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Propagation of traveling wave solutions for nonlinear evolution equation through the implementation of the extended modified direct algebraic method

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Abstract. In this work, different kinds of traveling wave solutions and uncategorized soliton wave solutions are obtained in a three dimensional (3-D) nonlinear evolution equations (NEEs) through the implementation of the modified extended direct algebraic method. Bright-singular and dark-singular combo solitons, Jacobi's elliptic functions, Weierstrass elliptic functions, constant wave solutions and so on are attained beside their existing conditions. Physical interpretation of the solutions to the 3-D modified KdV-Zakharov-Kuznetsov equation are also given.

§1 Introduction

Recently, some researchers have devoted themselves to the study of the exact solutions of several nonlinear evolution equations (NEEs), that play a critical part in explaining the characteristics of problems in areas of applied mathematics and mathematical physics. The nonlinear 3-D modified Korteweg-de Vries-Zakharov-Kuznetsov (mKdV-ZK) equation is a significant equation that has several physical occurrences also waves in nonlinear LC circuit through the mutual inductance between adjacent inductors, ion acoustic waves in plasma physics, fluid dynamics, nonlinear optic etc.[1, 2]. Attaining the exact and numerical solutions for NEEs shows a vital part in the research of physical occurrences and has progressively turned into critical and important responsibility. In contrast, the soliton and solitary wave solution of NEEs problems can help people understand these occurrences better than numerical solutions. Thus, the research of the soliton and solitary wave solutions of the NEEs plays a vital part in the learning of these physical occurrences, and various influential methods have been established to help find the exact solutions, for instance, the homotopy perturbation scheme, Bäcklund transformation, extended $\left(\frac{G'(\phi)}{G(\phi)}\right)$-expansion system, generalized algebraic scheme, Hirota's bilinear scheme, direct algebraic method, Jacobi elliptic function expansion scheme, inverse scattering

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scheme, Painlevé scheme, homogeneous balance scheme, etc. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] are all well-organized algorithms for finding soliton and solitary wave solutions of an excessive number of NEEs [16, 17, 18, 19, 20, 21]. Many other references on exact solutions to PDEs as Diversity of exact solutions to a (3+1)-dimensional nonlinear evolution equation and its reduction [22]; Backlund transformation, multiple wave solutions and lump solutions to a (3 + 1)-dimensional nonlinear evolution equation [23]; Constructing lump solutions to a generalized Kadomtsev-Petviashvili-Boussinesq equation [24]; Resonant behavior of multiple wave solutions to a Hirota bilinear equation [25]; Rational solutions to an extended Kadomtsev-Petviashvili-like equation with symbolic computation [26]; A note on rational solutions to a Hirota-Satsuma-like equation [27]. Many researchers consider exact travelling wave solutions to a nonlinear wave equation by reducing the PDE into an integrable ODE. Such an idea was systematically studied in the transformed rational function method theory and all one needs to do is to choose an integrable ODE. In most cases, one uses the Riccati equation as in the expansion method [28]. Recently, various studies show the remarkable richness of lump solutions (see, e.g., [29]) and a search for lump solutions to a combined fourth-order nonlinear PDE in (2+1)-dimensions, reference [30] for details on lumps in (2+1)-dimensions; and reference [31] for lumps in in (3+1)-dimensions and their interaction solutions with homoclinic and heteroclitic solutions (see, [32] for (2+1)-dimensional nonlinear dispersive wave equations).

Based on the motivation from the literature, this work implements the modified extended direct algebraic method to attain soliton and additional solutions of the leading nonlinear mKdV-ZK equation. Consequently, novel and uncategorized wave solutions have been attained. The following subsections provide a brief detail about the main equation and overview of the method.

