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The realization of positive definite matrices via planar networks and mixing-type sub-cluster algebras

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Abstract. As an improvement of the combinatorial realization of totally positive matrices via the essential positive weightings of certain planar network by S. Fomin and A. Zelevinsky [7], in this paper, we give a test method of positive definite matrices via the planar networks and the so-called mixing-type sub-cluster algebras respectively, introduced here originally.
This work firstly gives a combinatorial realization of all matrices through planar network, and then sets up a test method for positive definite matrices by LDU-decompositions and the horizontal weightings of all lines in their planar networks. On the other hand, mainly the relationship is built between positive definite matrices and mixing-type sub-cluster algebras.

§1 Introduction

The totally positive matrices and totally nonnegative matrices are very important classes of matrices, which were firstly studied by I. J. Schoenberg in [12], connecting total nonnegativity with variation diminishing property. Then for a given totally positive matrix, F. R. Gantmacher and M. G. Krein determined its distinct positive eigenvalues in [8]. The great progress was achieved by S. Fomin and A. Zelevinsky [7] in which they discussed the parametrization of all nonnegative matrices and the tests of a matrix for total positivity. In fact they established a combinatorial approach for testing total positivity and total nonnegativity based on two structures, the planar network and the double wiring diagram. One of their most important results is that every double wiring diagram gives rise to a total positivity criterion: an $n \times n$ matrix is totally positive if and only if its all n^2 chamber minors are positive. Later on S. Fomin and A. Zelevinsky introduced the concept of cluster algebra [6]. In the light of this important innovation, the criterion of total positivity could be realized by the extended cluster of the cluster algebra with the initial seed corresponding to a double wiring diagram.

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Now a natural question is to discuss problems of partial positivity (partial nonnegativity). Indeed, A. Browsowsky and S. Chepuri studied one case of partial positivity (partial nonnegativity), that is, k-positivity (respectively. k-nonnegativity) and gave related results in the language of cluster subalgebra [3]. In this paper, we will discuss another case of partial positivity, that is, the positive definite matrices from the viewpoint of mixing-type sub-cluster algebras. We will also present new criterion of a matrix to be positive definite using the tools of planar networks and double wiring diagrams.

This paper is arranged as follows: Section 2 is devoted to some concepts and results. Section 3 is the characterization of positive definite matrices via planar networks and Section 4 is the characterization of positive definite matrices via mixing-type sub-cluster algebras, while Theorem 4.4 shows the positive definite tests depending on double wiring diagrams.

§2 Preliminaries on some concepts and results

2.1 Totally positive matrices and positive definite matrices

A square matrix over the complex field $\mathbb{C}$ is said to be *totally positive* if all its minors are positive real numbers.

An $n \times n$ Hermitian matrix M is said to be *positive definite* if z^*Mz is always a positive real number for any $0 \neq z \in \mathbb{C}^n$, where $z^* = \bar{z}^\top$. The class of positive definite matrices plays an important role since it is closely related to positive-definite symmetric bilinear forms and inner product vector spaces.

It is well-known that a Hermitian matrix M is positive definite if and only if all leading principal minors of M are positive. Hence if a totally positive matrix M is Hermitian, then it is obviously positive definite. Therefore whether a Hermitian matrix is positive definite or not can be considered as a generalization of the problem about total positivity.

2.2 The matrices associated to planar networks

Recall that the so called *planar network* is a combinatorial method S. Fomin and A. Zelevinsky used in [7], to realize the totally positive matrices.

A *planar network* (Γ, ω) is an acyclic directed planar graph Γ whose edges e are assigned scalar weights $\omega(e)$. We always assume the edges of Γ to be directed from left to right. Assume each network has n sources and n sinks located at the left and right sides in the picture respectively, and numbered bottom-to-top.

Now we associate to every planar network (Γ, ω) a matrix $x(\Gamma, \omega)$, called the *weight matrix*, more precisely, $x(\Gamma, \omega)$ is an $n \times n$ matrix whose (i, j)-entry is the sum of weights of all paths from the source i to the sink j, where the weight of a (directed) path in Γ is the product of weights of all edges in this path. For examples, Figure 1 includes two planar networks.

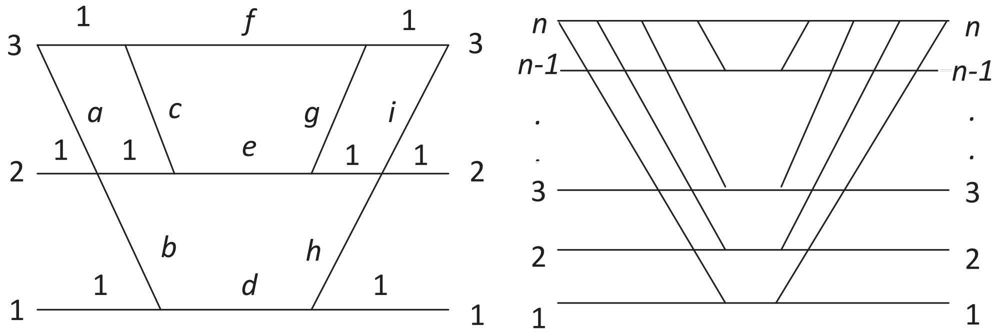

Figure 1

The corresponding weight matrix of the planar network on the left of Figure 1 is written as

$$\begin{pmatrix} d & dh & dhi \\ bd & bdh+e & bdhi+eg+ei \\ abd & abdh+ae+ce & abdhi+(a+c)e(g+i)+f \end{pmatrix}.$$

In order to calculate minors of the weight matrix, the following lemma is useful, where $\Delta_{I,J}(x)$ denotes the minor of a matrix x with the row set I and the column set J and the weight of a collection of directed paths in Γ is defined to be the product of their weights.

