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# Minimizers of curl prescribed full trace

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**Abstract.** This paper concerns the minimization problem of $L^2$ norm of curl of vector fields prescribed full trace on the boundary of a multiconnected bounded domain. The existence of the minimizers in $H^1$ are shown by orthogonal decompositions of vector function spaces and a constructed auxiliary variational problem. And the $H^2$ estimate of the type II divergence-free part of the minimizers is established by div-curl-gradient type estimates of vector fields.

## §1 Introduction

The variational problems involving operator curl are naturally proposed in the mathematical theory of electromagnetics, liquid crystals, superconductivity and Born-Infeld theory (see for instance [2,3,6,10,13]), while we were not intended to include all the abundant articles here.

This paper is devoted to the minimization problem of $L^2$ norm of curl of vector fields prescribed full trace in multiconnected bounded domains, proposed in [9, Problem 4.4],

$$a(\mathbf{u}^0) \triangleq \inf_{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0)} \int_\Omega |\operatorname{curl}\mathbf{u}|^2 \,\mathrm{d}x. \tag{1.1}$$

where $\mathbf{u}^0 \in H^{1/2}(\partial\Omega,\mathbb{R}^3)$, and the admissable space of vector fields

$$H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0) = \{\mathbf{u} \in H^1(\Omega,\mathbb{R}^3) :\ \mathbf{u} = \mathbf{u}^0 \text{ on } \partial\Omega\}.$$

Our major motivation is to study the effects of boundary condition and topology of the domain on the existence and regularity of minimizers of functionals involving operator curl. Let us mention several closely related papers here. For the case where $\Omega$ is a simply connected bounded domain: the existence of minimizers of (1.1) was proved by Pan and Qi [10]; Hölder continuity of the divergence-free weak solution of several linear (elliptic or parabolic) nonhomogeneous systems involving curl were studied well, see for instance [6,7,11,13]; the existence, uniqueness and regularity of the divergence-free weak solution of a parabolic p-curl system was studied by Yin etc. [12]; in addition if $\Omega$ has no holes either, the existence and interior Hölder

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regularity of critical points (usually not divergence-free) of functionals dominated by the $L^p$ ($p > 1$) norm of curl of vector fields under two types of boundary conditions were solved by the author and Pan [4]. For the case where $\Omega$ is a multiconnected bounded domain, Pan [9] has shown the existence and $H^2$ regularity of minimizers of $L^2$ norm of curl of vector fields prescribed tangential trace on the boundary.

The main difficulty of problem (1.1) is generated from the concurrence of full trace boundary condition and the nontrivial topology of the domain. As we know, a natural admissable space for the $L^2$ norm of $\operatorname{curl}\mathbf{u}$ is

$$\mathfrak{H}(\Omega, \operatorname{curl}) \triangleq \{\mathbf{u} \in L^2(\Omega, \mathbb{R}^3) : \ \operatorname{curl}\mathbf{u} \in L^2(\Omega, \mathbb{R}^3)\},$$

whose trace map is continuously extended from the tangential component of the vector field $\mathbf{u} \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$, see for instance [5, P. 204, Theorem 2]. And thus prescribing the full trace of the vector fields is unusual in a sense that $L^2$ norm of $\operatorname{curl}\mathbf{u}$ generally cannot control the normal component of $\mathbf{u}$ on the boundary of the domain. Besides, since the topology of the domain is closely related with the Neumann fields and Dirichlet fields (see subsection 2.1), its nontriviality would provide a variety of the minimizers of problem (1.1).

The paper is organized as follows. Notations and assumptions on the domain are collected in subsection 2.1, and decompositions of the spaces of vector fields and preliminary estimates of the vector fields are given in subsections 2.2 and 2.3. Section 3 is devoted to the existence and regularity of the minimizers.

# §2 Preliminaries

## 2.1 Notations

Throughout the paper, the bold typeface is applied to indicate vector functions, and the normal typeface is used for scalars. And we assume that $\Omega$ satisfies the following conditions:

(O1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with a $C^r (r \geq 2)$ boundary $\partial\Omega$ of dimension 2, and locally situated on one side of $\partial\Omega$; $\partial\Omega$ has a finite number of connected components denoted by $\Gamma_0, \cdots, \Gamma_m$, where $\Gamma_0$ denoting the boundary of the infinite connected components of $\mathbb{R}^3 \setminus \bar{\Omega}$.

(O2) The open set $\Omega$ which can be multiply connected, is made simply connected by $n$ regular cuts: $\Sigma_1, \cdots, \Sigma_n$. The cuts are of dimension 2 and of $C^r (r \geq 2)$ such that $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$, and nontangential to $\partial\Omega$ hence $\dot{\Omega} = \Omega \setminus \Sigma$ (with $\Sigma = \cup_{j=1}^n \Sigma_j$) is simply connected and Lipschitz.

