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Inverse nodal problem for the Sturm-Liouville operator with a weight

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Abstract. In this work, we consider the inverse nodal problem for the Sturm-Liouville problem with a weight and the jump condition at the middle point. It is shown that the dense nodes of the eigenfunctions can uniquely determine the potential on the whole interval and some parameters.

§1 Introduction

Consider the following Sturm-Liouville problem $L := L(P(x), r(x), h, H, b_1, b_2)$:

$$-y''(x) + P(x)y(x) = \lambda r(x)y(x), \quad x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi), \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \tag{2}$$

$$V(y) := y'(\pi) + Hy(\pi) = 0, \tag{3}$$

and the jump conditions

$$\begin{cases} y(\frac{\pi}{2} + 0) = b_1 y(\frac{\pi}{2} - 0), \\ y'(\frac{\pi}{2} + 0) = b_1^{-1} y'(\frac{\pi}{2} - 0) + b_2 y(\frac{\pi}{2} - 0). \end{cases} \tag{4}$$

Here λ is the spectral parameter, $P(x)$ is a real-valued function in $L^2(0, \pi)$, h, H, b_1, b_2 are real, $b_1 > 0$, $r(x) = 1$ for $0 < x < \frac{\pi}{2}$ and $r(x) = \beta^2$ for $\frac{\pi}{2} < x < \pi$ with $0 < \beta < 1$.

Generally, the boundary value problem with discontinuous conditions which are related to discontinuous material characters often appear in many aspects of natural sciences (see [1, 2, 4, 6, 10, 18]) and has been studied by many authors (see, e.g., [3, 15, 19–22, 24]).

We see that the equation (1) with the weight function $r(x)$ arises in studying interaction of electromagnetic waves with layered agent which have dielectric and magnetic characters (see [1, 2, 6, 10, 23]). Moreover, by using the classical Fourier method, one can reduce a class of

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problems for Sturm-Liouville equations on curves to the boundary value problem like the form (1)-(4) (see, e.g., [11, 23] and the references therein).

Ozkan et al. in [11] studied the so-called half-inverse problem $L(P(x), r(x), 0, 0, b_1, 0)$. Nabiev and Amirov considered a boundary value problem $L(P(x), r(x), h, H, 1, 0)$ and gave some integral representations of solutions for the equation (1) (see [10]). Shieh and Yurko in [14] studied the inverse nodal problems for $L(P(x), 1, h, H, b_1, b_2)$ and gave the uniqueness theorem and a procedure of the reconstructing the potential $P(x)$, h and H.

Inverse nodal problems consist in reconstructing operators from the given nodes of the eigenfunctions (see, e.g., [5, 7–9, 12, 13, 16, 17] and the references therein). In this paper, we consider the problem $L(P(x), r(x), h, H, b_1, b_2)$, where b_1 and b_2 are assumed to be known a priori, and prove that a set of nodal points of the eigenfunctions which is dense on $(0, \pi)$ can uniquely determine the potential $P(x)$ and h.

§2 Preliminaries

Denote $\phi(x) = \int_0^x \sqrt{r(t)}dt$, $q(x) = \frac{P(x)}{r(x)}$ and $\rho^2 = \lambda$, $\rho = \sigma + i\tau$. Let $\varphi(x, \lambda)$ be the solutions of the equation (1), satisfying the initial conditions $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = h$, and the jump condition (4).

Without loss of generality we assume that

$$\int_0^\pi \frac{P(x)}{\sqrt{r(x)}}dx = 0. \tag{5}$$

Then the following asymptotic relations hold as $|\rho| \to \infty$.

$$\begin{aligned} \varphi(x, \lambda) =& \cos(\rho\phi(x)) + (h + \frac{1}{2}\int_0^x \sqrt{r(t)}q(t)dt)\frac{\sin(\rho\phi(x))}{\rho} \\ &+ o(\frac{1}{\rho}\exp(|\tau|\phi(x))) \qquad \text{for } \ 0 < x < \frac{\pi}{2}, \end{aligned} \tag{6}$$

