Appl. Math. J. Chinese Univ.
2020, 35(2): 184-192

On the convergence for PNQD sequences with general moment conditions

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Abstract. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed pairwise negative quadrant dependent (PNQD) random variables and $\{a_n, n \geq 1\}$ be a sequence of positive constants with $a_n = f(n)$ and $f(\theta^k)/f(\theta^{k-1}) \geq \beta$ for all large positive integers k, where $1 < \theta \leq \beta$ and $f(x) > 0\,(x \geq 1)$ is a non-decreasing function on $[b, +\infty)$ for some $b \geq 1$. In this paper, we obtain the strong law of large numbers and complete convergence for the sequence $\{X, X_n, n \geq 1\}$, which are equivalent to the general moment condition $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$. Our results extend and improve the related known works in Baum and Katz [1], Chen at al. [3], and Sung [14].

§1 Introduction

Two random variables X and Y are said to be negative quadrant dependent (NQD) if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y),\ \forall\, x, y \in (-\infty, \infty).$$

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise negative quadrant dependent (PNQD) if every two random variables in the sequence is NQD. This definition was introduced by Lehmann [8]. Obviously, PNQD sequence includes many dependent random variables sequences, such as extended negatively dependent (END) random sequence and pairwise independent random sequence are the most special case.

It is more important for PNQD random variables since they have wide applications in mathematics and mechanic models, percolation theory, and reliability theory. For these reasons many authors have more and more interest in the study of PNQD and have established a series of useful results. Please refer to [2], [9-10], [12], [15-20], and so on.

Complete convergence is one of the most important problems in probability theory. The concept was introduced by Hsu and Robbins [6]. From then on, many authors have devoted

Received: 2016-07-06. Revised: 2019-09-22.
MR Subject Classification: 60F15.
Keywords: pairwise negative quadrant dependent (PNQD) random variable, strong law of large numbers, complete convergence, general moment condition.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-020-3480-0.
Supported by the National Natural Science Foundation of China (No. 11271161).
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their study to complete convergence, One can refer to [1], [3-5], [7], [13-14], [16-18], [20], and so forth.

It is also interesting to find the more generalized moment conditions such that the complete convergence holds. In fact, Gut and Stadtmüller [5] and Lanzinger [7] extended the Baum-Katz theorem under higher order moment conditions, Sung [14] obtained the complete convergence for pairwise independent random variables under some generalized moment conditions, Chen et al. [4] obtained an extension of the Baum-Katz theorem to i.i.d. random variables with general moment conditions, and so on. It is worth pointing out that Chen et al. [3] obtained the following result:

Theorem A. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed PNQD random variables with partial sums $S_n = \sum_{k=1}^n X_k$, $n \geq 1$, and $\{a_n, n \geq 1\}$ a sequence of real numbers with $0 < a_n/n \uparrow$. Then the following statements are equivalent:

$$\sum_{n=1}^{\infty} P(|X| > a_n) < \infty, \tag{1.1}$$

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq m\leq n} |S_m - mEXI(|X| \leq a_n)| > \varepsilon a_n) < \infty, \quad \forall\, \varepsilon > 0, \tag{1.2}$$

$$a_n^{-1} \sum_{k=1}^{n} (X_k - EX_k I(|X_k| \leq a_k)) \to 0 \ \text{ a.s.} \tag{1.3}$$

The goal of this paper is to extend and improve Theorem A to identically distributed PNQD random variables under the generalized condition (1.4).

In the following, let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $f(x) > 0$ $(x \geq 1)$ be a non-decreasing function on $[b, +\infty)$, where $b \geq 1, a_n = f(n), n \in \mathbb{N}$.

Now we state the main results. Some lemmas and the proofs of the main results will be detailed in the next section.