According to the above assumption, $\Omega$ is simply connected if the first Betti number $n = 0$, and has no holes if the second Betti number $m = 0$. In the following we denote the unit outer normal vector on $\partial\Omega$ by $\mathbf{n}$. Let us mention that the dimensions of the vector spaces of Neumann fields and Dirichlet fields

$$\begin{aligned}\mathbb{H}_1(\Omega) &\triangleq \{\mathbf{u}\in L^2(\Omega,\mathbb{R}^3):\ \operatorname{curl}\mathbf{u}=0,\operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\ \mathbf{n}\cdot\mathbf{u}=0 \text{ on } \partial\Omega\},\\ \mathbb{H}_2(\Omega) &\triangleq \{\mathbf{u}\in L^2(\Omega,\mathbb{R}^3):\ \operatorname{curl}\mathbf{u}=0,\operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\ \mathbf{n}\times\mathbf{u}=0 \text{ on } \partial\Omega\},\end{aligned} \tag{2.1}$$

are closely related to the Betti numbers of the domain $\Omega$, that is,

$$\dim \mathbb{H}_1(\Omega)=n,\quad \dim \mathbb{H}_2(\Omega)=m,$$

see for instance [5,1,9]. For simplicity in the following we also denote $\mathbb{H}_i(\Omega)$ by $\mathbb{H}_i$, $i=1,2$, and the orthogonal complement of $\mathbb{H}_i$ in $L^2(\Omega,\mathbb{R}^3)$ by $\mathbb{H}_i^\perp$ respectively.

We also denote the tangential component of a vector field $\mathbf{u}$ on $\partial\Omega$ by $\mathbf{u}_T$, that is, $\mathbf{u}_T=-\mathbf{n}\times(\mathbf{n}\times\mathbf{u})$ on $\partial\Omega$.

Here we collect some notations of the spaces of vector fields:

$$\begin{aligned}&H_0^1(\Omega,\mathbb{R}^3)=\{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3):\ \mathbf{u}=\mathbf{0} \text{ on } \partial\Omega\},\\ &H_n^1(\Omega,\operatorname{div}0,\mathbf{n}\cdot\mathbf{u}^0)=\{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0):\ \operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\ \mathbf{n}\cdot\mathbf{u}=\mathbf{n}\cdot\mathbf{u}^0 \text{ on } \partial\Omega\},\\ &H_t^1(\Omega,\operatorname{div}0,\mathbf{u}_T^0)=\{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0):\ \operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\ \mathbf{u}_T=\mathbf{u}_T^0 \text{ on } \partial\Omega\},\\ &H_{n0}^1(\Omega,\operatorname{div}0)=\{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0):\ \operatorname{div}\mathbf{u}=0 \text{ in } \Omega,\ \mathbf{n}\cdot\mathbf{u}=0 \text{ on } \partial\Omega\},\end{aligned} \tag{2.2}$$

where $\mathbf{u}_T^0$ is the tangential component of $\mathbf{u}^0$ on $\partial\Omega$.

## 2.2 Orthogonal decomposition of spaces of vector fields

To remove uncertainty originated from the topology of the domain $\Omega$ in the minimization problem (1.1), we will need a lemma on the orthogonal decomposition of $H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0)$.

**Lemma 2.1.** *The following orthogonal decompositions of $H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0)$ with respect to $L^2$ norm are established:*

$$H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0)=\mathcal{O}_i(\Omega,\mathbf{u}^0)\oplus\mathbb{H}_i,\quad i=1,2, \tag{2.3}$$

*where $\mathcal{O}_i(\Omega,\mathbf{u}^0)\triangleq H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0)\cap\mathbb{H}_i^\perp$, $i=1,2$ are both closed subsets in $L^2(\Omega,\mathbb{R}^3)$.*

*Proof.* The proofs of the two types of orthogonal decompositions are similar, and we only state the case where $i=1$. Since $\mathbb{H}_1$ is a closed linear subspace of $L^2(\Omega,\mathbb{R}^3)$, any $\mathbf{u}\in H^1(\Omega,\mathbb{R}^3,\mathbf{u}^0)$ can be decomposed uniquely in the form (see for instance [14, P. 82, Theorem 1])

$$\mathbf{u}=(\mathbf{u}-\mathbf{h})+\mathbf{h},$$

where $\mathbf{u}-\mathbf{h}\in\mathcal{O}_1(\Omega,\mathbf{u}^0)$, and $\mathbf{h}\in\mathbb{H}_1$ is obtained by

$$\|\mathbf{u}-\mathbf{h}\|_{L^2(\Omega,\mathbb{R}^3)}=\inf_{\mathbf{x}\in\mathbb{H}_1}\|\mathbf{u}-\mathbf{x}\|_{L^2(\Omega,\mathbb{R}^3)}.$$

□

## 2.3 Some inequalities for vector fields

We will need a variation of the Poincáre inequality in multiconnected domains. Although the proof is a classical one, we write it down for reader's convenience:

**Lemma 2.2.** *Assume that $\Omega$ satisfies (O1), (O2), then for any $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \cap \mathbb{H}_1^{\perp}$ it holds that*

$$\|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)} \le C\left(\|\operatorname{curl}\mathbf{v}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div}\mathbf{v}\|_{\mathrm{L}^2(\Omega,\mathbb{R}^3)} + \|\mathbf{n}\cdot\mathbf{v}\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right), \tag{2.4}$$