$$\begin{aligned} \varphi(x, \lambda) =& b^+\cos(\rho\phi(x)) + b^-\cos\rho(\pi - \phi(x)) \\ &+ \xi_1(x)\frac{\sin(\rho\phi(x))}{\rho} + \xi_2(x)\frac{\sin\rho(\pi - \phi(x))}{\rho} \\ &+ o(\frac{1}{\rho}\exp(|\tau|\phi(x))) \qquad \text{for } \ \frac{\pi}{2} < \ x < \pi, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \varphi'(x, \lambda) =& -\rho\sin(\rho\phi(x)) + (h + \frac{1}{2}\int_0^x \sqrt{r(t)}q(t)dt)\cos(\rho\phi(x)) \\ &+ o(\exp(|\tau|\phi(x))) \qquad \text{for } \ 0 < x < \frac{\pi}{2}, \end{aligned} \tag{8}$$

$$\begin{aligned} \varphi'(x, \lambda) =& \rho\beta(-b^+\sin(\rho\phi(x)) + b^-\sin\rho(\pi - \phi(x))) \\ &+ \beta\xi_1(x)\cos(\rho\phi(x)) - \beta\xi_2(x)\cos\rho(\pi - \phi(x)) \\ &+ o(\exp(|\tau|\phi(x))) \qquad \text{for } \ \frac{\pi}{2} < \ x < \pi, \end{aligned} \tag{9}$$

where $\tau = \Im\ \rho$, $b^{\pm} = \frac{1}{2}(b_1 \pm \frac{1}{\beta b_1})$ and

$$\xi_1(x) = b^+(h + \frac{1}{2}\int_0^x \sqrt{r(t)}q(t)dt) + \frac{b_2}{2\beta}, \tag{10}$$

$$\xi_2(x) = b^-(h - \frac{1}{2}\int_0^x \sqrt{r(t)}q(t)dt + \int_0^{\frac{\pi}{2}} \sqrt{r(t)}q(t)dt) - \frac{b_2}{2\beta}. \tag{11}$$

Denote $\Delta(\lambda) = V(\varphi)$. The function $\Delta(\lambda)$ is called the characteristic function of $L(P(x), r(x), h, H, b_1, b_2)$, which is entire in λ, and it has an at most countable set of zeros $\{\lambda_n\}_{n\geq 0}$. From (7) and (9), we get that as $|\rho| \to \infty$,

$$\begin{aligned}\Delta(\lambda) =&\beta\rho[b^+\sin(\rho\phi(\pi)) - b^-\sin\rho(\pi - \phi(\pi))] - \omega_1\cos(\rho\phi(\pi)) \\ &- \omega_2\cos\rho(\pi - \phi(\pi)) + o(\exp(|\tau|\phi(\pi))),\end{aligned} \tag{12}$$

where

$$\omega_1 = b^+(\beta h + H + \frac{\beta}{2}\int_0^\pi q(t)\sqrt{r(t)}dt) + \frac{b_2}{2},$$

$$\omega_2 = b^-(H - \beta h + \frac{\beta}{2}\int_0^\pi q(t)\sqrt{r(t)}dt - \beta\int_0^{\frac{\pi}{2}} q(t)\sqrt{r(t)}dt) - \frac{b_2}{2}.$$

Lemma 1. *The boundary value problem $L(P(x), r(x), h, H, b_1, b_2)$ possesses real eigenvalues $\{\lambda_n\}_{n\geq 0}$. Moreover, for $n \to +\infty$, there holds*

$$\rho_n = \rho_n^0 + \frac{\theta_n}{\rho_n^0} + \frac{\kappa_n}{\rho_n^0}, \tag{13}$$

where $\rho_n = \sqrt{\lambda_n}, \{\kappa_n\} \in l_2$, and $\rho_n^0 = \sqrt{\lambda_n^0}$ are zeros of the function

$$\Delta_0(\lambda) := \beta\rho[b^+\sin(\rho\phi(\pi)) - b^-\sin\rho(\pi - \phi(\pi))], \tag{14}$$

and

$$\theta_n = \frac{\omega_1\cos(\rho_n^0\phi(\pi)) + \omega_2\cos\rho_n^0(\pi - \phi(\pi))}{2\dot{\Delta}_0(\lambda_n^0)}, \tag{15}$$

$$\begin{aligned}\dot{\Delta}_0(\lambda_n^0) =&\Big(\frac{d}{d\lambda}\Delta_0(\lambda)\Big)\Big|_{\lambda=\lambda_n^0} = \frac{\beta}{2}\Big[b^+\phi(\pi)\cos(\rho_n^0\phi(\pi)) \\ &- b^-(\pi - \phi(\pi))\cos\rho_n^0(\pi - \phi(\pi))\Big].\end{aligned} \tag{16}$$

