal. The nonnegative function ϕ and ψ are assumed to satisfy

$$\phi, \psi \in C^2([0,\infty)) \text{ with } \psi(0) = 0, \tag{2}$$

as well as

$$0 < \psi(s) \leq K(s+1)^\alpha \text{ for all } s > 0, \tag{3}$$

$$K_1(s+1)^m \leq \phi(s) \leq K_2(s+1)^m \text{ for all } s > 0, \tag{4}$$

with some $K > 0, K_2 \geq K_1 > 0, \alpha > 0$, and $m \in \mathbb{R}$.

The dynamical behavior of solutions to the corresponding attractive chemotaxis systems obtained on replacing ψ by $-\psi$ has been studied much more clearly. For example, the optimal condition $\alpha < \frac{2}{N}$ was determined for the attractive question concerning global existence versus blow-up by Tao and Winkler [10].

Generally speaking, the repulsion mechanism benefits the global existence of solution. However it seems difficult to gain the possible contribution of repulsions for the fully parabolic systems of chemorepulsion models like (1) by using the current tools. So, it is not surprising that the mathematical analysis to the fully parabolic repulsion chemotaxis models is still relatively weak.

When $\psi = 1$ and $m = 0$ in (1), Cieślak at el [1] asserted the global existence of smooth solutions and convergence to the steady states for $N = 2$, as well as the global existence of weak solutions for $N = 3, 4$. The global existence of smooth solutions of (1) for $N \geq 3$ remained open in a long time, until Tao [9] proved that the system (1) possesses nonnegative bounded smooth solutions for the linear diffusion case with $m = 0$ whenever $\alpha < \frac{4}{N+2}$. His strategy includes a combined estimate on $\int_\Omega u^p + \int_\Omega |\nabla v|^{\frac{2p}{2-\alpha}}$ (instead of dealing with u and v separately), for which the estimate to $\int_\Omega u^{2-\alpha}$ is necessary. The present paper will extend Tao's result to the nonlinear diffusion case of (1). Our result is the following theorem.

Theorem 1 *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$, ϕ and ψ satisfy (2)–(4) with $K > 0$, $K_2 \geq K_1 > 0$, $m \in \mathbb{R}$. If*

$$\alpha - m < \frac{4}{N+2}, \tag{5}$$

then for any $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in C^1(\bar{\Omega})$, there exists a couple (u, v) of nonnegative bounded functions in $C^0(\bar{\Omega} \times [0,\infty)) \cup C^{2,1}(\bar{\Omega} \times (0,\infty))$ solving (1) classically.

Remark 1 The known boundedness condition $\alpha < \frac{4}{N+2}$ for the chemorepulsion model with nonlinear sensitivity and linear diffusion [9] can be obtained by letting $m = 0$ in Theorem 1.

Remark 2 Comparing with the optimal boundedness vs. blow-up condition $\alpha < \frac{2}{N}$ for the corresponding attractive chemotaxis model [10], it can be found that $\alpha < \frac{2}{N} < \frac{4}{N+2}$ whenever $N \geq 3$. This shows the positive contribution of the repulsion mechanism to the boundedness of solutions.

§2 Local existence

The local existence of solutions to (1) can be addressed by methods involving standard parabolic regularity theory in a suitable fixed point approach (refer e.g. [5] for details).

Lemma 2.1 *Let ϕ and ψ satisfy (2)–(4) with $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in C^1(\bar{\Omega})$ nonnegative. Then there exist $T_{\max} \in (0,\infty]$ and a pair (u,v) of nonnegative functions from $C^0(\bar{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max}))$ solving (1) classically in $\Omega \times (0,T_{\max})$ with*

$$u > 0 \quad \text{and} \quad v \geq 0 \quad \text{in } \bar{\Omega} \times (0,T_{\max}). \tag{6}$$

Moreover, if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{7}$$

The next lemma can be obtained by a direct calculation.

Lemma 2.2 *The solution (u,v) of (1) satisfies the following properties*

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}, \tag{8}$$

$$\|v(\cdot,t)\|_{L^1(\Omega)} = \|v_0\|_{L^1(\Omega)} \mathrm{e}^{-t} + \|v_0\|_{L^1(\Omega)}(1-\mathrm{e}^{-t}) \tag{9}$$

for all $t \in (0,T_{\max})$.