*where $C$ is a positive constant depending only on $\Omega$.*

*Proof.* We argue by contradiction. Were the stated estimate false, there would exist for each integer $m = 1, 2, \cdots$ a function $\mathbf{v}_m \in H^1(\Omega, \mathbb{R}^3) \cap \mathbb{H}_1^{\perp}$ satisfying

$$\|\mathbf{v}_m\|_{H^1(\Omega,\mathbb{R}^3)} > m\left(\|\operatorname{curl}\mathbf{v}_m\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div}\mathbf{v}_m\|_{\mathrm{L}^2(\Omega,\mathbb{R}^3)} + \|\mathbf{n}\cdot\mathbf{v}_m\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right). \tag{2.5}$$

We renormalize by defining

$$\mathbf{w}_m = \frac{\mathbf{v}_m}{\|\mathbf{v}_m\|_{H^1(\Omega,\mathbb{R}^3)}}, \quad m = 1, 2, \cdots,$$

then

$$\mathbf{w}_m \in H^1(\Omega, \mathbb{R}^3) \cap \mathbb{H}_1^{\perp}, \quad \|\mathbf{w}_m\|_{H^1(\Omega,\mathbb{R}^3)} = 1.$$

After passing to a subsequence we may assume that $\mathbf{w}_m \to \mathbf{w}_0$ weakly in $H^1(\Omega, \mathbb{R}^3)$ and strongly in $L^2(\Omega, \mathbb{R}^3)$, and thus $\mathbf{w}_0 \in \mathbb{H}_1^{\perp}$. Moreover, (2.5) implies that as $m \to \infty$ we have

$$\|\operatorname{curl}\mathbf{w}_m\|_{L^2(\Omega,\mathbb{R}^3)} \to 0, \quad \|\operatorname{div}\mathbf{w}_m\|_{\mathrm{L}^2(\Omega,\mathbb{R}^3)} \to 0, \quad \|\mathbf{n}\cdot\mathbf{w}_m\|_{\mathrm{H}^{1/2}(\partial\Omega)} \to 0, \tag{2.6}$$

which shows $\mathbf{w}_0 \in \mathbb{H}_1$. And hence $\mathbf{w}_0 = \mathbf{0}$, say,

$$\|\mathbf{w}_m\|_{L^2(\Omega,\mathbb{R}^3)} \to 0. \tag{2.7}$$

Applying the div-curl-gradient inequality in [5, Corollary 1] to $\mathbf{w}_m \in H^1(\Omega, \mathbb{R}^3)$, together with (2.6) and (2.7) we derive that

$$\begin{aligned} 1 = \|\mathbf{w}_m\|_{H^1(\Omega,\mathbb{R}^3)} \le & C\big(\|\mathbf{w}_m\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{curl}\mathbf{w}_m\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div}\mathbf{w}_m\|_{\mathrm{L}^2(\Omega,\mathbb{R}^3)} \\ & + \|\mathbf{n}\cdot\mathbf{w}_m\|_{H^{1/2}(\partial\Omega)}\big) \to 0, \quad \text{as } m \to \infty. \end{aligned}$$

This contradiction establishes the estimate (2.4). □

Here we list a similar lemma and omit the proof:

**Lemma 2.3.** *Assume that $\Omega$ satisfies (O1), (O2), then for any $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \cap \mathbb{H}_2^{\perp}$ it holds that*

$$\|\mathbf{v}\|_{H^1(\Omega,\mathbb{R}^3)} \le C\left(\|\operatorname{curl}\mathbf{v}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div}\mathbf{v}\|_{\mathrm{L}^2(\Omega,\mathbb{R}^3)} + \|\mathbf{v}_T\|_{\mathrm{H}^{1/2}(\Omega,\mathbb{R}^3)}\right), \tag{2.8}$$

*where $C$ is a positive constant depending only on $\Omega$.*

# §3 Main results

## 3.1 Existence of the minimizers

Following the idea introduced by [9], firstly we study an auxiliary minimization problem in the orthogonal subspace $\mathcal{O}_1(\Omega, \mathbf{u}^0)$ of $H^1(\Omega, \mathbb{R}^3, \mathbf{u}^0)$:

$$b(\mathbf{u}^0) \triangleq \inf_{\mathbf{u}\in\mathcal{O}_1(\Omega,\mathbf{u}^0)} \int_\Omega |\operatorname{curl}\mathbf{u}|^2\,\mathrm{d}x. \tag{3.1}$$

Denote $\mathring{H}^2(\Omega) \triangleq \{\phi \in H^2(\Omega) : \int_\Omega \phi\,\mathrm{d}x = 0\}$, and we prove the following lemma.