Theorem 1.1. Let $1 < \theta \leq \beta$ and $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of identically distributed PNQD random variables with partial sums $S_n = \sum_{k=1}^n X_k, n \in \mathbb{N}$. If there exists some positive integer number M_0 such that

$$f(\theta^k)/f(\theta^{k-1}) \geq \beta, \quad \forall k \geq [M_0, \infty) \cap \mathbb{N}. \tag{1.4}$$

Then (1.1) $\sim$ (1.3) are equivalent.

Corollary 1.2. Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of identically distributed PNQD random variables with partial sums $S_n = \sum_{k=1}^n X_k, n \in \mathbb{N}$, and $\{a_n, n \in \mathbb{N}\}$ a sequence of real numbers with $0 < a_n/n \uparrow \infty$. Then (1.1) and the following statements are equivalent:

$$a_n^{-1} S_n \to 0 \ \ a.s.$$

$$a_n^{-1} \sum_{i=1}^{n} |X_i| \to 0 \ \ a.s.$$

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq m\leq n} |S_m| > \varepsilon a_n\right) < \infty, \quad \forall\, \varepsilon > 0.$$

Remark 1.1. When $0 < a_n/n \uparrow$, let $f(x) = a_n, x \in [n, n+1), n \geq 1$, and $\theta = \beta = 2$, then (1.4) holds. Therefore, Theorem A is obtained by Theorem 1.1.

Remark 1.2. Let $f(x) = 2^{n-1}, x \in [2^{n-1}, 2^n), n \in \mathbb{N}$. Thus, $f(x)$ is a non-decreasing function on $[1, \infty)$. Obviously, the condition $a_n/n \uparrow$ does not hold, hence, the results of Theorem A can't be obtained. But, Let $\theta = 2, \beta = \theta$, then, (1.4) holds and the results of Theorem A are obtained by Theorem 1.1.

Remark 1.3. When Corollary 1.2 is compared with Theorem 2.5 of Sung [14], the condition $a_{2n}/a_n = O(1)$ in Theorem 2.5 is removed.

Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance, $I(A)$ denotes the indicator function of the event A, $[x]$ denotes the integer part of x.

§2 Lemmas and Proofs

In this section, we prove the main results. To do this, the following lemmas are needed.

Lemma 2.1. Let $\theta > 1$ and X be a random variable. Then (1.1) is equivalent to

$$\sum_{j=1}^{\infty} \theta^j P(|X| > a_{[\theta^j]}) < \infty. \tag{2.1}$$

Proof. Without loss of generality, we assume that $f(x)$ is a non-decreasing function on $[1, \infty)$. First we prove that (1.1)⇒(2.1). Since $\theta > 1$, there exists a positive integer M_1 such that $\theta^{n+1} - \theta^n > 4$ for all $n \geq M_1$, so that we have

$$\begin{aligned}\sum_{n=1}^{\infty} P(|X| > a_n) &\geq \sum_{j=M_1+1}^{\infty} \sum_{n \in [\theta^{j-1}, \theta^j) \bigcap \mathbb{N}} P(|X| > a_n) \\ &\geq 2^{-1}(\theta - 1)\theta^{-1} \sum_{j=M_1+1}^{\infty} \theta^j P(|X| > a_{[\theta^j]}).\end{aligned} \tag{2.2}$$

Hence (2.1) holds.

The proof that (2.1)⇒(1.1) is essentially the same as that given in (2.2),

$$\sum_{n \in [\theta^{M_1}, \infty) \bigcap \mathbb{N}} P(|X| > a_n) \leq \sum_{j=M_1+1}^{\infty} \theta^j P(|X| > a_{[\theta^{j-1}]}) = \theta \sum_{j=M_1}^{\infty} \theta^j P(|X| > a_{[\theta^j]}).$$

Therefore, (1.1) holds by (2.1).