We have also an elementary estimate for $|\nabla v|^2$:

Lemma 2.3 *Let ϕ, ψ satisfy (2)–(4). Then there exists $C > 0$ such that the solution (u,v) of (1) satisfies*

$$\int_\Omega |\nabla v(\cdot,t)|^2 \leq C \tag{10}$$

for all $t \in (0,T_{\max})$.

Proof. Conditions (2) and (3) ensure that the integral

$$\Phi(s) := \int_1^s \int_1^\sigma \frac{1}{\psi(\tau)} d\tau d\sigma, \quad s > 0$$

is well-defined. A straightforward computation shows

$$\begin{aligned} \frac{d}{dt}\int_\Omega \Phi(u) &= \int_\Omega \Phi'(u) u_t = \int_\Omega \Phi'(u) \Big(\nabla \cdot \big(\phi(u)\nabla u\big) + \nabla \cdot \big(\psi(u)\nabla v\big) \Big) \\ &= -\int_\Omega \Phi''(u)\phi(u)|\nabla u|^2 - \int_\Omega \Phi''(u)\psi(u)\nabla u \cdot \nabla v \\ &= -\int_\Omega \frac{\phi(u)}{\psi(u)}|\nabla u|^2 - \int_\Omega \nabla u \cdot \nabla v \quad \text{for all } \ t \in (t_0,T_{\max}), \end{aligned} \tag{11}$$

where the fact that $\Phi''(u) = \frac{1}{\psi(u)}$ is used. Multiply the second equation of (1) by $-\Delta v$ and integrate by part to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 = \int_\Omega \nabla u \cdot \nabla v \quad \text{ for all } \ t \in (t_0,T_{\max}). \tag{12}$$

By adding (11) to (12) and integrating from 0 to t, we obtain

$$\begin{aligned} &\int_\Omega \Phi(u(\cdot,t)) + \frac{1}{2}\int_\Omega |\nabla v(\cdot,t)|^2 + \int_0^t \int_\Omega \frac{\phi(u)}{\psi(u)}|\nabla u|^2 + \int_0^t \int_\Omega |\Delta v|^2 + \int_0^t \int_\Omega |\nabla v|^2 \\ \leq &\int_\Omega \Phi(u_0) + \frac{1}{2}\int_\Omega |\nabla v_0|^2. \end{aligned}$$

This completes the proof. $\square$

§3 Global boundedness of solutions

Now we deal with the global boundedness of (1) to prove Theorem 1.

We begin with an estimate on component u.

Lemma 3.1 *Let the conditions of Theorem 1 hold, and (u,v) be a solution ensured by Lemma 2.1. Then for any $p\in(1,+\infty)$, there exists $C>0$ such that for all $t\in(0,T_{\max})$,*

$$\frac{d}{dt}\int_\Omega(u+1)^p+\frac{p(p-1)K_1}{2}\int_\Omega(u+1)^{p+m-2}|\nabla u|^2\le C\int_\Omega(u+1)^{p+2\alpha-m-2}|\nabla v|^2. \tag{13}$$

Proof. For arbitrary $p>1$, take $p(u+1)^{p-1}$ as a test function for the first equation in (1) and integrate by part to obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega(u+1)^p&=p\int_\Omega(u+1)^{p-1}\nabla\cdot(\phi(u)\nabla u)+p\int_\Omega(u+1)^{p-1}\nabla\cdot(\psi(u)\nabla v)\\&=-p(p-1)\int_\Omega(u+1)^{p-2}\phi(u)|\nabla u|^2-p(p-1)\int_\Omega(u+1)^{p-2}\psi(u)\nabla u\cdot\nabla v.\end{aligned} \tag{14}$$

By the Cauchy inequality with (2)–(4), we have

$$\begin{aligned}&-p(p-1)\int_\Omega(u+1)^{p-2}\psi(u)\nabla u\cdot\nabla v\\&\le p(p-1)K\int_\Omega(u+1)^{p+\alpha-2}|\nabla u\cdot\nabla v|\\&\le\frac{p(p-1)K_1}{2}\int_\Omega(u+1)^{p+m-2}|\nabla u|^2+\frac{K^2p(p-1)}{2K_1}\int_\Omega(u+1)^{p+2\alpha-m-2}|\nabla v|^2.\end{aligned} \tag{15}$$

Consequently,

$$\frac{d}{dt}\int_\Omega(u+1)^p+\frac{p(p-1)K_1}{2}\int_\Omega(u+1)^{p+m-2}|\nabla u|^2\le C\int_\Omega(u+1)^{p+2\alpha-m-2}|\nabla v|^2$$

with $C:=\frac{K^2p(p-1)}{2K_1}$. $\square$

Next, we make an estimate for the component v.