**Lemma 3.1.** *Assume that $\Omega$ satisfies (O1), (O2), and $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. Then $b(\mathbf{u}^0)$ is achieved, and the minimizer $\mathbf{u} \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$ can be decomposed orthogonally into*

$$\mathbf{u} = \mathbf{v} + \nabla\phi, \tag{3.2}$$

*where $\mathbf{v} \in H^1_{n0}(\Omega, \mathrm{div}0) \cap \mathbb{H}_1^{\perp}$, and $\phi \in \mathring{H}^2(\Omega)$ satisfying $\nabla\phi = \mathbf{u}^0 - \mathbf{v}$ on $\partial\Omega$. Moreover, choosing a minimizer $\mathbf{u}^* \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$, the set $\mathcal{M}_b$ of all the minimizers of problem* (3.1) *is formed as*

$$\mathcal{M}_b = \{\mathbf{u}^* + \nabla\zeta : \zeta \in H^2(\Omega),\, \nabla\zeta = 0 \text{ on } \partial\Omega\}. \tag{3.3}$$

*Proof.* Step 1. Let $\{\mathbf{u}_j\}_{j=1}^{\infty}$ be a minimizing sequence of the auxiliary minimization problem (3.1). We decompose $\mathbf{u}_j \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$ as

$$\mathbf{u}_j = \mathbf{v}_j + \nabla\phi_j,$$

where $\mathbf{v}_j \in H^1_{n0}(\Omega, \mathrm{div}0) \cap \mathbb{H}_1^{\perp}$, and $\phi_j \in \mathring{H}^2(\Omega)$ is the unique solution of

$$\begin{cases} \Delta\phi_j = \mathrm{div}\,\mathbf{u}_\mathrm{j} & \text{in } \Omega, \\ \dfrac{\partial\phi_j}{\partial\mathbf{n}} = \mathbf{n}\cdot\mathbf{u}^0 & \text{on } \partial\Omega, \end{cases}$$

which is obtained by the following minimization problem

$$\inf_{\phi \in H^1(\Omega)} \int_\Omega |\nabla\phi - \mathbf{u}_j|^2 \,\mathrm{d}x. \tag{3.4}$$

Hence there exists a positive constant $C$ such that

$$b(\mathbf{u}^0) + C \geq \int_\Omega |\,\mathrm{curl}\,\mathbf{u}_j|^2 \,\mathrm{d}x = \int_\Omega |\,\mathrm{curl}\,\mathbf{v}_j|^2 \,\mathrm{d}x. \tag{3.5}$$

Applying Lemma 2.2 to $\mathbf{v}_j$, together with (3.5) we derive that

$$\|\mathbf{v}_j\|_{H^1(\Omega,\mathbb{R}^3)} \leq C\|\,\mathrm{curl}\,\mathbf{v}_j\|_{L^2(\Omega,\mathbb{R}^3)} \leq C.$$

So after passing to a subsequence, we may assume that $\mathbf{v}_j \to \mathbf{v}^*$ weakly in $H^1(\Omega, \mathbb{R}^3)$ and weakly in $H^{1/2}(\partial\Omega, \mathbb{R}^3)$. And thus

$$\nabla\phi_j = \mathbf{u}_j - \mathbf{v}_j \to \mathbf{u}^0 - \mathbf{v}^* \quad \text{weakly in } H^{1/2}(\partial\Omega, \mathbb{R}^3). \tag{3.6}$$

Observing that uniform control of $H^2$ norm of $\phi_j$ is absent, we bring in a sequence of minimization problems

$$\lambda_j \triangleq \inf_{\psi \in \mathcal{D}_j} \int_\Omega |\nabla^2\psi|^2 \,\mathrm{d}x, \quad j = 1, 2, \cdots,$$

where $\mathcal{D}_j = \{\psi \in \mathring{H}^2(\Omega) :\, \nabla\psi = \nabla\phi_j \text{ on } \partial\Omega\}$. For each $j$, the minimizer exists and is unique, denoted by $\psi_j$, since the functional is an equivalent norm for $\psi \in \mathcal{D}_j$ and strictly convex.

From (3.6) we know that $\nabla\psi_j \to \mathbf{u}^0 - \mathbf{v}^*$ weakly in $H^{1/2}(\partial\Omega, \mathbb{R}^3)$, according to the trace theorem of $H^1(\Omega, \mathbb{R}^3)$ we know that $\{\lambda_j\}$ is bounded. And thus $\{\psi_j\}$ is bounded in $H^2(\Omega, \mathbb{R}^3)$. So after passing to a subsequence, we may assume that $\psi_j \to \psi^*$ weakly in $H^2(\Omega)$, and $\nabla\psi^* = \mathbf{u}^0 - \mathbf{v}^*$ on $\partial\Omega$.

Now set $\mathbf{u}^* = \mathbf{v}^* + \nabla\psi^*$, and hence $\mathbf{u}^* \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$. Moreover,

$$b(\mathbf{u}^0) \leq \int_\Omega |\,\mathrm{curl}\,\mathbf{u}^*|^2 \,\mathrm{d}x = \int_\Omega |\,\mathrm{curl}\,\mathbf{v}^*|^2 \,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega |\,\mathrm{curl}\,\mathbf{v}_j|^2 \,\mathrm{d}x = b(\mathbf{u}^0),$$

that is, $b(\mathbf{u}^0)$ is achieved.