Lemma 2.2. Let $1 < \theta \leq \beta, 0 < \delta < 1, X$ be a random variable, and (1.1) and (1.4) hold, then

$$\sup_{n \geq 1} \frac{n}{a_n} E|X| I(|X| \leq a_n) < \infty, \tag{2.3}$$

and

$$\lim_{n \to \infty} \frac{n}{a_n} E|X| I(a_{[n^\delta]} < |X| \leq a_n) = 0. \tag{2.4}$$

Proof. For any $j > k \geq M_0$, by (1.4) and $f(x)$ is a non-decreasing function on $[1, \infty)$, we have

$$\frac{a_{[\theta^k]}}{a_{[\theta^j]}} = \frac{f([\theta^k])}{f([\theta^j])} \leq \frac{f(\theta^k)}{f(\theta^{j-1})} \leq \beta^{-(j-k-1)}. \tag{2.5}$$

For $j \geq M_0+2$, we have by $1<\theta \leq \beta$ and (2.5) that

$$\begin{aligned}
\frac{\theta^j}{a_{[\theta^j]}}E|X|I(|X|\leq a_{[\theta^j]}) &= \frac{\theta^j}{a_{[\theta^j]}}\left(\sum_{k=M_0+1}^{j} E|X|I(a_{[\theta^{k-1}]}<|X|\leq a_{[\theta^k]}) + E|X|I(|X|\leq a_{[\theta^{M_0}]}\right) \\
&\leq \frac{\theta^j}{a_{[\theta^j]}}\left(\sum_{k=M_0+1}^{j} a_{[\theta^k]}EI(a_{[\theta^{k-1}]}<|X|\leq a_{[\theta^k]}) + a_{[\theta^{M_0}]}\right) \\
&\leq \beta\sum_{k=M_0}^{j}\left(\frac{\theta}{\beta}\right)^{j-k}\theta^k EI(a_{[\theta^{k-1}]}<|X|\leq a_{[\theta^k]}) + \beta\theta^{M_0}\left(\frac{\theta}{\beta}\right)^{j-M_0} \\
&\leq \beta\theta\sum_{k=1}^{\infty}\theta^k P(|X|>a_{[\theta^k]}) + \beta\theta^{M_0}. \qquad (2.6)
\end{aligned}$$

For any $n \in [\theta^{M_0+2},\infty)\cap\mathbb{N}$, there exists a corresponding positive integer number j such that $\theta^j \leq n < \theta^{j+1}$. Hence, we have

$$\begin{aligned}
\frac{n}{a_n}E|X|I(|X|\leq a_n) &= \frac{n}{a_n}\left\{E|X|I(|X|\leq a_{[\theta^j]}) + E|X|I(a_{[\theta^j]}<|X|\leq a_n)\right\} \\
&\leq \frac{\theta^{j+1}}{a_{[\theta^j]}}E|X|I(|X|\leq a_{[\theta^j]}) + \theta^{j+1}P(|X|>a_{[\theta^j]}).
\end{aligned}$$

Thus, (2.3) holds by (2.6) and Lemma 2.1. For any j large enough such that $[j\delta]\geq M_0+2$, by $1<\theta\leq\beta$ and similar to the proof of (2.6), we have that

$$\frac{\theta^j}{a_{[\theta^j]}}E|X|I(a_{[\theta^{[j\delta]}]}<|X|\leq a_{[\theta^j]}) \leq \beta\sum_{k=[j\delta]+1}^{j}\theta^k P(|X|>a_{[\theta^{k-1}]}).$$

Hence, for any positive integer number n such that $\theta^j \leq n < \theta^{j+1}$ for some $j \in N$ and $[j\delta]\geq M_0+2$, we have that

$$\begin{aligned}
\frac{n}{a_n}E|X|I(a_{[n^\delta]}<|X|\leq a_n) &= \frac{n}{a_n}\left\{E|X|I(a_{[n^\delta]}<X\leq a_{[\theta^j]}) + E|X|I(a_{[\theta^j]}<|X|\leq a_n)\right\} \\
&\leq \frac{\theta^{j+1}}{a_{[\theta^j]}}E|X|I(a_{[\theta^{[j\delta]}]}<|X|\leq a_{[\theta^j]}) + \theta^{j+1}P(|X|>a_{[\theta^j]}).
\end{aligned}$$

Therefore, (2.4) holds by Lemma 2.1.