Lemma 2.1. (Lemma 1, [7]) A minor $\Delta_{I,J}(x)$ of the weight matrix of a planar network is equal to the sum of weights of all collections of vertex-disjoint paths that connect the sources labeled by I with the sinks labeled by J.

Remark 2.2. *For the planar network in the above example, or more generally the planar network* (Γ, ω) *of the form "*$\backslash\backslash--//$*", the leading principal minors of the weight matrix* $x(\Gamma,\omega)$ *can be calculated by Lemma 2.1 as*

$$\Delta_{1,1} = \prod_{i=1}^{t_1} \omega_{1i}, \Delta_{12,12} = \prod_{i=1}^{t_1} \omega_{1i} \prod_{i=1}^{t_2} \omega_{2i}, \cdots, \Delta_{12\cdots n, 12\cdots n} = \prod_{i=1}^{t_1} \omega_{1i} \prod_{i=1}^{t_2} \omega_{2i} \cdots \prod_{i=1}^{t_n} \omega_{ni} \qquad (1)$$

where the horizontal weightings $\omega_{k1}, \omega_{k2}, \cdots, \omega_{kt_k}$ *appear in Line* k *for* $k = 1, 2, \cdots, n$*, respectively. Indeed, the collections of vertex-disjoint paths that connect the sources labeled by* $\{12\cdots s\}$ *with the sinks labeled by* $\{12\cdots s\}$*,* $1 \le s \le n$ *must be horizontal, then the results (1) follows.*

In order to state the one-to-one correspondence between the set of totally positive matrices and that of all positive weightings of a fixed planar network, we need more notations.

Denote the planar network on the right of Figure 1 as Γ_0 and call an edge of Γ_0 *essential* if it is either slanted or one of the n horizontal edges in the middle of Γ_0. Note that Γ_0 has exactly $2 \times (1 + 2 + \cdots + (n-1)) + n = n^2$ essential edges. Moreover, a weighting ω of Γ_0 is called *essential* if $\omega(e) \neq 0$ for any essential edge e and $\omega(e) = 1$ for all other edges.

The following theorem in [7] characterizes all totally positive $n \times n$ matrices via the essential positive weightings of the planar networks Γ_0.

Theorem 2.3. (Theorem 5, [7]) The map $\omega \mapsto x(\Gamma_0, \omega)$ restricts to a bijection between the set of all essential positive weightings of Γ_0 and that of all totally positive $n \times n$ matrices.

2.3 Cluster algebras

The original motivation of cluster algebras was to give the combinatorial characterizations of the dual canonical basis of the quantized enveloping algebra and of the total positivity for algebraic groups. Now, it plays a prominent role in cluster theory and arises in connection with integrable systems, algebraic Lie theory, Poisson geometry, total positivity, representation theory, combinatorics and etc.

Throughout this paper, we only restrict our attention to cluster algebras given by the so-called *cluster quivers* Q, which are directed (multi-)graphs with no loops and no 2-cycles. Some vertices of Q are denoted as *mutable* and the remaining ones are called *frozen*. The terminology needed in this paper is collected in the following definition, see [6].

Definition 2.4. *(i) Let v be a mutable vertex of a cluster quiver Q. The* quiver mutation *of Q at v is an operation that produces another quiver $Q' = \mu_v(Q)$ via the following three steps:*

(1) add a new edge $u \to w$ for each pair of edges $u \to v$, $v \to w$ in Q, except in the case when both u, w are frozen.

(2) reverse the direction of all edges adjacent to v.

(3) remove 2-cycles until none remains.

(ii) Let $\mathcal{F} \supset \mathbb{R}$ be any field. We say that a pair $t = (Q, \mathbf{z})$ is a seed *in $\mathcal{F}$ if Q is a cluster quiver and $\mathbf{z}$, called the* extended cluster*, is a set consisting of some algebraically independent elements of $\mathcal{F}$, as many as the vertices of the quiver Q.*

(iii) The elements of $\mathbf{z}$ corresponding to the mutable vertices in Q are called cluster variables *and those corresponding to the frozen vertices in Q are called* frozen variables.

(iv) The seed mutation *μ_z at a cluster variable z transforms the seed $t = (Q, \mathbf{z})$ into a new seed $t' = (Q', \mathbf{z}') = \mu_z(Q, \mathbf{z})$, where Q' is the quiver resulted after mutating Q at the vertex corresponding to z and $\mathbf{z}' = \mathbf{z} \cup \{z'\} \backslash \{z\}$, where the new variable z' is given by the* exchange relation at z , $zz' = \prod_{z \leftarrow x} x + \prod_{z \to y} y$. *One sometimes calls this relation the* exchange relation at *v for the vertex v of Q corresponding to z.*

(v) Two seeds $t = (Q, \mathbf{z})$ and $t' = (Q', \mathbf{z}')$ are said to be mutation-equivalent *if one of them can be obtained from the other after a sequence of seed mutations.*

(vi) The cluster algebra $\mathcal{A}(Q, \mathbf{z})$ *generated by an* initial seed $t = (Q, \mathbf{z})$ *is defined as the subring of $\mathcal{F}$ generated by the elements of extended clusters which are mutation-equivalent to t.*

§3 Characterization of positive definite matrices via planar networks

3.1 Realization of matrices via generalized elementary Jacobi matrices

In this section, we first realize all $n \times n$ matrices via certain planar networks and then give a way to test whether a given Hermitian matrix is positive definite or not.