Proof. Using the Rouch's theorem, it follows from (12) and (16) that

$$\rho_n = \rho_n^0 + \eta_n, \qquad n \to +\infty, \tag{17}$$

here $\eta_n = o(1)$. The following is the proof for more accurate asymptotic of ρ_n. According to $\Delta(\lambda_n) = 0$ and (17), we obtain

$$b^+\sin(\rho_n\phi(\pi)) - b^-\sin\rho_n(\pi - \phi(\pi)) = O(\frac{1}{\rho_n^0}). \tag{18}$$

Note that

$$\Delta_0(\lambda_n^0) := \beta\rho_n^0(b^+\sin(\rho_n^0\phi(\pi)) - b^-\sin\rho_n^0(\pi - \phi(\pi))) = 0. \tag{19}$$

Substituting ρ_n into (18), and direct calculation yield

$$\eta_n(b^+\phi(\pi)\cos(\rho_n^0\phi(\pi)) - b^-(\pi - \sigma(\pi))\cos\rho_n^0(\pi - \phi(\pi))) = O(\frac{1}{\rho_n^0}) + O(\eta_n^2). \tag{20}$$

Combining (16) and (19), we obtain

$$\eta_n\dot{\Delta}_0(\lambda_n^0)=O(\frac{1}{\rho_n^0})+O(\eta_n^2). \tag{21}$$

Moreover, we can conclude that

$$\eta_n=O(\frac{1}{\rho_n^0}). \tag{22}$$

Substituting (12) into the formula $\Delta(\lambda_n)=0$ and using (22), we can get that $\eta_n=\frac{\theta_n}{\rho_n^0}+\frac{\kappa_n}{\rho_n^0}$. This completes the proof. □

§3 Main results

Together with the problem $L:=L(P(x),r(x),h,H)$ we consider a boundary value problem $\tilde{L}=L(\tilde{P}(x),\tilde{r}(x),\tilde{h},\tilde{H})$ of the same form but with the different coefficients $\tilde{P}(x),\tilde{r}(x),\tilde{h},\tilde{H}$. We agree that if a certain symbol υ denotes an object related to L, then $\tilde{\upsilon}$ denote the analogous object related to $\tilde{L}$. Note that $\varphi(x,\lambda_n)=\varphi_n(x)$ are real-valued functions.

Lemma 2. *the following asymptotic formulaes hold as $n\to+\infty$.*

$$\begin{aligned}\varphi_n(x)=&\cos(\rho_n^0x)+\frac{1}{2\rho_n^0}(2h-2\theta_nx+\int_0^x\sqrt{r(t)}q(t)dt)\sin(\rho_n^0x)\\&+o(\frac{1}{\rho_n^0})\qquad\qquad \textit{for}\ \ 0<x<\frac{\pi}{2},\end{aligned} \tag{23}$$

$$\begin{aligned}\varphi_n(x)=&[b^++b^-\cos(\rho_n^0\pi)]\cos(\rho_n^0\phi(x))+\frac{1}{\rho_n^0}[\xi_1(x)-\cos(\rho_n^0\pi)\xi_2(x)\\&-\theta_n(b^+\phi(x)-b^-(\pi-\phi(x))\cos(\rho_n^0\pi))]\sin(\rho_n^0\phi(x))\\&+o(\frac{1}{\rho_n^0})\qquad\qquad \textit{for}\ \ \frac{\pi}{2}<x<\pi,\end{aligned} \tag{24}$$

where $\xi_1(x)$ and $\xi_2(x)$ are given as (10) and (11).

Proof. Substituting (13) into (6) and (7), we can get (23) and (24). □

Refer to [11], we get $\rho_n^0=\frac{2n}{1+\beta}[1+o(1)]$, then $[\frac{1+\beta}{2}\rho_n^0]=n$.

Lemma 3. *For sufficiently large n, the eigenfunction $\varphi_n(x)$ has exactly n nodes $\{x_n^j:j=\overline{1,n}\}$ in the interval $(0,\pi)$, which are simple.*

Proof. Firstly, we consider the case $x_n^j\in(0,\frac{\pi}{2})$. It follows from (23) that the zeros of $\varphi_n(x)$ is

$$x_n^j=(j-\frac{1}{2})\frac{\pi}{\rho_n^0}+o(1). \tag{25}$$

Since $0<x_n^j<\frac{\pi}{2}$, we can get $j=\overline{1,[\frac{\rho_n^0}{2}]}$.