Lemma 2.3. Let $1<\theta\leq\beta$ and X be a random variable. If (1.1) and (1.4) hold, then

$$\sum_{n=1}^{\infty} a_n^{-2}E|X|^2 I(|X|\leq a_n) < \infty. \qquad (2.7)$$

Proof. By $1 < \theta \le \beta$, (2.3) and similar to the proof of Lemma 2.1, we have

$$\begin{aligned}
&\sum_{n\in[\theta^{M_0},\infty)\bigcap\mathbb{N}} a_n^{-2}E|X|^2I(|X|\le a_n) \le C\sum_{j=M_0+1}^{\infty}\theta^j\cdot(a_{[\theta^{j-1}]})^{-2}E|X|^2I(|X|\le a_{[\theta^j]})\\
&\le C\sum_{j=M_0+1}^{\infty}\theta^j\left\{\sum_{k=M_0+1}^{j}\left(\frac{a_{[\theta^k]}}{a_{[\theta^{j-1}]}}\right)^2 EI(a_{[\theta^{k-1}]}<|X|\le a_{[\theta^k]})+\left(\frac{a_{[\theta^{M_0}]}}{a_{[\theta^{j-1}]}}\right)^2\right\}\\
&\le C\sum_{j=M_0+1}^{\infty}\theta^j\left\{\sum_{k=M_0+1}^{j}\beta^{-2(j-k-2)}P(|X|>a_{[\theta^{k-1}]})+\beta^{-2(j-M_0-2)}\right\}\\
&\le C\sum_{k=M_0+1}^{\infty}\beta^{2k}P(|X|>a_{[\theta^{k-1}]})\sum_{j=k}^{\infty}\left(\frac{\theta}{\beta^2}\right)^j + C\sum_{j=M_0+1}^{\infty}\left(\frac{\theta}{\beta^2}\right)^j\\
&\le C\sum_{k=M_0+1}^{\infty}\theta^kP(|X|>a_{[\theta^{k-1}]})+C.
\end{aligned}$$

Hence, (2.7) holds by Lemma 2.1.

Lemma 2.4. Let $1<\theta\le\beta, 0<\delta<1/2$. If (1.4) holds, then

$$\sum_{n=1}^{\infty}a_n^{-2}(\log n)^2\left(a_{[n^\delta]}\right)^2<\infty. \tag{2.8}$$

Proof. By positive series convergence criterion and (2.5), we have

$$\begin{aligned}
&\sum_{n\in[\theta^{[M_0/\delta]},\infty)\bigcap\mathbb{N}} a_n^{-2}(\log n)^2\left(a_{[n^\delta]}\right)^2=\sum_{j=[M_0/\delta]+1}^{\infty}\sum_{n\in[\theta^{j-1},\theta^j)\bigcap\mathbb{N}}a_n^{-2}(\log n)^2\left(a_{[n^\delta]}\right)^2\\
&\le(\theta-1)(\log\theta)^2\sum_{j=[M_0/\delta]+1}^{\infty}\theta^{j-1}j^2\left(\frac{a_{[\theta^{\delta j}]}}{a_{[\theta^{j-1}])}}\right)^2\\
&\le(\theta-1)(\log\theta)^2\beta^4\sum_{j=[M_0/\delta]+1}^{\infty}j^2\theta^{j-1}\beta^{-2(1-\delta)j}<\infty.
\end{aligned}$$

Therefore, (2.8) holds.