1.1 The main equation

The main mKdV-ZK equation in three dimensional form states [33] as

$$\frac{\partial q_1}{\partial \iota} + e q_1^2 \frac{\partial q_1}{\partial \zeta} + d\frac{\partial^3 q_1}{\partial \zeta^3} + f\frac{\partial}{\partial \zeta}\left(\frac{\partial^2 q_1}{\partial \tau^2} + \frac{\partial^2 q_1}{\partial \eta^2}\right) = 0, \tag{1}$$

where, $d, e,$ and f are given by

$$d = \frac{1}{D}, \qquad e = \frac{E}{D}, \qquad f = \frac{G}{D}, \qquad D = 2\sum_{\gamma} \frac{\sigma_{p\gamma}^2 (V - U_{\gamma 0})}{\left((V-U_{\gamma 0})^2 - \psi_{T\gamma}^2\right)^2},$$

$$G = 1 + \sum_{\gamma} \frac{\sigma_{p\gamma}^2 (V-U_{\gamma 0})^4}{\Omega_\gamma^2 \left((V-U_{\gamma 0})^2 - \psi_{T\gamma}^2\right)^2},$$

$$E = \sum_{\gamma} \frac{\sigma_{p\gamma}^2 r_\gamma^2 \left(15(V-U_{\gamma 0})^4 + F_1 (V-U_{\gamma 0})^2 \psi_{T\gamma}^2 + F_2 \psi_{T\gamma}^4\right)}{2m^2 \left((V-U_{\gamma 0})^2 - \psi_{T\gamma}^2\right)^5} - \sum_{\delta} \frac{r_\delta}{2\Lambda_{G\delta}^2 \lambda^2 T_\delta^2},$$

where the plasma frequencies $\sigma_{p\gamma}^2$, the Debye lengths $\Lambda_{G\delta}^2$ and the thermal velocities $\psi_{T\gamma}$ for the species γ are defined as $\sigma_{p\gamma}^2 = \frac{N_\gamma r_\gamma^2}{\epsilon_0 m}$, $\Lambda_{G\delta}^2 = \frac{\epsilon_0 \lambda T_\delta}{N_\delta r_\delta^2}$ and $\psi_{T\gamma}^2 = \frac{\nu_\gamma p_\gamma}{N_\gamma m}$, respectively.

1.2 A summary of the method

We assume that the given nonlinear evolution equation of $q(\iota,\tau,\eta,\zeta)$ is in the form

$$H(q,q_\zeta,q_\iota,q_\tau,q_\eta,q_{\zeta\zeta},q_{\iota\iota},q_{\zeta\zeta},\dots)=0, \tag{2}$$

where H is the polynomial in its parameters. The nature of the method can be given in the subsequent stages [34, 35, 35]:

Stage 1: To obtain the solutions of Eq.(2) take $q_1(\iota,\tau,\eta,\zeta)=q(\phi),\phi=\lambda\iota+\mu\tau+\rho\eta-\sigma\zeta$ and convert Eq.(2) to ODE:

$$R(q,q',q'',q''',\dots)=0, \tag{3}$$

where $'$ (prime) represent the derivative w.r.t ϕ.

Stage 2: By introducing the solution $q(\phi)$ of Eq.(3) in a limited series way given by [36]

$$q(\phi)=\sum_{j=-n}^{n} b_j\Psi^j(\phi), \tag{4}$$

where b_j (real constants with $b_n\neq 0$) and n (nonnegative integer) to be calculated. $\Psi(\phi)$ represents the solution of the following equation

$$\Psi'(\phi)=\sqrt{h_0+h_1\Psi(\phi)+h_2\Psi^2(\phi)+h_3\Psi^3(\phi)+h_4\Psi^4(\phi)+h_5\Psi^5(\phi)+h_6\Psi^6(\phi)}, \tag{5}$$

where h_j are constants.

Stage 3: The value n is attained by applying the principle of homogeneous balance on Eq.(3).

Stage 4: By [37], using Mathematica to solve the system and based on the value of the parameters h_j we can attain the solutions of Eq.(1).