For any $n \times n$ matrix M, we attempt to find a planar network (Γ, ω) with weight matrix $x(\Gamma, \omega) = M$. For this purpose, we first consider the case of invertible matrices.

Since any invertible matrix can be written as a product of elementary matrices, let us recall the definition of elementary Jacobi matrices and their connection with the planar networks as in [7].

Definition 3.1. *(i) Let $E_{i,j}$ be the $n \times n$ matrix whose (i, j)-entry is 1 and all other entries are 0. For $i = 1, 2, \cdots, n-1$ and $t \in \mathbb{C}$, let $x_i(t) = I + tE_{i,i+1}$, $x_{\bar{i}}(t) = I + tE_{i+1,i}$. For $i = 1, 2, \cdots, n$ and $t \in \mathbb{C}\backslash\{0\}$, let $x_{\textcircled{1}}(t) = I + (t-1)E_{i,i}$.*

(ii) The elementary Jacobi matrices *are matrices $x \in GL_n(\mathbb{C})$ that differ from the identity matrix I_n in a single entry located either on the main diagonal or immediately above or below it, more precisely, those matrices are of the forms $x_i(t), x_{\bar{i}}(t), x_{\textcircled{1}}(t')$, $\forall t \in \mathbb{C}, t' \in \mathbb{C}\backslash\{0\}$.*

Lemma 3.2. (Theorem 12, [7]) The three classes of elementary Jacobi matrices are weight matrices of the following three "chip"s (See Figure 2 below), respectively. In each "chip", all edges but one have weight 1 and the distinguished edge has weight t. Slanted edges connect horizontal levels i and $i+1$, counting from the bottom. Additionally, the concatenation of these "chip"s corresponds to multiplying their weight matrices.

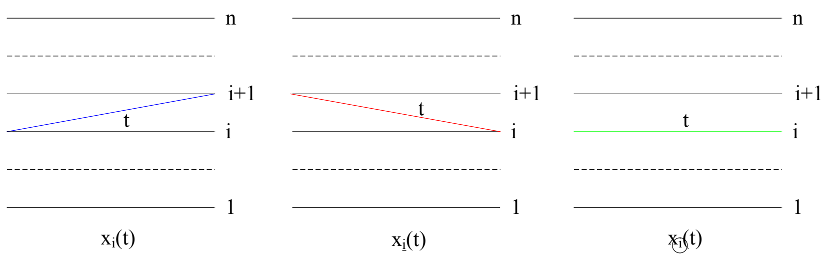

Figure 2

Therefore, any invertible matrix generated by some elementary Jacobi matrices can be realized as the weight matrix for a planar network. Then we have the following characterization for $GL_n(\mathbb{C})$.

Proposition 3.3. For any integer $n > 1$, the general linear group $GL_n(\mathbb{C})$ can be generated by all elementary Jacobi matrices, that is,

$$GL_n(\mathbb{C}) = \langle x_i(t), x_{\bar{i}}(t), x_{\textcircled{1}}(t'), t \in \mathbb{C}, t' \in \mathbb{C}\backslash\{0\}\rangle.$$

Proof. Clearly, $GL_n(\mathbb{C}) \supseteq \langle x_i(t), x_{\bar{i}}(t), x_{①}(t'), t \in \mathbb{C}, t' \in \mathbb{C}\backslash\{0\}\rangle$.

Since any matrix in GL_n can be written as a product of elementary matrices, it suffices to show that any one of elementary matrices of three types can be generated by some elementary Jacobi matrices.

The case 1: The type of row multiplication.

Obviously, all elementary matrices of this type have the form $x_{①}(t)$ for $t \in \mathbb{C}\backslash\{0\}$.

The case 2: The type of row addition.

For any $1 \leq i, j \leq n$, $i \neq j$, without loss of generality we only consider the case $i < j$. We will use the induction method on the number $j - i$.

If $j - i = 1$, then the result is trivial due to $x_i(t) = I + tE_{i,i+1}$.

In general, assume that the conclusion is true for $j - i \leq k$, then considering the case for $j - i = k + 1$, we have:

$$\begin{aligned} I + tE_{i,i+k+1} &= I + E_{i+k,i+k+1} + tE_{i,i+k+1} - E_{i+k,i+k+1} \\ &= (I + tE_{i,i+k})(I + E_{i+k,i+k+1})(I - tE_{i,i+k})(I - E_{i+k,i+k+1}) \end{aligned}$$

due to $E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}$, where $\delta_{j,k}$ is the Kronecker symbol. By induction, it means that any such elementary matrix can be generated by some elementary Jacobi matrices.

The case 3: The type of row switching.

The elementary matrix for exchanging the rows i and j is just equal to the matrix $I + E_{i,j} + E_{j,i} - E_{i,i} - E_{j,j}$, moreover, which can be represented as

$$\begin{aligned} I + E_{i,j} + E_{j,i} - E_{i,i} - E_{j,j} &= I + E_{j,i} - E_{i,j} - E_{i,i} - E_{j,j} - 2E_{j,j} + 2E_{i,j} + 2E_{j,j} \\ &= (I - E_{i,j})(I + E_{j,i})(I - E_{i,j})(I - 2E_{j,j}). \end{aligned}$$

Then the result follows similarly as in case (2). □

Since all elementary Jacobi matrices are invertible, a non-invertible matrix cannot be represented as their product. So, in order to consider all matrices, we need to generalize the notion of elementary Jacobi matrices.