Lemma 3.2 *Let the conditions of Theorem 1 hold, and (u,v) be a solution ensured by Lemma 2.1. Then for any $q\in(2,+\infty)$, there exists $C>0$ such that for all $t\in(0,T_{\max})$,*

$$\frac{1}{q}\frac{d}{dt}\int_\Omega|\nabla v|^{2q}+\frac{2(q-1)}{q^2}\int_\Omega\left|\nabla|\nabla v|^q\right|^2\le C\int_\Omega(u+1)^2|\nabla v|^{2q-2}+C. \tag{16}$$

Proof. We know from the second equation of (1) with the identity $2\nabla v\cdot\nabla\Delta v=\Delta|\nabla v|^2-2|D^2v|^2$ that

$$\begin{aligned}(|\nabla v|^2)_t&=2\nabla v\cdot\nabla\Delta v-2|\nabla v|^2+2\nabla u\cdot\nabla v\\&=\Delta|\nabla v|^2-2|D^2v|^2-2|\nabla v|^2+2\nabla u\cdot\nabla v.\end{aligned}$$

Testing by $|\nabla v|^{2q-2}$, we get

$$
\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} &= 2\int_\Omega |\nabla v|^{2q-2}\nabla v\cdot\nabla(\Delta v - v + u)\\
&= \int_\Omega |\nabla v|^{2q-2}\Delta|\nabla v|^2 - 2\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 - 2\int_\Omega |\nabla v|^{2q}\\
&\quad - 2\int_\Omega u\nabla\cdot(|\nabla v|^{2q-2}\nabla v)\\
&= -(q-1)\int_\Omega |\nabla v|^{2q-4}\big|\nabla|\nabla v|^2\big|^2 + \int_{\partial\Omega}|\nabla v|^{2q-2}\frac{\partial|\nabla v|^2}{\partial\nu} - 2\int_\Omega |\nabla v|^{2q}\\
&\quad - 2\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 - 2\int_\Omega u|\nabla v|^{2q-2}\Delta v - 2\int_\Omega u\nabla v\cdot\nabla(|\nabla v|^{2q-2})\\
&= -\frac{4(q-1)}{q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2 + \int_{\partial\Omega}|\nabla v|^{2q-2}\frac{\partial|\nabla v|^2}{\partial\nu} - 2\int_\Omega |\nabla v|^{2q}\\
&\quad - 2\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 - 2\int_\Omega u|\nabla v|^{2q-2}\Delta v - 2\int_\Omega u\nabla v\cdot\nabla(|\nabla v|^{2q-2}) \qquad (17)
\end{aligned}
$$

for all $t\in(0,T_{\max})$. Due to $|\Delta v|\le\sqrt{N}|D^2v|$, we have by Young's inequality that

$$
\begin{aligned}
-2\int_\Omega u|\nabla v|^{2q-2}\Delta v &\le 2\sqrt{N}\int_\Omega u|\nabla v|^{2q-2}|D^2v|\\
&\le \int_\Omega |\nabla v|^{2q-2}|D^2v|^2 + 4N\int_\Omega u^2|\nabla v|^{2q-2} \qquad (18)
\end{aligned}
$$

for all $t\in(0,T_{\max})$. Furthermore, with the help of Cauchy-Schwarz's inequality,

$$
\begin{aligned}
-2\int_\Omega u\nabla v\cdot\nabla(|\nabla v|^{2q-2}) &= -2(q-1)\int_\Omega u|\nabla v|^{2q-4}\nabla v\cdot\nabla(|\nabla v|^2)\\
&\le \frac{q-1}{2}\int_\Omega |\nabla v|^{2q-4}\big|\nabla|\nabla v|^2\big|^2 + 2(q-1)\int_\Omega u^2|\nabla v|^{2q-2}\\
&= -\frac{2(q-1)}{q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2 + 2(q-1)\int_\Omega u^2|\nabla v|^{2q-2}. \qquad (19)
\end{aligned}
$$