When $x_n^j\in(\frac{\pi}{2},\pi)$, it follows from (24) that

$$x_n^j=(j-\frac{1}{2})\frac{\pi}{\rho_n^0\beta}-\frac{\pi}{2\beta}+\frac{\pi}{2}+o(1). \tag{26}$$

Using the similar method, we can get that $j=\overline{[\frac{\rho_n^0}{2}]+1,[\frac{1+\beta}{2}\rho_n^0]}$. In conclusion, we can get the results. □

Denote by $X_L := \{x_n^j\}_{n\geq 1, j=\overline{1,n}}$ the set of nodal points of the boundary value problem L, $\alpha_n^j := \frac{(j-\frac{1}{2})\pi}{\rho_n^0}$ and $\chi_n^j := \frac{(j-\frac{1}{2})\pi}{\beta\rho_n^0} + (1-\frac{1}{\beta})\frac{\pi}{2}$.

Lemma 4. *As $n \to +\infty$, the following asymptotic formulas for nodes hold:*

$$x_n^j = \alpha_n^j + \frac{1}{2(\rho_n^0)^2}[2h - 2\theta_n\alpha_n^j + \int_0^{\alpha_n^j}\sqrt{r(t)}q(t)dt] + o(\frac{1}{(\rho_n^0)^2}) \quad \text{for } 0 < x < \frac{\pi}{2} \text{ and } j = \overline{1, [\frac{\rho_n^0}{2}]}, \tag{27}$$

and

$$x_n^j = \chi_n^j + \frac{1}{\beta(\rho_n^0)^2}\frac{e_n + a_n + c_n}{A_n} + o(\frac{1}{(\rho_n^0)^2}) \quad \text{for } \frac{\pi}{2} < x < \pi \text{ and } j = \overline{[\frac{\rho_n^0}{2}] + 1, n}, \tag{28}$$

where

$$\begin{aligned} e_n =& (\frac{b^+}{2} - \frac{b^-}{2}\cos(\rho_n^0\phi(\chi_n^j)) + \frac{b^-}{2}\cos(\rho_n^0\pi))\int_0^{\chi_n^j}\sqrt{r(t)}q(t)dt, \\ a_n =& b^-h\cos(\rho_n^0\phi(\chi_n^j)) + b^-\cos(\rho_n^0\phi(\chi_n^j))\int_0^{\frac{\pi}{2}}\sqrt{r(t)}q(t)dt \\ &- \frac{b_2}{2\beta}\cos(\rho_n^0\phi(\chi_n^j)) - b^+\beta\theta_n\chi_n^j - b^-\beta\theta_n\chi_n^j\cos(\rho_n^0\pi), \end{aligned} \tag{29}$$

$$\begin{aligned} c_n =& b^+h + \frac{b_2}{2\beta} - b^-h\cos(\rho_n^0\pi) - b^-\cos(\rho_n^0\pi)\int_0^{\frac{\pi}{2}}\sqrt{r(t)}q(t)dt \\ &+ \frac{b_2}{2\beta}\cos(\rho_n^0\pi) - \frac{\pi}{2}b^+\theta_n + \frac{\pi}{2}\beta b^+\theta_n + \theta_n b^-\phi(\pi)\cos(\rho_n^0\pi), \end{aligned} \tag{30}$$

and

$$A_n = b^+ + b^-\cos(\rho_n^0\pi). \tag{31}$$

Proof. Taking (23) and (24) into account, we can obtain the results. □

Noting that X_L is dense on $(0,\pi)$, and θ_n is bounded. It follows from these formulae that the following results are valid.