Definition 3.4. *Define a* generalized elementary Jacobi matrix *in $M_n(\mathbb{C})$ to be a matrix that differs from the identity matrix I in a single entry located either on the main diagonal or immediately above or below it, more precisely, that is one of the form $x_i(t), x_{\bar{i}}(t), x_{①}(t)$ for $t \in \mathbb{C}$.*

Proposition 3.5. For any positive integer n, any $n \times n$ matrix M over $\mathbb{C}$ can be generated by some generalized elementary Jacobi matrices, that is, $M_n(\mathbb{C}) = \langle x_i(t), x_{\bar{i}}(t), x_{①}(t), \ \forall t \in \mathbb{C}\rangle$.

Proof. We know that $M = M_1 \begin{pmatrix} I_r & \\ & O_{n-r} \end{pmatrix} M_2$, where both M_1, M_2 are invertible, r is the rank of M and O_{n-r} is the zero matrix of order $n - r$. The result follows from Proposition 3.3 and $\begin{pmatrix} I_r & \\ & O_{n-r} \end{pmatrix} = \prod_{s=r+1}^{n} x_{ⓢ}(0)$. □

By Lemma 3.2 and Proposition 3.5, we get the following

Corollary 3.6. For any matrix M, there exists a planar network (Γ,ω) with weight matrix $x(\Gamma,\omega)=M$.

3.2 The criterion for positive definite matrices via planar networks

We have already seen by Corollary 3.6, that for any matrix M, there exists a planar network (Γ,ω) such that its weight matrix $x(\Gamma,\omega)=M$, this correspondence is neither injective nor surjective in general. Thus, by Lemma 2.1 we can read the leading principal minors $\Delta_{\{1\},\{1\}},\cdots,\Delta_{[1,n],[1,n]}$ directly from the planar network (Γ,ω). From the basic fact that a Hermitian matrix is positive definite if and only if all of its leading principal minors are positive, we can obtain an algorithm for checking whether a given matrix M is positive definite or not, which gives us a new viewpoint to observe positive definite matrices.

Algorithm 3.7. (1) Decompose $M=M_1M_2\cdots M_t$ for some generalized elementary Jacobi matrices $M_1,M_2,\cdots,M_t$;

(2) By the correspondences between "chips" and generalized elementary Jacobi matrices, draw the planar network (Γ,ω) with $x(\Gamma,\omega)=M$;

(3) Calculate the leading principal minors directly from (Γ,ω) by the method given in Lemma 2.1;

(4) Check whether a given matrix M is positive definite or not.

Additionally, by [11], we obtain that for any $n\times n$ matrix M, M can be decomposed as $M=U_1LU_2$ or $M=L_1UL_2$ with L,L_1,L_2 being lower triangular matrices and U,U_1,U_2 being upper triangular matrices. Thus the planar network has the forms "// \\ //" or "\\ // \\", respectively. Furthermore, we have the following

Proposition 3.8. For any $n\times n$ invertible matrix M, the following statements are equivalent.

(i) M has a LDU-decomposition, i.e., $M=LDU$ with L a unit lower triangular matrix, U a unit upper triangular matrix and D a diagonal matrix.

(ii) The corresponding planar network (Γ,ω) of M is of the form "\\ − −//".

(iii) All the leading principal minors of M are nonsingular.

Proof. Firstly, the equivalence of (i) and (iii) can be found in [9].

(i)$\Longrightarrow$(ii): Let $M=LDU$ be a LDU-decomposition. According to the proof of Proposition 3.3, we have the decompositions $L=L_1L_2\cdots L_r$, $U=U_1U_2\cdots U_s$, $D=D_1D_2\cdots D_t$, where $L_1,\cdots,L_r$ are in the form $x_{\bar{i}}(\gamma)$, $U_1,\cdots,U_s$ are in the form $x_i(\gamma)$ and $D_1,\cdots,D_t$ are in the form $x_{①}(\gamma)$ respectively. Thus the corresponding planar network (Γ,ω) of M is in the form "\\ − −//".

(ii)$\Longrightarrow$(iii): Since the corresponding planar network (Γ,ω) of M is in the form "\\ − −//" and M is invertible, by Remark 2.2 we have

$$\Delta_{12\cdots n,12\cdots n}=\prod_{i=1}^{t_1}\omega_{1i}\prod_{i=1}^{t_2}\omega_{2i}\cdots\prod_{i=1}^{t_n}\omega_{ni}=\det(M)\neq 0$$

where the horizontal weightings $\omega_{k1}, \omega_{k2}, \cdots, \omega_{kt_k}$ appear in Line k for $k = 1, 2, \cdots, n$ respectively. Hence all the leading principal minors

$$\Delta_{1,1} = \prod_{i=1}^{t_1} \omega_{1i}, \Delta_{12,12} = \prod_{i=1}^{t_1} \omega_{1i} \prod_{i=1}^{t_2} \omega_{2i}, \cdots, \Delta_{12\cdots n,12\cdots n} = \prod_{i=1}^{t_1} \omega_{1i} \prod_{i=1}^{t_2} \omega_{2i} \cdots \prod_{i=1}^{t_n} \omega_{ni}$$

are nonsingular. □

It is easy to see that if an $n \times n$ matrix has an LDU-decomposition with $D = \text{diag}(d_1, d_2, \cdots, d_n)$, $M = LDU$, then the leading principal minors of M are $\Delta_{12\cdots k,12\cdots k} = d_1 d_2 \cdots d_k$ for $k = 1, 2, \cdots, n$. Moreover , if M is invertible, then by Proposition 3.8 and Remark 2.2, we have $d_k = \prod_{i=1}^{t_k} \omega_{ki}$ for all $1 \le k \le n$. We then have the following

Corollary 3.9. An invertible Hermitian matrix M with LDU-decomposition is positive definite if and only if the weightings $d_k = \prod_{i=1}^{t_k} \omega_{ki}$ of Line k in the corresponding planar network (Γ, ω) of M are positive for all $1 \le k \le n$.