It is known from (3.10) of [6] (also in [12]) that the following inequality

$$
\int_{\partial\Omega}|\nabla v|^{2q-2}\frac{\partial|\nabla v|^2}{\partial\nu} \le C\big\|\nabla|\nabla v|^q\big\|_{L^2(\Omega)}^{2a} + C \qquad (20)
$$

is true with $a := \frac{\frac{1}{2}-\frac{1}{2N}-\frac{q}{2}-\frac{r}{N}}{\frac{1}{2}-\frac{1}{N}-\frac{q}{2}}\in(0,1)$, $r\in(0,\frac{1}{2})$ and some $C>0$. Now, combining (17)–(20), we obtain by Young's inequality that

$$
\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + \frac{2(q-1)}{q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2 + 2\int_\Omega |\nabla v|^{2q} + \int_\Omega |\nabla v|^{2q-2}|D^2v|^2\\
&\le C\int_\Omega (u+1)^2|\nabla v|^{2q-2} + C.
\end{aligned}
$$

The proof is complete. □

In addition, we need a direct consequence of Young's inequality:

Lemma 3.3 *Let $\beta,\gamma>0$ with $\beta+\gamma<1$. Then for all $\varepsilon>0$, there exists $c>0$ such that*

$$a^\beta b^\gamma \le \varepsilon(a+b)+c \quad \text{for all } a,b\ge 0.$$

Proof of Theorem 1. Throughout this proof, denote by C_i positive constants depending on some of α, N, $|\Omega|$, K, K_1, K_2, u_0 and v_0, $i=1,\dots,14$. By adding (13) to (16), we know

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega (u+1)^p+\frac{1}{q}\int_\Omega |\nabla v|^{2q}\Big)\\&\quad+\frac{2p(p-1)K_1}{(p+m)^2}\int_\Omega \big|\nabla (u+1)^{\frac{p+m}{2}}\big|^2+\frac{2(q-1)}{q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2\\&\le C_1\int_\Omega (u+1)^{p+2\alpha-m-2}|\nabla v|^2+C_1\int_\Omega (u+1)^2|\nabla v|^{2q-2}+C_1.\end{aligned}\tag{21}$$

Let $q>q_0:=\max\{8N,\frac{8N(1-\alpha+m)-(4N-8)}{(N+2)(1-\alpha+m)-(N-2)}\}$. The condition $0<\alpha-m<\frac{4}{N+2}$ with $N\ge 3$ implies

$$\begin{aligned}\frac{2(1-\alpha+m)}{N-2}(Nq-N+2)+1-\frac{2}{N}-m&>\frac{2q-8}{q(N+2)-8N}(Nq-N+2)+1-\frac{2}{N}-m\\&\ge \max\{3-m,\frac{q(N-2)}{4N}-m\}.\end{aligned}\tag{22}$$

Take $p\in(p_0(q),p_1(q))$ with

$$\begin{aligned}p_0(q)&:=\frac{2q-8}{q(N+2)-8N}(Nq-N+2)+1-\frac{2}{N}-m,\\p_1(q)&:=\frac{2(1-\alpha+m)}{N-2}(Nq-N+2)+1-\frac{2}{N}-m.\end{aligned}$$

Obviously, $p_0(q),p_1(q)\to+\infty$ as $q\to+\infty$. By the Hölder inequality,

$$\int_\Omega (u+1)^{p+2\alpha-m-2}|\nabla v|^2\le\Big(\int_\Omega (u+1)^{\frac{N}{N-2}(p+2\alpha-m-2)}\Big)^{\frac{N-2}{N}}\Big(\int_\Omega |\nabla v|^N\Big)^{\frac{2}{N}}.\tag{23}$$

Noticing

$$\frac{2}{p+m}\le\frac{2}{p+m}\frac{N}{N-2}(p+2\alpha-m-2)\le\frac{2N}{N-2}$$

due to $N\ge3$ and $\alpha-m\in(0,1)$, we obtain by the Gagliardo-Nirenberg inequality that

$$\begin{aligned}\Big(\int_\Omega (u+1)^{\frac{N}{N-2}(p+2\alpha-m-2)}\Big)^{\frac{N-2}{N}}&=\big\|(u+1)^{\frac{p+m}{2}}\big\|^{\frac{2(p+2\alpha-m-2)}{p+m}}_{L^{\frac{2}{p+m}\frac{N}{N-2}(p+2\alpha-m-2)}(\Omega)}\\\le C_2\big\|\nabla (u+1)^{\frac{p+m}{2}}\big\|^{\frac{2(p+2\alpha-m-2)}{p+m}a}_{L^2(\Omega)}&\big\|(u+1)^{\frac{p+m}{2}}\big\|^{\frac{2(p+2\alpha-m-2)}{p+m}(1-a)}_{L^{\frac{2}{p+m}}(\Omega)}\\+C_2\big\|(u+1)^{\frac{p+m}{2}}\big\|^{\frac{2(p+2\alpha-m-2)}{p+m}}_{L^{\frac{2}{p+m}}(\Omega)}&\\\le C_3\Big(\big\|\nabla (u+1)^{\frac{p+m}{2}}\big\|^{\frac{2(p+2\alpha-m-2)}{p+m}a}_{L^2(\Omega)}+1\Big),&\end{aligned}\tag{24}$$