Step 2. Let $\mathbf{u} \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$ be a minimizer of problem (3.1). The decomposition (3.2) is obtained by a similar way as (3.4) in step 1.

Step 3. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$ be minimizers of problem (3.1). Then we set $\mathbf{w} = \frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$, so $\mathbf{w} \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$, and using Cauchy's inequality we derive that

$$b(\mathbf{u}^0) \leq \int_\Omega |\operatorname{curl} \mathbf{w}|^2 \, \mathrm{d}x = \frac{1}{4}\int_\Omega (|\operatorname{curl} \mathbf{u}_1|^2 + |\operatorname{curl} \mathbf{u}_2|^2) \, \mathrm{d}x + \frac{1}{2}\int_\Omega \operatorname{curl} \mathbf{u}_1 \cdot \operatorname{curl} \mathbf{u}_2 \, \mathrm{d}x$$
$$\leq \frac{1}{2}\int_\Omega (|\operatorname{curl} \mathbf{u}_1|^2 + |\operatorname{curl} \mathbf{u}_2|^2) \, \mathrm{d}x = b(\mathbf{u}^0).$$

And hence $\operatorname{curl} \mathbf{u}_1 = K \operatorname{curl} \mathbf{u}_2$ a.e. in $\Omega$ for some constant $K > 0$. So we have

$$b(\mathbf{u}^0) = \int_\Omega |\operatorname{curl} \mathbf{u}_1|^2 \, \mathrm{d}x = K^2 \int_\Omega |\operatorname{curl} \mathbf{u}_2|^2 \, \mathrm{d}x = K^2 b(\mathbf{u}^0),$$

which tells that $K = 1$, and $\operatorname{curl}(\mathbf{u}_1 - \mathbf{u}_2) = 0$ a.e. in $\Omega$. According to the Hodge decomposition introduced by [5, P. 226, Corollary. 6]), there exists a unique (up to an additive constant) $\zeta \in H^2(\Omega)$ satisfying $\nabla\zeta = \mathbf{0}$ on $\partial\Omega$ such that $\mathbf{u}_1 - \mathbf{u}_2 = \nabla\zeta$. And thus (3.3) is established. □

**Remark 3.2.** *If we consider the auxiliary minimization problem in $\mathcal{O}_2(\Omega, \mathbf{u}^0)$ instead of $\mathcal{O}_1(\Omega, \mathbf{u}^0)$ under the same condition of Lemma 3.1, by a similar argument (using Lemma 2.3 and Hodge decomposition of $L^2(\Omega, \mathbb{R}^3)$ given in [5, P. 225, (1.61)] instead), the minimizers exist and each minimizer $\mathbf{u} \in \mathcal{O}_2(\Omega, \mathbf{u}^0)$ can be decomposed orthogonally into*

$$\mathbf{u} = \mathbf{v} + \nabla\phi,$$

*where $\mathbf{v} \in H^1_t(\Omega, \mathrm{div}0, \mathbf{u}^0_T) \cap \mathbb{H}_2^\perp$, and $\phi \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying $\nabla\phi = \mathbf{u}^0 - \mathbf{v}$ on $\partial\Omega$. Moreover, choosing a minimizer $\mathbf{u}^* \in \mathcal{O}_2(\Omega, \mathbf{u}^0)$, the set $\mathcal{M}'_b$ of all the minimizers in $\mathcal{O}_2(\Omega, \mathbf{u}^0)$ is formed as*

$$\mathcal{M}'_b = \{\mathbf{u}^* + \nabla\zeta : \zeta \in H^2_0(\Omega)\}. \tag{3.7}$$

Now we are ready to solve the original minimization problem (1.1):

**Theorem 3.3.** *Assume that $\Omega$ satisfies (O1), (O2), and $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. Then $a(\mathbf{u}^0) = b(\mathbf{u}^0)$. Moreover, choosing a minimizer $\mathbf{u}^* \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$, the set $\mathcal{M}_a$ of all the minimizers of problem* (1.1) *is formed as*

$$\mathcal{M}_a = \left\{\mathbf{u}^* + \mathbf{h} + \nabla\xi : \mathbf{h} \in \mathbb{H}_1, \xi \in H^2(\Omega),\ \nabla\xi + \mathbf{h} = \mathbf{0} \text{ on } \partial\Omega\right\}. \tag{3.8}$$

*Proof.* Step 1. Let $\mathbf{u} \in \mathcal{O}_1(\Omega, \mathbf{u}^0)$ be a minimizer of problem (3.1), then it is also a minimizer of problem (1.1) due to the orthogonal decomposition (2.3) of $H^1(\Omega, \mathbf{u}^0)$. And thus $a(\mathbf{u}^0)$ is achieved.