We end this section with the following example to illustrate the mentioned algorithm of checking positive definite matrices.

Example 3.10. Let $M = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 8 \\ 4 & 8 & 18 \end{pmatrix}$. We know that M can be decomposed as

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= x_{\bar{1}}(2) x_{\bar{2}}(-1) x_{\bar{1}}(-4) x_{\bar{2}}(1) x_{\bar{1}}(4) x_{\textcircled{2}}(2) x_{\textcircled{3}}(2) x_1(4) x_2(1) x_1(-4) x_2(-1) x_1(2) \end{aligned}$$

Next, the corresponding planar network (Γ, ω) is shown as in Figure 3, where the weightings $\omega(e) = 1$ for all unlabelled edges e.

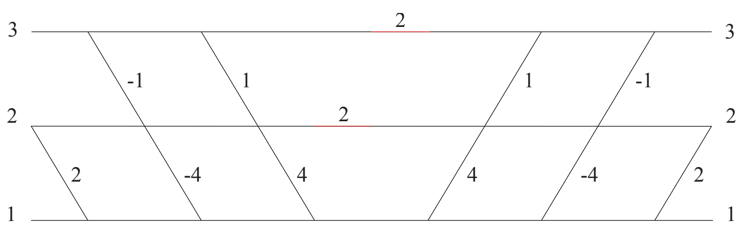

Figure 3

Finally, we have $d_1 = 1, d_2 = 2, d_3 = 2$. Now by Corollary 3.9, we know that M is positive definite.

§4 Characterization of positive definite matrices via mixing-type sub-cluster algebras

In [5], S. Fomin showed that total positivity is one of his motivations of studying cluster algebras. In brief, we take a complex variety X together with a family $\mathbf{\Delta}$ of "important" regular functions on X. The corresponding totally positive variety $X_{>0}$ is the set of points at which all of these functions take positive values, that is, $X_{>0} = \{x \in X : \Delta(x) > 0 \text{ for all } \Delta \in \mathbf{\Delta}\}$. If X is the affine space of matrices of a given size, e.g., if $X = M_n(\mathbb{C})$, and $\mathbf{\Delta}$ is the set of all minors, then we recover the classical notation of totally positive matrices.

The question arises naturally is that for any element $x \in X$, should we check all the inequalities $\Delta(x) > 0$ for all $\Delta \in \mathbf{\Delta}$ to test whether $x \in X_{>0}$ or not?

In the above example, there are $\binom{2n}{n} - 1$ minors in total. However, S. Fomin and A. Zelevinsky in [7] told us that it is enough to check certain n^2 minors in order to make sure that a given matrix is totally positive. These minors are the chamber minors of the corresponding double wiring diagram. This is because all the minors of $M_n(\mathbb{C})$ can be represented by rational functions of these n^2 minors with positive coefficients.

4.1 Double wiring diagram and its associated quiver

Recall that the *double wiring diagram* defined in [7] is a diagram that consists of two families of n piecewise-straight lines (each family is colored with one of two colors), the crucial requirement is that each pair of lines with the same color intersect exactly once. Moreover, the lines in a double wiring diagram are numbered separately within each color.

Now we give the definition of *chamber* and *chamber minor* of a double wiring diagram as follows.

Definition 4.1. *Let Ω be a double wiring diagram and $\square$ be a region of Ω. We then assign to $\square$ a pair of subsets I, J of the set $[1, n] = \{1, \cdots, n\}$ such that each subset indicates the line numbers of the corresponding color passing below the $\square$. If $|I| = |J|$, then the region $\square$ is called a* chamber *and in this case, the minor $\Delta_{I,J}$ of an $n \times n$ matrix $x = (x_{ij})$ is called the* chamber minor *of x.*

For example, we show all the chambers of the double wiring diagram in Figure 4 and the corresponding nine chamber minors (the total number is always n^2) are x_{31}, x_{21}, x_{11}, x_{13}, $\Delta_{23,12}$, $\Delta_{12,12}$, $\Delta_{12,13}$, $\Delta_{12,23}$ and $\Delta_{123,123} = \det(x)$.

In [4], or more generally in Section 2 of [2], we can associate a quiver $Q = Q(\Omega)$ to a double wiring diagram Ω as follows:

(i) The vertices of Q are the chambers of Ω.

(ii) There is an edge between two chambers c and c' of Q in the following cases:

(1) They are adjacent chambers in the same row. If the color of the crossing between them is blue, the edge is directed to the left. Otherwise, it is directed to the right.