with

$$a:=\frac{\frac{N}{2}(p+m)\Big(1-\frac{1}{\frac{N}{N-2}(p+2\alpha-m-2)}\Big)}{1-\frac{N}{2}+\frac{N}{2}(p+m)}\in(0,1).$$

On the other hand, $N \geq 3$ and $q > q_0$ imply $\frac{2}{q} \leq \frac{N}{q} \leq \frac{2N}{N-2}$. By the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}
&\Big(\int_\Omega |\nabla v|^N\Big)^{\frac{2}{N}} = \big\||\nabla v|^q\big\|^{\frac{2}{q}}_{L^{\frac{N}{q}}(\Omega)} \\
&\leq C_4\Big(\big\|\nabla|\nabla v|^q\big\|^{\frac{2}{q}b}_{L^2(\Omega)}\big\||\nabla v|^q\big\|^{\frac{2}{q}(1-b)}_{L^{\frac{2}{q}}(\Omega)} + \big\||\nabla v|^q\big\|^{\frac{2}{q}(1-b)}_{L^{\frac{2}{q}}(\Omega)}\Big) \\
&\leq C_5\Big(\big\|\nabla|\nabla v|^q\big\|^{\frac{2}{q}b}_{L^2(\Omega)} + 1\Big) \qquad (25)
\end{aligned}$$

with

$$b := \frac{Nq\big(\frac{1}{2} - \frac{1}{N}\big)}{1 - \frac{N}{2} + \frac{Nq}{2}} \in (0,1).$$

Similarly, by the Hölder inequality,

$$\int_\Omega (u+1)^2|\nabla v|^{2q-2} \leq \Big(\int_\Omega (u+1)^{\frac{q}{4}}\Big)^{\frac{8}{q}}\Big(\int_\Omega |\nabla v|^{\frac{q(2q-2)}{q-8}}\Big)^{\frac{q-8}{q}}. \qquad (26)$$

Notice $\frac{2}{p+m} < \frac{q}{2(p+m)} < \frac{2N}{N-2}$ due to $q > q_0$ and $p > \frac{q(N-2)}{4N} - m$. Again by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
&\Big(\int_\Omega (u+1)^{\frac{q}{4}}\Big)^{\frac{8}{q}} = \big\|(u+1)^{\frac{p+m}{2}}\big\|^{\frac{4}{p+m}}_{L^{\frac{q}{2(p+m)}}(\Omega)} \\
&\leq C_6\Big(\big\|\nabla(u+1)^{\frac{p+m}{2}}\big\|^{\frac{4}{p+m}c}_{L^2(\Omega)}\big\|(u+1)^{\frac{p+m}{2}}\big\|^{\frac{4}{p+m}(1-c)}_{L^{\frac{2}{(p+m)}}(\Omega)} + \big\|(u+1)^{\frac{p+m}{2}}\big\|^{\frac{4}{p+m}}_{L^{\frac{2}{(p+m)}}(\Omega)}\Big) \\
&\leq C_7\Big(\big\|\nabla(u+1)^{\frac{p+m}{2}}\big\|^{\frac{4}{p+m}c}_{L^2(\Omega)} + 1\Big) \qquad (27)
\end{aligned}$$

with

$$c := \frac{\frac{N}{2}(m+p)\Big(1 - \frac{4}{q}\Big)}{1 - \frac{N}{2} + \frac{N}{2}(m+p)} \in (0,1).$$