Step 2. The Euler-Lagrange equation of the minimizer $\mathbf{u} \in H^1(\Omega, \mathbb{R}^3, \mathbf{u}^0)$ of (1.1) is deduced by the conventional method: for any $\mathbf{b} \in H^1_0(\Omega, \mathbb{R}^3)$ we have

$$\int_\Omega \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{b} \, \mathrm{d}x = 0. \tag{3.9}$$

Step 3. Let $\mathbf{u}_1, \mathbf{u}_2$ are both minimizers of (1.1), and set $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2 \in H^1_0(\Omega, \mathbb{R}^3)$. Using

(3.9) we derive that

$$\begin{aligned}&\int_\Omega |\operatorname{curl}\mathbf{u}_1|^2\,\mathrm{d}x = \int_\Omega |\operatorname{curl}\mathbf{u}_2 + \operatorname{curl}\mathbf{w}|^2\,\mathrm{d}x\\ =&\int_\Omega (|\operatorname{curl}\mathbf{u}_2|^2 + 2\operatorname{curl}\mathbf{u}_2\cdot\operatorname{curl}\mathbf{w} + |\operatorname{curl}\mathbf{w}|^2)\,\mathrm{d}x = \int_\Omega (|\operatorname{curl}\mathbf{u}_2|^2 + |\operatorname{curl}\mathbf{w}|^2)\,\mathrm{d}x\\ =&\int_\Omega (|\operatorname{curl}\mathbf{u}_1|^2 + |\operatorname{curl}\mathbf{w}|^2)\,\mathrm{d}x,\end{aligned}$$

which tells that $\operatorname{curl}\mathbf{w} = \mathbf{0}$ in $\Omega$. According to the orthogonal decomposition of $L^2(\Omega, \mathbb{R}^3)$ introduced by [5, P. 226, Corollary 6]), there exists a unique orthogonal decomposition

$$\mathbf{w} = \mathbf{h} + \nabla\xi,$$

where $\mathbf{h} \in \mathbb{H}_1$, and $\xi \in H^2(\Omega)$ unique to within an additive constant. Moreover, since $\mathbf{w} = \mathbf{0}$ on $\partial\Omega$, the decomposition (3.8) of the minimizers is established. □

**Remark 3.4.** *By a similar argument as Theorem 3.3 (using Hodge decomposition of $L^2(\Omega, \mathbb{R}^3)$ given in [5, P. 225, (1.61)] instead), choosing a minimizer $\mathbf{u}^* \in \mathcal{O}_2(\Omega, \mathbf{u}^0)$, the set $\mathcal{M}_a$ of all the minimizers of problem* (1.1) *can be formed alternatively as*

$$\mathcal{M}_a = \left\{\mathbf{u}^* + \mathbf{h} + \nabla\xi : \mathbf{h} \in \mathbb{H}_2, \xi \in H^2(\Omega) \cap H_0^1(\Omega), \nabla\xi + \mathbf{h} = \mathbf{0} \text{ on } \partial\Omega\right\}. \tag{3.10}$$

**Remark 3.5.** *(i) From Lemma 3.1 and Theorem 3.3 we know that each minimizer $\mathbf{u} \in H^1(\Omega, \mathbb{R}^3, \mathbf{u}^0)$ of problem* (1.1) *can be decomposed orthogonally into*

$$\mathbf{u} = \mathbf{v} + \nabla\eta + \mathbf{h}, \tag{3.11}$$

*where $\mathbf{v} \in H^1_{n0}(\Omega, \operatorname{div}0) \cap \mathbb{H}_1^\perp$, $\eta \in H^2(\Omega)$ unique within an additive constant satisfying $\nabla\eta = \mathbf{u}^0 - (\mathbf{v} + \mathbf{h})$ on $\partial\Omega$, and $\mathbf{h} \in \mathbb{H}_1$.*

*(ii) Due to Remark 3.2 and Remark 3.4, an alternative choice of orthogonal decomposition of each minimizer $\mathbf{u} \in H^1(\Omega, \mathbb{R}^3, \mathbf{u}^0)$ of problem* (1.1) *is formed as*

$$\mathbf{u} = \mathbf{v} + \nabla\eta + \mathbf{h}, \tag{3.12}$$

*where $\mathbf{v} \in H^1_t(\Omega, \operatorname{div}0, \mathbf{u}^0_T) \cap \mathbb{H}_2^\perp$, $\eta \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfies $\nabla\eta = \mathbf{u}^0 - (\mathbf{v} + \mathbf{h})$ on $\partial\Omega$, and $\mathbf{h} \in \mathbb{H}_2$.*

For convenience (3.11) and (3.12) are called as type I and type II decomposition of the minimizers of problem (1.1) respectively. Let us mention that for each type all the minimizers own the same divergence-free part $\mathbf{v}$.