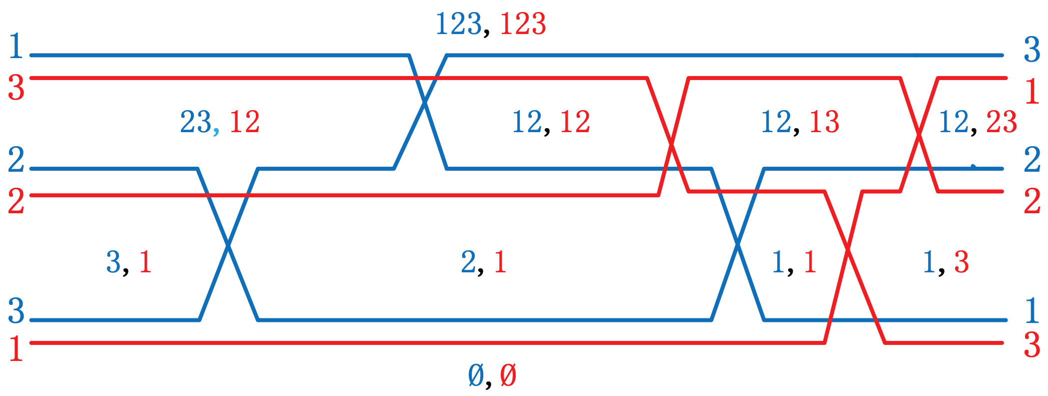

Figure 4

(2) If c' has left and right boundaries of different color and lies completely above (or below) c. If the left boundary of c' is blue, the edge is directed from c to c'. Otherwise, it is directed from c' to c.

(3) If the left boundary of c' is above c and the right boundary of c is below c' and both boundaries have the same color. If such common color is blue, the edge is directed from c to c'; otherwise, it is directed from c' to c.

(4) If the right boundary of c' is above c and the left boundary of c is below c' and both boundaries have the same color. If such common color is blue, the edge is directed from c' to c; otherwise, it is directed from c to c'.

Then according to (i) and (ii), we get a cluster quiver Q from Ω.

The mutable vertices of $Q(\Omega)$ are the bounded chambers in Ω . Then the frozen vertices are the chambers which are unbounded. The following figure on the left shows the quiver associated to the double wiring diagram in Figure 4, where the vertices surrounded by $\square$ are frozen and the others are mutable.

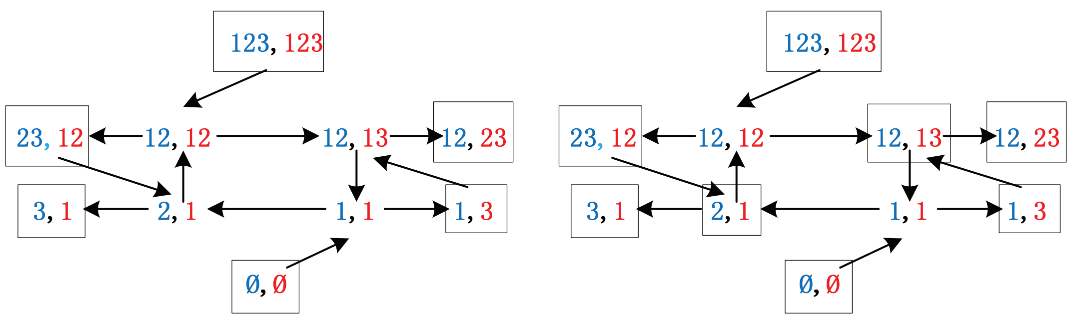

Figure 5

In [2], A. Berenstein, S. Fomin and A. Zelevinsky showed that the $\binom{2n}{n} - 1$ minors of a matrix M correspond to cluster variables or frozen variables in the cluster algebra $\mathcal{A}(\Omega)$ generated by the initial seed $(Q(\Omega), \mathbf{z}(\Omega))$ associated to a double wiring diagram Ω, where $\mathbf{z}(\Omega)$

is the collection of all chamber minors in Ω , thus it is enough to check the positivity of these n^2 chamber minors in $\mathbf{z}(\Omega)$ to ensure that M is totally positivity.

4.2 Mixing-type sub-cluster algebras and positive definite matrices

In what follows we replace the "totally positive matrices" by other classes of matrices which share some properties about "partial positivity", such as positive definite matrix, P-matrix and so on.

In order to give a similar result for positive definite matrices, we first introduce some notation. Let $\mathcal{A} = \mathcal{A}(Q, \mathbf{z})$ be the cluster algebra generated by an initial seed $t = (Q, \mathbf{z})$ as in Definition 2.4. Let $Q_0 = Q_{0,m} \cup Q_{0,f}$ and $\mathbf{z} = \mathbf{z}_m \cup \mathbf{z}_f$, where $Q_0, Q_{0,m}, Q_{0,f}$ represents the vertices, the mutable vertices, the frozen vertices of Q respectively and $\mathbf{z}_m, \mathbf{z}_f$ correspond to the mutable part, the frozen part of $\mathbf{z}$ respectively. In the light of the previous notation we will refer to the initial seed as $t = (Q, z_m, z_f)$. Now we raise the definition of the so-called *mixing-type sub-seed.*

Definition 4.2. *Keeping our foregoing notation. Let* $t = (Q, \mathbf{z}_m, \mathbf{z}_f)$ *be a given seed, we assume* $I_0 \subseteq \mathbf{z}_m$,$I_1 \subseteq \mathbf{z}$, *with* $I_0 \cap I_1 = \phi$ *and* $I_1 = I_1' \cup I_1''$ *for* $I_1' = \mathbf{z}_m \cap I_1$ *and* $I_1'' = \mathbf{z}_f \cap I_1$. *Denoting* $\mathbf{z}_m' = \mathbf{z}_m \backslash (I_0 \cup I_1')$, $\mathbf{z}_f' = (\mathbf{z}_f \cup I_0) \backslash I_1$,$\mathbf{z}' = \mathbf{z}_f' \cup \mathbf{z}_m'$. Q' *is the subgraph of* Q *with vertices corresponding to* $\mathbf{z}'$, *then the seed* $t_{I_0,I_1} = (Q', \mathbf{z}_m', \mathbf{z}_f')$ *is called* (I_0, I_1) **mixing-type sub-seed** *or* (I_0, I_1) **-type sub-seed** *of the seed* $t = (Q, \mathbf{z}_m, \mathbf{z}_f)$.