Lemma 2.5. [2]. Let $\{X_n, n\in\mathbb{N}\}$ be a sequence of PNQD random variables with $Var(X_n)<\infty\,(n\in\mathbb{N})$, and $\{b_n, n\in\mathbb{N}\}$ a sequence of real numbers with $0<b_n\uparrow\infty$. If

$$\sup_{n\ge1}\frac{1}{b_n}\sum_{j=1}^{n}E|X_j-EX_j|<\infty \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{1}{b_n^2}Var(X_n)<\infty,$$

then

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=1}^{n}(X_j-EX_j)=0,\ a.s..$$

Proof of Theorem 1.1. Firstly, we prove that (1.1)⇒(1.2). Set

$$X_{nk}=-a_nI(X_k<-a_n)+X_kI(|X_k|\le a_n)+a_nI(X_k>a_n), 1\le k\le n, n\in\mathbb{N}.$$

Then $EX_{nk}=-a_nP(X<-a_n)+EXI(|X|\le a_n)+a_nP(X>a_n)$. By (1.1) and $0<a_n\uparrow$, we have

$$a_n^{-1}\cdot n\left|-a_nP(X<-a_n)+a_nP(X>a_n)\right|\le nP(|X|>a_n)\to0$$

as $n \to \infty$. Therefore, to prove (1.2), it is enough to show that

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\le m\le n} |\sum_{k=1}^{m}(X_k - EX_{nk})| > \varepsilon a_n) < \infty, \quad \forall\, \varepsilon > 0. \tag{2.9}$$

Note that

$$\begin{aligned}&(\max_{1\le m\le n} |\sum_{k=1}^{m}(X_k - EX_{nk})| > \varepsilon a_n)\\ &\subset \bigcup_{k=1}^{n}(|X_k| > a_n)\bigcup(\max_{1\le m\le n} |\sum_{k=1}^{m}(X_{nk} - EX_{nk})| > \varepsilon a_n).\end{aligned}$$

Hence by (1.1), to prove (2.9), it is enough to prove that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\le m\le n} |\sum_{k=1}^{m}(X_{nk} - EX_{nk})| > \varepsilon a_n) < \infty. \tag{2.10}$$

For any fixed $\delta \in (0, 1/2)$, set

$$\begin{aligned}X_{nk}^{(1)} &= -a_{[n^\delta]} I(X_k < -a_{[n^\delta]}) + X_k I(|X_k| \le a_{[n^\delta]}) + a_{[n^\delta]} I(X_k > a_{[n^\delta]}),\\ X_{nk}^{(2)} &= (X_k - a_{[n^\delta]}) I(a_{[n^\delta]} < X_k \le a_n) + (a_n - a_{[n^\delta]}) I(X_k > a_n),\\ X_{nk}^{(3)} &= (X_k + a_{[n^\delta]}) I(-a_n \le X_k < -a_{[n^\delta]}) - (a_n - a_{[n^\delta]}) I(X_k < -a_n).\end{aligned}$$

Then $X_{nk} = X_{nk}^{(1)} + X_{nk}^{(2)} + X_{nk}^{(3)}$, and $\{X_{nk}^{(1)}, 1 \le k \le n\}$, $\{X_{nk}^{(2)}, 1 \le k \le n\}$, $\{X_{nk}^{(3)}, 1 \le k \le n\}$ are all PNQD by Lemma 1.1 of Wu [17] for every $n \ge 2$. Hence to prove (2.10), it is enough to prove that for all $\varepsilon > 0$ and $i = 1, 2, 3$,

$$I_i = \sum_{n=1}^{\infty} n^{-1} P(\max_{1\le m\le n} |\sum_{k=1}^{m}(X_{nk}^{(i)} - EX_{nk}^{(i)})| > \varepsilon a_n) < \infty.$$

By the Markov inequality and Lemma 1.2 of Wu [17] and Lemma 2.4,

$$\begin{aligned}I_1 &\le C\sum_{n=1}^{\infty} n^{-1} \cdot a_n^{-2} E \max_{1\le m\le n} |\sum_{k=1}^{m}(X_{nk}^{(1)} - EX_{nk}^{(1)})|^2\\ &\le C\sum_{n=1}^{\infty} n^{-1} \cdot a_n^{-2} (\log n)^2 \sum_{k=1}^{n} E|X_{nk}^{(1)}|^2\\ &\le C\sum_{n=1}^{\infty} (\log n)^2 a_n^{-2} (a_{[n^\delta]})^2 < \infty.\end{aligned}$$