Theorem 1. *Fix $x \in (0,\pi)$. There exist a convergent subsequences $\{\theta_{n_k}\}$ of $\{\theta_n\}$ satisfying $\lim_{k\to\infty}\theta_{n_k} = \theta_0$ and a subsequences $\{x_{n_k}^{j_{n_k}}\} \in X_L$ such that $\lim_{k\to\infty} x_{n_k}^{j_{n_k}} = x$. Then the following finite limit exists:*

$$\lim_{k\to\infty}\frac{2(a_{n_k} + c_{n_k})}{(b^+ - b^-\cos(\rho_{n_k}^0\phi(\chi_{n_k}^{j_{n_k}})) + b^-\cos(\rho_{n_k}^0\pi))} \overset{def}{=} f(x),$$

$$A(x) := \lim_{k\to\infty} 2(\rho_{n_k}^0)^2(x_{n_k}^{j_{n_k}} - \alpha_{n_k}^{j_{n_k}}) \quad \text{for } 0 < x < \frac{\pi}{2}, \tag{32}$$

and

$$B(x) := \lim_{k\to\infty}\frac{2\beta(\rho_{n_k}^0)^2(x_{n_k}^{j_{n_k}} - \chi_{n_k}^{j_{n_k}})A_{n_k}}{(b^+ - b^-\cos(\rho_{n_k}^0\phi(\chi_{n_k}^{j_{n_k}})) + b^-\cos(\rho_{n_k}^0\pi))} \quad \text{for } \frac{\pi}{2} < x < \pi. \tag{33}$$

Moreover,

$$A(x) := \int_0^x \sqrt{r(t)}q(t)dt - 2\theta_0 x + 2h \qquad \text{for } 0 < x < \frac{\pi}{2}, \tag{34}$$

$$B(x) := \int_0^x \sqrt{r(t)}q(t)dt + f(x) \qquad \text{for } \frac{\pi}{2} < x < \pi, \tag{35}$$

where θ_0 *and* $f(x)$ *are above-mentioned.*

Next, we provide a uniqueness theorem and a constructive procedure for the solution of the inverse nodal problem.

Theorem 2. *Let* $X \subset X_L$ *be a subset of nodes which is dense on* $(0,\pi)$*. If* $X = \tilde{X}$*, then* $q(x) = \tilde{q}(x)$ *a.e. on* $(0,\pi)$*,* $\beta = \tilde{\beta}$ *and* $h = \tilde{h}$*. The parameters* β*,* h *and the function* $q(x)$ *can be reconstructed by the following formulae:*

$$(i)\ \beta = \lim_{n\to\infty} \frac{2nx_n^j}{(j-\frac{1}{2})\pi} - 1;$$

$$(ii)\ h = \frac{A(0)}{2}; \tag{36}$$

$$(iii)\ H = \frac{C_6}{b^+C_4 + b^-C_5} \quad (\text{when } b^+C_4 + b^-C_5 \neq 0); \tag{37}$$

$$(iv)\ q(x) = A'(x) + \frac{2(K-d)}{\pi} \qquad \text{for } 0 < x < \frac{\pi}{2}; \tag{38}$$

$$(v)\ q(x) = \frac{1}{\beta}[B'(x) - f'(x)] \qquad \text{for } \frac{\pi}{2} < x < \pi, \tag{39}$$

where $d = A(\frac{\pi}{2}) - A(0) + B(\pi) - B(\frac{\pi}{2})$*,* $K = f(\pi) - f(\frac{\pi}{2})$ *and the representations of* C_4*,* C_5 *and* C_6 *are given in the following proof.*

Proof. Refer to [11], we can get that $\rho_n^0 = \frac{2n}{1+\beta}(1+o(1))$. Note that for $j = 1, 2, \cdots, [\rho_n^0/2]$

$$\begin{aligned} x_n^j =& \alpha_n^j + O(\frac{1}{(\rho_n^0)^2}) = \frac{(j-\frac{1}{2})\pi(1+\beta)}{2n}(1+o(1)) + O(\frac{1}{n^2}) \\ =& \frac{(j-\frac{1}{2})\pi(1+\beta)}{2n} + o(1), \end{aligned}$$

then $x_n^j \frac{2n}{(j-\frac{1}{2})\pi} = 1 + \beta + o(1)$. Fix j, it yields

$$\lim_{n\to\infty}(x_n^j \frac{2n}{(j-\frac{1}{2})\pi}) = 1 + \beta, \tag{40}$$

i.e.,

$$\beta = \lim_{n\to\infty} \frac{2nx_n^j}{(j-\frac{1}{2})\pi} - 1. \tag{41}$$

Then, we can get the representation of $r(x)$ from (41).