In case $I_1 = \phi$, *the sub-seed* $t_{I_0,\phi}$ *is called* **a pure sub-seed**, *and in case* $I_0 = \phi$, *the sub-seed* t_{ϕ,I_1} *is called* **a partial sub-seed** *of the seed* t.

We now give the definition of the sub-algebra related to a (I_0, I_1) mixing-type sub-seed.

Definition 4.3. *Let* $\mathcal{A} = \mathcal{A}(t)$ *be the cluster algebra with the initial seed* $t = (Q, \mathbf{z}_m, \mathbf{z}_f)$, *and let* t_{I_0,I_1} *be a* (I_0, I_1) *mixing-type sub-seed of the seed* t, *the cluster algebra* $\mathcal{A}_{I_0,I_1} = \mathcal{A}(t_{I_0,I_1})$ *with the initial seed* t_{I_0,I_1} *is called a* (I_0, I_1) **mixing-type sub-cluster algebra**.

Remark 4.4. *In general the definition of the mixing-type sub-seed was initially set for seeds with sign-skew symmetric matrices and the definition of the mixing-type sub-cluster algebra was considered up to isomorphism between cluster algebras, but we restrict these definitions in a way that keeps their essence and serves directly the purposes of this paper, for more detailed information about the general case we refer to* [1],[10].

Definition 4.5. *Let* $t_{I_0,I_1} = (Q', \mathbf{z}_m', \mathbf{z}_f')$ *be a* (I_0, I_1) *mixing-type sub-seed of the seed* $t = (Q, \mathbf{z}_m, \mathbf{z}_f)$. *Then an* **isolated vertex** *is a vertex from the set* $\mathbf{z}'$ *which was originally satisfying one or more of three following cases:*

1. *a frozen vertex of the set* $\mathbf{z}_f$ *only connected to vertices of the set* I_0.

2. *a vertex of the set* I_0 *only connected to vertices of the same set .*

3. *a vertex of the set* $\mathbf{z}$ *only connected to vertices of the set* I_1.

The variables of the extended clusters of the cluster algebra $\mathcal{A}_{I_0,I_1} = \mathcal{A}(t_{I_0,I_1})$ *corresponding to isolated vertices are called* **isolated variables**.

Remark 4.6. *The isolated variables play no role in the exchange relation that produces new cluster variables since they are either corresponding to mutable vertices which are not connected to any other vertices or corresponding to frozen vertices connected, and if connected, only to other frozen vertices.*

Example 4.7. Let $\mathcal{A} = \mathcal{A}(\Omega) = \mathcal{A}(Q, \mathbf{z})$, where Ω is the double wiring diagram of Figure 4 and $Q = Q(\Omega)$ is the cluster quiver on the left side of Figure 5 and $\mathbf{z}$ is the extended cluster in correspondence with its vertices. Then by choosing $I_0 = \{12, 13;\ 2, 1\}, I_1 = \phi$, the corresponding (I_0, ϕ) mixing-type sub-cluster algebra is the cluster algebra $\mathcal{A}_{I_0,I_1} = \mathcal{A}(Q', \mathbf{z}')$, where Q' is the cluster quiver on the right side of Figure5 and $\mathbf{z}'$ is the extended cluster in correspondence with its vertices .The isolated vertices of the quiver Q' are $\{12, 23; 3, 1\}$.

From this example on the 3×3 Hermitian matrix M, it is easy to see that in order to test whether M is positive definite or not, it suffices to check only the positivity of the non isolated variables of any extended cluster from the mixing-type sub-cluster algebra $\mathcal{A}(Q', \mathbf{z}')$.

In what follows we aim to produce new criterions for positive definite matrices depending on the a mixing-type sub-cluster algebra of a cluster algebra obtained from a double wiring diagram.

Definition 4.8. *A* **positive definite test** $\mathcal{S}$ *is a set of minors of a given Hermitian matrix which guarantees by the positivity of its elements that this matrix is positive definite.*

Remark 4.9. *We remark that for a matrix* M *and a positive definite test* $\mathcal{S}$*, by definition if all the minors in* $\mathcal{S}$ *of* M *is positive, then* M *is a positive definite matrix. However, the converse is not true, that is, if some minor in* $\mathcal{S}$ *of* M *is* ≤ 0*, then* M *may also be positive definite.*

For example, let $\mathcal{S} = \{2, 2; 1, 2; 2, 1; 12, 12\}$ *be a positive definite test and* $M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. *Although* $M_{21} = -1 < 0$*, it is easy to see that* M *is also positive definite.*

Finally, for any $n \times n$ Hermitian matrix M we can obtain the following conclusion.

Theorem 4.10. Let $n > 0$ be an integer. Then one can construct a double wiring diagram Ω with a cluster algebras $\mathcal{A} = \mathcal{A}(Q(\Omega), \mathbf{z})$ and a subset I_0 of $Q(\Omega)_{0,m}$ such that the non isolated variables of any extended cluster of the (I_0, ϕ) mixing-type sub-cluster algebra give rise to a positive definite test for any given $n \times n$ Hermitian matrix M.