## 3.2 Estimates of the minimizers

In this subsection we use the type II decomposition to improve the estimate of minimizers of problem (1.1). Since $\operatorname{curl}\mathbf{u}$ is dominated by $\operatorname{curl}\mathbf{v}$, and $\mathbf{v} \in H^1_t(\Omega, \operatorname{div}0, \mathbf{u}^0_T) \cap \mathbb{H}_2^\perp$ is a weak solution of

$$\begin{cases} \operatorname{curl}^2\mathbf{H} = \mathbf{0}, \quad \operatorname{div}\mathbf{H} = 0 & \text{in } \Omega,\\ \mathbf{H}_T = \mathbf{u}^0_T & \text{on } \partial\Omega,\end{cases} \tag{3.13}$$

derived from (3.9) and (3.12), so it is natural to obtain the following theorem.

**Theorem 3.6.** *Assume that $\Omega$ satisfies (O1), (O2), and $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. Let $\mathcal{P}$ be the projection of the minimizers $\mathbf{u}$ of problem (1.1) into the orthogonal subspace $H^1_t(\Omega, \mathrm{div}0, \mathbf{u}^0_T) \cap \mathbb{H}_2^{\perp}$. Then*

$$\|\mathcal{P}(\mathbf{u})\|_{H^1(\Omega,\mathbb{R}^3)} \le C\|\mathbf{u}^0_T\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}, \tag{3.14}$$

*where $C$ is a positive constant depending only on $\Omega$. Furthermore, if $r \ge 3$ and $\mathbf{u}^0 \in H^{3/2}(\Omega, \mathbb{R}^3)$, then $\mathcal{P}(\mathbf{u}) \in H^2(\Omega, \mathbb{R}^3)$, and there exists a positive constant $C$ depending only on $\Omega$ such that*

$$\|\mathcal{P}(\mathbf{u})\|_{H^2(\Omega,\mathbb{R}^3)} \le C\|\mathbf{u}^0_T\|_{H^{3/2}(\partial\Omega,\mathbb{R}^3)}. \tag{3.15}$$

*Proof.* Step 1. Let $\mathbf{u} \in H^1(\Omega, \mathbf{u}^0)$ be a minimizer of problem (1.1), and let $\mathcal{L}$ be the mapping from $H^{1/2}(\partial\Omega, \mathbb{R}^3)$ onto $H^1_t(\Omega, \mathrm{div}0, \mathbf{u}^0_T) \cap \mathbb{H}_2^{\perp}$ such that $\mathcal{L}(\mathbf{u}^0_T) = \mathbf{v}$, where $\mathbf{v} = \mathcal{P}(\mathbf{u})$. So we have $\mathcal{L}(\lambda\mathbf{u}^0_T) = \lambda\mathcal{L}(\mathbf{u}^0_T)$ for any $\lambda \in \mathbb{R}$.

To prove (3.14), we only need to show that for any $\mathbf{u}^0 \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ it holds that

$$\|\mathcal{L}(\mathbf{u}^0_T)\|_{H^1(\Omega,\mathbb{R}^3)} \le C\|\mathbf{u}^0_T\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}, \tag{3.16}$$

where $C > 0$ is a constant depending only on $\Omega$. Suppose (3.16) is not true. For each integer $m = 1, 2, \cdots$ there exists a vector field $\mathbf{u}^0_m \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ such that

$$\|\mathcal{L}(\mathbf{u}^0_{m,T})\|_{H^1(\Omega,\mathbb{R}^3)} > m\|\mathbf{u}^0_{m,T}\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}.$$

Set $\mathbf{w}^0_m = \mathbf{u}^0_m/\|\mathcal{L}(\mathbf{u}^0_{m,T})\|_{H^1(\Omega,\mathbb{R}^3)}$, then $\|\mathbf{v}_m\|_{H^1(\Omega,\mathbb{R}^3)} = \|\mathcal{L}(\mathbf{w}^0_{m,T})\|_{H^1(\Omega,\mathbb{R}^3)} = 1$ and

$$1 = \|\mathcal{L}(\mathbf{w}^0_{m,T})\|_{H^1(\Omega,\mathbb{R}^3)} > m\|\mathbf{w}^0_{m,T}\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}.$$

And thus as $m \to \infty$ we have

$$\|\mathbf{v}_{m,T}\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)} = \|\mathbf{w}^0_{m,T}\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)} \to 0. \tag{3.17}$$

Moreover, according to the trace theorem of $H^1(\Omega, \mathbb{R}^3)$, for each $\mathbf{w}^0_{m,T} \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ there exists a $\mathbf{w}_m \in H^1(\Omega, \mathbb{R}^3)$ such that $\mathbf{w}_m = \mathbf{w}^0_{m,T}$ on $\partial\Omega$ and

$$\|\mathbf{w}_m\|_{H^1(\Omega,\mathbb{R}^3)} \le C\|\mathbf{w}^0_{m,T}\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}, \tag{3.18}$$

where $C$ is a positive constant independent of $\mathbf{w}^0_{m,T}$. Since $\mathbf{v}_m \in H^1_t(\Omega, \mathrm{div}0, \mathbf{w}^0_{\mathrm{m},T}) \cap \mathbb{H}_2^{\perp}$ is the divergence-free part of a minimizer $\mathbf{u}_m$ of problem (1.1) in $H^1(\Omega, \mathbb{R}^3, \mathbf{w}^0_m)$, it holds that

$$\|\operatorname{curl}\mathbf{v}_m\|_{L^2(\Omega,\mathbb{R}^3)} = \|\operatorname{curl}\mathbf{u}_m\|_{L^2(\Omega,\mathbb{R}^3)} \le \|\operatorname{curl}\mathbf{w}_m\|_{L^2(\Omega,\mathbb{R}^3)} \le \sqrt{2}\|\mathbf{w}_m\|_{H^1(\Omega,\mathbb{R}^3)}. \tag{3.19}$$