Letting $x = 0$, it follows from (34) that $A(0) = 2h$, i.e. $h = \frac{A(0)}{2}$. Substituting $x = \frac{\pi}{2}$ into (34), we can get that $A(\frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \sqrt{r(t)}q(t)dt - \pi\theta_0 + 2h$. Deriving (34), we have

$$\sqrt{r(x)}q(x) = A'(x) + 2\theta_0 \qquad \text{for } 0 < x < \frac{\pi}{2}. \tag{42}$$

Similarly, letting $x = \frac{\pi}{2}$, it follows from (35) that $B(\frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \sqrt{r(t)}q(t)dt + f(\frac{\pi}{2})$. Letting

$x=\pi$, we can get that $B(\pi)=\int_0^\pi \sqrt{r(t)}q(t)dt+f(\pi)$. Deriving (35), we have

$$\sqrt{r(x)}q(x)=B'(x)-f'(x) \qquad \text{for } \frac{\pi}{2}<x<\pi. \tag{43}$$

From above formulaes we can get that

$$A(\frac{\pi}{2})-A(0)=\int_0^{\frac{\pi}{2}} \sqrt{r(t)}q(t)dt-\pi\theta_0, \tag{44}$$

$$B(\pi)-B(\frac{\pi}{2})=\int_{\frac{\pi}{2}}^{\pi} \sqrt{r(t)}q(t)dt+K. \tag{45}$$

Then

$$d=A(\frac{\pi}{2})-A(0)+B(\pi)-B(\frac{\pi}{2})=\int_0^{\pi} \sqrt{r(t)}q(t)dt-\pi\theta_0+K. \tag{46}$$

Taking $\int_0^\pi \sqrt{r(t)}q(t)dt=0$ into account, it follows that

$$\theta_0=\frac{K-d}{\pi}. \tag{47}$$

So, from (42), we have

$$\sqrt{r(x)}q(x)=A'(x)+\frac{2(K-d)}{\pi} \qquad \text{for } 0<x<\frac{\pi}{2},$$

that is,

$$q(x)=A'(x)+\frac{2(K-d)}{\pi} \qquad \text{for } 0<x<\frac{\pi}{2}. \tag{48}$$

Similarly, it follows from (43) that

$$q(x)=\frac{B'(x)}{\beta}-\frac{f'(x)}{\beta} \qquad \text{for } \frac{\pi}{2}<x<\pi. \tag{49}$$

Next, we will calculate H. From (35), we know that $B(\pi)=f(\pi)$. In virtue of Theorem 1, there exists

$$\lim_{k\to\infty}\frac{2(a_{n_k}+c_{n_k})}{(b^+-b^-\cos(\rho^0_{n_k}\phi(\chi^{j_{n_k}}_{n_k}))+b^-\cos(\rho^0_{n_k}\pi))}=f(x).$$

Because

$$\lim_{k\to\infty}\frac{1}{(b^+-b^-\cos(\rho^0_{n_k}\phi(\chi^{j_{n_k}}_{n_k}))+b^-\cos(\rho^0_{n_k}\pi))}\overset{def}{=}g(x),$$

we can get that

$$\lim_{k\to\infty}(a_{n_k}+c_{n_k})=\frac{f(x)g(x)}{2}. \tag{50}$$

Combining the expressions of a_{n_k} and c_{n_k} as (29) and (30), and $\lim\limits_{k\to\infty}\theta_{n_k}=\theta_0$, we can get that

$$\lim_{k\to\infty}a_{n_k}=C_1(x)-b^+\beta\theta_0x-b^-\beta\theta_0x\cos(\rho^0_{n_k}\pi) \tag{51}$$

and

$$\lim_{k\to\infty}c_{n_k}=C_2(x)-\frac{\pi}{2}b^+\theta_0+\frac{\pi}{2}\beta b^+\theta_0+\theta_0b^-\phi(\pi)\cos(\rho^0_{n_k}\pi), \tag{52}$$

where

$$C_1(x)=\lim_{k\to\infty}\Big(b^-h\cos(\rho^0_{n_k}\phi(\chi^j_{n_k}))+b^-\cos(\rho^0_{n_k}\phi(\chi^j_{n_k}))\int_0^{\frac{\pi}{2}}\sqrt{r(t)}q(t)dt$$
$$-\frac{b_2}{2\beta}\cos(\rho^0_{n_k}\phi(\chi^j_{n_k}))\Big)$$

and

$$C_2(x) = \lim_{k\to\infty}\Big(b^+h + \frac{b_2}{2\beta} - b^- h\cos(\rho^0_{n_k}\pi) - b^-\cos(\rho^0_{n_k}\pi)\int_0^{\frac{\pi}{2}}\sqrt{r(t)}q(t)dt + \frac{b_2}{2\beta}\cos(\rho^0_{n_k}\pi)\Big).$$