Proof. Firstly, we construct the double wiring diagram Ω as follows: for each row between two wires, all the blue crossings are on the left side of the red crossings. Then there are $n-1$ such rows and for each row, there exists a unique chamber with a blue left boundary and a red right boundary. Thus we get $n-1$ such chambers, which correspond by one-to-one to the $n-1$ minors $\Delta_{1,1}, \Delta_{12,12}, \cdots, \Delta_{12\cdots(n-1),12\cdots(n-1)}$.

Now we choose I_0 such that $I_0 = Q(\Omega)_{0,m}\backslash\{1,1;\ 12,12;\ \cdots;\ 12\cdots(n-1), 12\cdots(n-1)\}$. Since the intial seed of the (I_0, ϕ) mixing-type sub-cluster algebra contains all the leading principal minors of M as cluster variables (mutable vertices), and by the fact that every variable in any extended cluster can be expressed as a free subtraction rational function with the non isolated variables of the initial extended cluster, the result follows. □

Remark 4.11. *In the proof of the previous theorem, there is only one mutable vertex in each row of the double wiring diagram of the mixing-type sub-cluster algebra, the thing that ensures having isolated vertices, hence the number of variables that requested to be positive in the positive definite test is less than* n^2 *and this the main difference between the total positivity test and the positive definite test.*

§5 Further Remarks as Summary

For any $n \times n$ matrix M, on one hand, all its $\binom{2n}{n} - 1$ minors share the cluster algebra structure via a double wiring diagram Ω and its associated quiver $Q(\Omega)$. Thus in order to check some "positivity" properties about the minors of M, it suffices to check the "positivity" of some minors which appears in any extended cluster in the corresponding mixing-type sub-cluster algebra. On the other hand, for the matrix M, there exists a planar network (Γ,ω) with weight matrix $x(\Gamma,\omega) = M$ and we can calculate any minor $\Delta_{I,J}$ of M for $I, J \subseteq [1, n]$ by Lemma 2.1. Therefore, we can summarise the results of this paper by the following Figure.

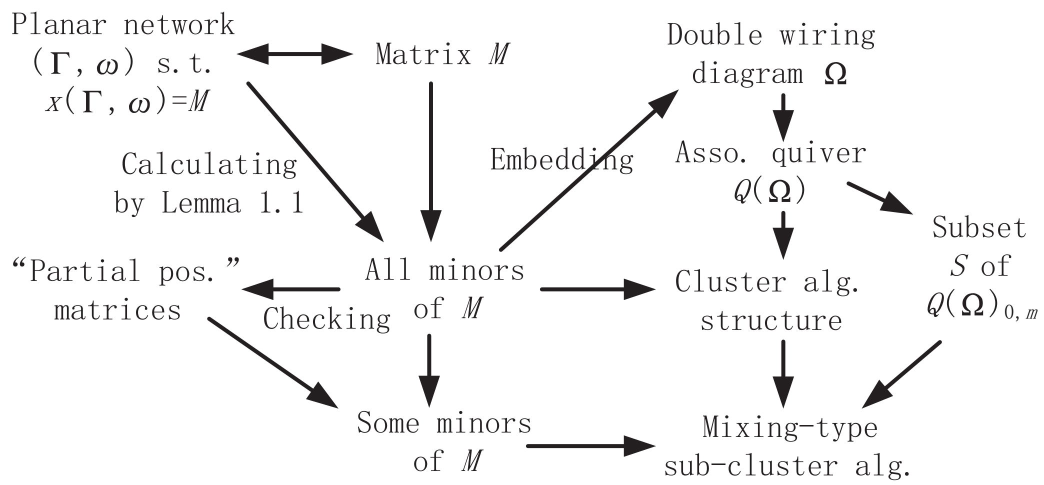

Figure 6

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References

[1] I Assem, G Dupont, R Schiffler. *On a category of cluster algebras*, J Pure Appl Algebra, 2014, 218: 553-582.

[2] A Berenstein, S Fomin, A Zelevinsky. *Cluster Algebras III: Upper Bounds and Double Bruhat Cells*, Duke Math J, 2005, 126: 1-52.

[3] A Browsowsky, S Chepuri. *Parametrization of K-nonnegative matrices: Cluster algebra and k-positivity test*, arXiv:1712.05037v1.

[4] C Cuenca. http://www.math.umn.edu/∼reiner/REU/Cuenca2013.pdf.

[5] S Fomin. *Total positivity and cluster algebras (English summary)*, Proceedings of the International Congress of Mathematicians, Volume II, 125-145, Hindustan Book Agency, New Delhi, 2010.

[6] S Fomin, A Zelevinsky. *Cluster algebras I: Foundations*, J Amer Math Soc, 2002, 15: 497-529.

[7] S Fomin, A Zelevinsky. *Total Positivity: Tests and Parametrizations*, Math Intell, 2000, 22: 23-33.

[8] F R Gantmacher, M G Krein. *Sur les matrices osciilatoires*, C R Acad Sci(Paris), 1935, 201: 577-579.

[9] R A Horn, C R Johnson. *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

[10] M Huang, F Li, Y Yang. *On structure of cluster algebras of geometric type I: in view of sub-seeds and seed homomorphisms*, Sci China Math, 2018, 61: 831-854.

[11] P Okunev, C R Johnson. *Necessary and sufficient conditions for existence of the LU factorization of an arbitrary matrix*, 1997.

[12] I J Schoenberg. *Uber variationsvermindernde lineare transformationen*, Math Z, 1930, 32: 312-328.

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