On the other hand, we have by the Gagliardo-Nirenberg inequality for $q > q_0$ that

$$\begin{aligned}
&\Big(\int_\Omega |\nabla v|^{\frac{q(2q-2)}{q-8}}\Big)^{\frac{q-8}{q}} = \big\||\nabla v|^q\big\|^{\frac{2q-2}{q}}_{L^{\frac{2q-2}{q-8}}(\Omega)} \\
&\leq C_8\Big(\big\|\nabla|\nabla v|^q\big\|^{\frac{2q-2}{q}d}_{L^2(\Omega)}\big\||\nabla v|^q\big\|^{\frac{2q-2}{q}(1-d)}_{L^{\frac{2}{q}}(\Omega)} + \big\||\nabla v|^q\big\|^{\frac{2q-2}{q}}_{L^{\frac{2}{q}}(\Omega)}\Big) \\
&\leq C_9\Big(\big\|\nabla|\nabla v|^q\big\|^{\frac{2q-2}{q}d}_{L^2(\Omega)} + 1\Big), \qquad (28)
\end{aligned}$$

with

$$d := \frac{Nq\Big(\frac{1}{2} - \frac{q-8}{q(2q-2)}\Big)}{1 - \frac{N}{2} + \frac{Nq}{2}} \in (0,1).$$

Combining (23)–(28), we know by using Young's inequality that

$$\begin{aligned}&C_1\int_\Omega (u+1)^{p+2\alpha-m-2}|\nabla v|^2+C_1\int_\Omega (u+1)^2|\nabla v|^{2q-2}\\ \le\ & C_{10}\Big(\int_\Omega \big|\nabla (u+1)^{\frac{p+m}{2}}\big|^2\Big)^{\beta_1}\Big(\int_\Omega \big|\nabla |\nabla v|^q\big|^2\Big)^{\gamma_1}\\ &+C_{10}\Big(\int_\Omega \big|\nabla (u+1)^{\frac{p+m}{2}}\big|^2\Big)^{\beta_2}\Big(\int_\Omega \big|\nabla |\nabla v|^q\big|^2\Big)^{\gamma_2}+C_{11},\end{aligned} \tag{29}$$

where

$$\begin{aligned}\beta_1+\gamma_1&=\frac{p+2\alpha-m-2}{p+m}a+\frac{1}{q}b\\ &=\frac{\frac{N}{2}\Big(p+2\alpha-m-2-\frac{N-2}{N}\Big)}{1-\frac{N}{2}+\frac{N}{2}(p+m)}+\frac{\frac{N}{2}\big(1-\frac{2}{N}\big)}{1-\frac{N}{2}+\frac{Nq}{2}}\in(0,1)\end{aligned}$$

and

$$\begin{aligned}\beta_2+\gamma_2&=\frac{2}{p+m}c+\frac{q-1}{q}d\\ &=\frac{\frac{N}{2}\Big(2-\frac{8}{q}\Big)}{1-\frac{N}{2}+\frac{N}{2}(p+m)}+\frac{\frac{N}{2}\big(q-1-\frac{q-8}{q}\big)}{1-\frac{N}{2}+\frac{Nq}{2}}\in(0,1)\end{aligned}$$

due to the choice of p. Therefore, by Lemma 3.3 with (29), we obtain

$$\begin{aligned}&C_1\int_\Omega (u+1)^{p+2\alpha-m-2}|\nabla v|^2+C_1\int_\Omega (u+1)^2|\nabla v|^{2q-2}\\ \le\ &\frac{p(p-1)K_1}{(p+m)^2}\int_\Omega \big|\nabla (u+1)^{\frac{p+m}{2}}\big|^2+\frac{q-1}{q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2+C_{12}.\end{aligned} \tag{30}$$

Substituting (30) into (21) yields

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega (u+1)^p+\frac{1}{q}\int_\Omega |\nabla v|^{2q}\Big)\\ &\quad+\frac{2p(p-1)K_1}{(p+m)^2}\int_\Omega \big|\nabla (u+1)^{\frac{p+m}{2}}\big|^2+\frac{(q-1)}{q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2\le C_{13}.\end{aligned} \tag{31}$$

Now letting $y(t):=\int_\Omega (u+1)^p+\frac{1}{q}\int_\Omega |\nabla v|^{2q}$, we have from (31) by the Gagliardo-Nirenberg inequality that

$$\frac{d}{dt}y(t)+C_{13}y^h(t)\le C_{14}\quad\text{for all}\quad t\in(0,T_{\max})$$

with some $h>0$. Hence, an ODE comparison argument yields the boundedness of $y(t)$ for all $t\in(0,T_{\max})$. This concludes

$$\|u(\cdot,t)\|_{L^p(\Omega)}\le C\ \text{and}\ \|\nabla v(\cdot,t)\|_{L^{2q}(\Omega)}\le C. \tag{32}$$

Finally, we can use the well-known Moser-Alikakos iteration technique (see Lemma A.1 of [10]) to arrive at

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le C\quad\text{for all}\quad t\in(0,T_{\max}).$$

This completes the proof by Lemma 2.1. □

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