From the definition of $X_{nk}^{(2)}$, Lemma 2.2 and (1.1), we have

$$\begin{aligned}&a_n^{-1} \max_{1\le m\le n} |\sum_{k=1}^{m} EX_{nk}^{(2)}| = a_n^{-1}\sum_{k=1}^{n} X_{nk}^{(2)}\\ &\quad = a_n^{-1} \cdot nE\left\{(X - a_{[n^\delta]}) I(a_{[n^\delta]} < X \le a_n) + (a_n - a_{[n^\delta]}) I(X > a_n)\right\}\\ &\quad \le a_n^{-1} \cdot n\left\{E|X| I(a_{[n^\delta]} < |X| \le a_n) + a_n P(|X| > a_n)\right\} \to 0, \quad n \to \infty.\end{aligned}$$

Therefore, to prove $I_2 < \infty$, it is enough to prove that for all $\varepsilon > 0$

$$I_2' = \sum_{n=1}^{\infty} n^{-1} P(|\sum_{k=1}^{n}(X_{nk}^{(2)} - EX_{nk}^{(2)})| > \varepsilon a_n) < \infty.$$

By the Markov inequality, Lemma 1.2 of Wu [17], (1.1) and Lemma 2.3,

$$\begin{aligned}
I_2' &\le C\sum_{n=1}^{\infty} n^{-1}\cdot a_n^{-2}E|\sum_{k=1}^{n}(X_{nk}^{(2)}-EX_{nk}^{(2)})|^2\\
&\le C\sum_{n=1}^{\infty} n^{-1}\cdot a_n^{-2}\left(\sum_{k=1}^{n}E|X_{nk}^{(2)}|^2\right)\\
&\le C\sum_{n=1}^{\infty} a_n^{-2}\left\{E|X|^2I(|X|\le a_n)+a_n^2P(|X|>a_n)\right\}<\infty.
\end{aligned}$$

By the same argument as $I_2<\infty$, we have $I_3<\infty$. Thus, (1.2) holds.

Secondly, we prove that (1.2)$\Rightarrow$ (1.1). Let $\{X', X_n', n\in\mathbb{N}\}$ be an independent copy of $\{X, X_n, n\in\mathbb{N}\}$, then $\{X', X_n', n\in\mathbb{N}\}$ and $\{X-X', X_n-X_n', n\in\mathbb{N}\}$ are sequences of identically distributed PNQD random variables by Theorem 1 of Su and Wang [13], respectively. By (1.2),

$$\sum_{n=1}^{\infty} n^{-1}P\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}X_k'-mEXI(|X|\le a_n)\right|>\varepsilon a_n\right)<\infty,\quad \forall\,\varepsilon>0.$$

Hence

$$\sum_{n=1}^{\infty} n^{-1}P\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(X_k-X_k')\right|>\varepsilon a_n\right)<\infty,\quad \forall\,\varepsilon>0.$$

Note that for all $1\le k\le n, n\in\mathbb{N}$,

$$|X_k-X_k'|=\left|\sum_{j=1}^{k}(X_j-X_j')-\sum_{j=1}^{k-1}(X_j-X_j')\right|\le 2\max_{1\le m\le n}\left|\sum_{k=1}^{m}(X_k-X_k')\right|.$$

Thus

$$\sum_{n=1}^{\infty} n^{-1}P\left(\max_{1\le k\le n}|X_k-X_k'|>\varepsilon a_n\right)<\infty,\quad \forall\,\varepsilon>0. \tag{2.11}$$

Since

$$\begin{aligned}
\sum_{m=[\theta^j]+1}^{[\theta^{j+1}]} m^{-1}P(\max_{1\le k\le m}|X_k-X_k'|>\varepsilon a_m) &\ge \sum_{m=[\theta^j]+1}^{[\theta^{j+1}]}\frac{1}{[\theta^{j+1}]}P(\max_{1\le k\le [\theta^j]}|X_k-X_k'|>\varepsilon a_{[\theta^{j+1}]})\\
&\ge CP(\max_{1\le k\le [\theta^j]}|X_k-X_k'|>\varepsilon a_{[\theta^{j+1}]}).
\end{aligned}$$