Substituting $x=\pi$ into (7), we can obtain that

$$\frac{f(\pi)g(\pi)}{2} = C_1(\pi) + C_2(\pi) - \Big(b^+\beta\pi + b^-\beta\pi\cos(\rho^0_{n_k}\pi) + \frac{\pi}{2}b^+ - \frac{\pi}{2}\beta b^+ - b^-\phi(\pi)\cos(\rho^0_{n_k}\pi)\Big)\theta_0.$$

So,

$$\theta_0 = \frac{C_1(\pi) + C_2(\pi) - \frac{f(\pi)g(\pi)}{2}}{\Big(b^+\beta\pi + b^-\beta\pi\cos(\rho^0_{n_k}\pi) + \frac{\pi}{2}b^+ - \frac{\pi}{2}\beta b^+ - b^-\phi(\pi)\cos(\rho^0_{n_k}\pi)\Big)}.$$

Considering $\lim\limits_{k\to\infty}\theta_{n_k} = \theta_0$, that is,

$$\lim_{k\to\infty}\frac{\omega_1\cos(\rho^0_{n_k}\phi(\pi)) + \omega_2\cos\rho^0_{n_k}(\pi-\phi(\pi))}{\beta\Big[b^+\phi(\pi)\cos(\rho^0_{n_k}\phi(\pi)) - b^-(\pi-\phi(\pi))\cos\rho^0_{n_k}(\pi-\phi(\pi))\Big]} = \theta_0. \tag{53}$$

Let

$$\lim_{k\to\infty}\frac{1}{\beta\Big[b^+\phi(\pi)\cos(\rho^0_{n_k}\phi(\pi)) - b^-(\pi-\phi(\pi))\cos\rho^0_{n_k}(\pi-\phi(\pi))\Big]} \overset{def}{=} C_3.$$

Combining the expression of ω_1 and ω_2 in (12) and substituting them into (53), we can arrive at

$$\lim_{k\to\infty}\Big(\Big[b^+(\beta h + H) + \frac{b_2}{2}\Big]\cos(\rho^0_{n_k}\phi(\pi)) + \Big[b^-(H - \beta h - \beta\int_0^{\frac{\pi}{2}}q(t)\sqrt{r(t)}dt) - \frac{b_2}{2}\Big]\cos(\rho^0_{n_k}(\pi-\phi(\pi)))\Big) = C_3\theta_0. \tag{54}$$

Then,

$$\Big[b^+(\beta h + H) + \frac{b_2}{2}\Big]C_4 + \Big[b^-(H - \beta h - \beta\int_0^{\frac{\pi}{2}}q(t)\sqrt{r(t)}dt) - \frac{b_2}{2}\Big]C_5 = C_3\theta_0,$$

where

$$C_4 = \lim_{k\to\infty}\cos(\rho^0_{n_k}\phi(\pi)), \qquad C_5 = \lim_{k\to\infty}\cos(\rho^0_{n_k}(\pi-\phi(\pi))).$$

Thus, when $b^+C_4 + b^-C_5 \neq 0$, we can get that

$$H = \frac{C_6}{b^+C_4 + b^-C_5},$$

where

$$C_6 = C_3\theta_0 - (b^+\beta h + \frac{b_2}{2})C_4 + (b^-\beta h + b^-\beta\int_0^{\frac{\pi}{2}}q(t)\sqrt{r(t)}dt + \frac{b_2}{2})C_5.$$

If $X = \tilde{X}$, from (32), (33) and (36), we have

$$A(x) = \tilde{A}(x), \quad B(x) = \tilde{B}(x), \quad h = \tilde{h}. \tag{55}$$

According to (55) and $\beta = \tilde{\beta}$, we can get that

$$f(x) = \tilde{f}(x), \quad g(x) = \tilde{g}(x). \tag{56}$$

It follows from (41)-(56), we get the results. This completes the proof. □

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