Applying Lemma 2.3 to $\mathbf{v}_m \in H^1_t(\Omega, \mathrm{div}0, \mathbf{w}^0_{\mathrm{m},T}) \cap \mathbb{H}_2^{\perp}$, and using (3.17)–(3.19) we obtain that

$$1 = \|\mathbf{v}_m\|_{H^1(\Omega,\mathbb{R}^3)} \le C(\Omega)(\|\operatorname{curl}\mathbf{v}_m\|_{L^2(\Omega,\mathbb{R}^3)} + \|\mathbf{v}_{m,T}\|_{H^{1/2}(\partial\Omega,\mathbb{R}^3)}) \to 0, \quad \text{as } m \to \infty,$$

which is a contradiction. And hence (3.16) is established.

Step 2. The proof of (3.15) is similar as step 7 of the proof of [9, Theorem 1.3], so we omit it here. □

From Remark 3.5, Theorem 3.6 and the regularity result on $\mathbb{H}_2$ shown in [5, P. 222, Propositon 3], we know that each minimizer of problem (1.1) is composed of a "good" regularity part and a "bad" regularity part in gradient form.

**Corollary 3.7.** *Assume that $\Omega$ satisfies (O1), (O2) with $r \ge 3$, and $\mathbf{u}^0 \in H^{3/2}(\partial\Omega, \mathbb{R}^3)$. Let $\mathbf{u}$ be a minimizer of problem (1.1). Then we have the following orthogonal decomposition*

$$\mathbf{u} = \mathbf{A} + \nabla\eta, \tag{3.20}$$

*where* $\mathbf{A} \in H^2_t(\Omega, \mathrm{div}0, \mathbf{u}^0_T)$*, and* $\eta \in H^2(\Omega) \cap H^1_0(\Omega)$ *satisfies* $\nabla\eta = \mathbf{u}^0 - (\mathbf{v} + \mathbf{h})$ *on* $\partial\Omega$*.*

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## References

[1] J Bolik, W von Wahl. *Estimating* $\nabla\mathbf{u}$ *in terms of* div $\mathbf{u}$*,* curl $\mathbf{u}$*, either* $(\nu, \mathbf{u})$ *or* $\nu \times \mathbf{u}$ *and the topology*, Math Methods in the Appl Sci, 1997, 20(9): 734-744.

[2] S J Chapman. *A hierarchy of models for type-II superconductors*, SIAM Review, 2000, 42(4): 555-598.

[3] J Chen, X B Pan. *Functionals with operator curl in an extended magnetostatic Born-Infeld model*, SIAM J Math Anal, 2013, 45(4): 2253-2284.

[4] J Chen, X B Pan. *Quasilinear systems involving curl*, Proc Roy Soc Edinburgh Sect A: Mathematics, 2018, 148(2): 243-279.

[5] R Dautray, J L Lions. *Mathematical Analysis and Numerical Methods for Science and Technology (Vol. 3): Spectral Theory and Applications*, New York: Springer-Verlag, 1990.

[6] K Kang, S Kim. *On the Hölder continuity of solutions of a certain system related to Maxwell's equations*, SIAM J Math Anal, 2002, 34(1): 87-100.

[7] K Kang, S Kim. *Elliptic systems with measurable coefficients of the type of Lamé system in three dimensions*, J Diff Equation, 2011, 251: 2466-2493.

[8] F Miranda, J F Rodrigues, L Santos. *On a p-curl system arising in electromagnetism*, Discrete Contin Dyn Syst Ser S, 2012, 5(3): 605-629.

[9] X B Pan. *Minimizing curl in a multiconnected domain*, J Math Phys, 2009, 50(3): art no 033508.

[10] X B Pan, Y W Qi. *Asymptotics of minimizers of variational problems involving curl functional*, J Math Phys, 2000, 41(7): 5033-5063.

[11] H M Yin. *Regularity of solutions to Maxwell's system in quasi-stationary electromagnetic fields and applications*, Comm in PDE, 1997, 22(7&8): 1029-1053.

[12] H M Yin, B Q Li, J Zou. *A degenerate evolution system modeling Bean's critical-state type-II superconductors*, Discrete Contin Dyn Syst, 2002, 8(3): 781-794.

[13] H M Yin. *Regularity of weak solution to Maxwells equations and applications to microwave heating*, J Diff Equation, 2004, 200: 137-161.

[14] K Yosida. *Functional Analysis*,Berlin: Springer-Verlag, 6th ed, 1999.

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