Therefore, we have by (2.11) that

$$\lim_{j\to\infty}P(\max_{1\le k\le [\theta^j]}|X_k-X_k'|>\varepsilon a_{[\theta^{j+1}]})=0.$$

Hence, by Lemma 1.4 of Wu [17], we have for j large enough that

$$\begin{aligned}
[\theta^j]P(|X-X'|>\varepsilon a_{[\theta^{j+1}]}) &= \sum_{k=1}^{[\theta^j]}P(|X_k-X_k'|>\varepsilon a_{[\theta^{j+1}]})\\
&\le CP(\max_{1\le k\le [\theta^j]}|X_k-X_k'|>\varepsilon a_{[\theta^{j+1}]}).
\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we have

$$\sum_{j=1}^{\infty}\theta^jP(|X-X'|>\varepsilon a_{[\theta^j]})<\infty,\quad \forall\,\varepsilon>0.$$

Therefore, by Lemma 2.1, we have

$$\sum_{n=1}^{\infty} P(|X - X'| > \varepsilon a_n) < \infty, \quad \forall\, \varepsilon > 0. \tag{2.13}$$

Note that $0 < a_n \uparrow \infty$, by the weak symmetrization inequality (see Loève [17]), we have for n large enough that

$$\begin{aligned} P(|X| > a_n) &= P(|X - med(X) + med(X)| > a_n) \\ &\le P(|X - med(X)| > a_n/2) \le 2P(|X - X'| > a_n/2). \end{aligned}$$

Hence, (1.1) holds by (2.13).

Thirdly, we prove that (1.1)⇒(1.3). Let $Y_n = -a_n I(X_n < -a_n) + X_n I(|X_n| \le a_n) + a_n I(X_n > a_n)$ for $n \in \mathbb{N}$. Then we have by Lemma 2.2 and (1.1) that

$$\sup_{n\ge1} \frac{1}{a_n} \sum_{j=1}^{n} E|Y_j - EY_j| \le \sup_{n\ge1} \frac{2}{a_n} \sum_{j=1}^{n} E|Y_j| \le \sup_{n\ge1} \frac{2n}{a_n} \{E|X|I(|X| \le a_n) + a_n P(|X| > a_n)\} < \infty.$$

We also have by Lemma 2.3 and (1.1) that

$$\sum_{n=1}^{\infty} a_n^{-2} Var(Y_n) \le \sum_{n=1}^{\infty} a_n^{-2} E(Y_n)^2 \le \sum_{n=1}^{\infty} a_n^{-2} EX^2 I(|X| \le a_n) + \sum_{n=1}^{\infty} P(|X| > a_n) < \infty.$$

Therefore, by Lemma 2.5,

$$\frac{1}{a_n} \sum_{j=1}^{n} (Y_j - EY_j) \to 0 \quad a.s.. \tag{2.14}$$

By Lemma 1.3 of Wu [17], (1.1) implies

$$\frac{1}{a_n} \sum_{j=1}^{n} |X_j| I(|X_j| > a_j) \to 0 \quad a.s.. \tag{2.15}$$

By the Kronecker Lemma (see Loève [17]) and (1.1), we have

$$\frac{1}{a_n} \sum_{j=1}^{n} a_j P(|X_j| > a_j) \to 0 \quad a.s.. \tag{2.16}$$

Thus, (1.3) holds by (2.14)∼(2.16).

Finally, we prove that (1.3)⇒ (1.1). The proof of (1.3)⇒ (1.1) is similar to that in Theorem 2.3 of Sung [14], and so we omit it.

Proof of Corollary 1.2. By Lemma 2.4 of Sung [14] and Theorem 1.1 and the same method as in Theorem 2.3 of Sung [14], Corollary 1.2 is obtained.

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