The organization of this work is as follows. In Section 1, the introduction of the work is given. The method is implemented in Section 2 to construct more or less new exact soliton, period, Jacobi elliptic function and constant wave solution of the three- dimensional mKdV-ZK model and the physical explanation of the results. The conclusion of the work is finally stated in Section 3.

§2 Implementation of the procedure to the 3-D mKdV-ZK equation

By assuming the transformational wave solution as

$$q_1(\iota,\tau,\eta,\zeta)=q(\phi)=\sum_{j=-n}^{n} b_j\Psi^j(\phi), \qquad \phi=\lambda\iota+\mu\tau+\rho\eta-\sigma\zeta. \tag{6}$$

By putting Eq.(6) in Eq.(1) leads to

$$-\sigma q'+e\lambda q^2q'+(d\lambda^3+f\lambda(\mu^2+\rho^2))q'''=0. \tag{7}$$

Now, balancing q''' (highest order derivative) and q^2q' (nonlinear term), we attain $n=1$ and the solution to Eq.(7) is considered as

$$q(\phi)=\frac{b_{-1}}{\Psi}+d_0+d_1\Psi. \tag{8}$$

Now, Eq.(5) and (8) are substituted into Eq.(7) and applying [**?**], then Eq.(1) solutions can be stated as follows:

Case 1: $h_0 = h_1 = h_3 = f_5 = f_6 = 0$

$$b_{-1} = 0, \quad b_0 = 0, \quad b_1 = \pm\frac{i\sqrt{6h_4(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}}, \quad \lambda = \lambda, \quad \mu = \mu,$$

$$\rho = \rho, \quad \sigma = \lambda h_2\left(d\lambda^2 + f\mu^2 + f\rho^2\right).$$

In this case, the following solutions of Eq.(1) are obtained:

(1): When $h_2 > 0$ and $h_4 < 0$, we get

$$q_{1,1}(\phi) = \pm\frac{i\sqrt{-6h_2(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}}\operatorname{sech}\left(\sqrt{h_2}\phi\right). \tag{9}$$

(2): When $h_2 < 0$ in addition to $h_4 > 0$, we get double triangular periodic solutions

$$q_{1,2}(\phi) = \pm\frac{i\sqrt{-6h_2(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}}\sec\left(\sqrt{-h_2}\phi\right), \tag{10}$$

and

$$q_{1,3}(\phi) = \pm\frac{i\sqrt{-6h_2(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}}\csc\left(\sqrt{-h_2}\phi\right). \tag{11}$$

Case 2: $h_1 = h_3 = h_5 = h_6 = 0, \ h_0 = \dfrac{h_2^2}{4h_4}$

(a):
$$b_{-1} = \pm\frac{ih_2\sqrt{\frac{3}{2}(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{eh_4}}, \quad b_0 = 0, \quad b_1 = 0, \quad \lambda = \lambda, \quad \mu = \mu,$$

$$\rho = \rho, \quad \sigma = h_2\lambda\left(d\lambda^2 + f\mu^2 + f\rho^2\right).$$

(b):
$$b_{-1} = \pm\frac{ih_2\sqrt{\frac{3}{2}(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{eh_4}}, \quad b_0 = 0, \quad b_1 = \pm\frac{i\sqrt{6h_4(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}},$$

$$\lambda = \lambda, \quad \mu = \mu, \quad \rho = \rho, \quad \sigma = -2h_2\lambda\left(d\lambda^2 + f\mu^2 + f\rho^2\right).$$

(c):
$$b_{-1} = \pm\frac{ih_2\sqrt{\frac{3}{2}(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{eh_4}}, \quad b_0 = 0, \quad b_1 = \pm\frac{i\sqrt{6h_4(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}},$$

$$\lambda = \lambda, \quad \mu = \mu, \quad \rho = \rho, \quad \sigma = 4h_2\lambda\left(d\lambda^2 + f\mu^2 + f\rho^2\right).$$

(d):
$$b_{-1} = 0, \quad b_0 = 0, \quad b_1 = \pm\frac{i\sqrt{6h_4(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}}, \quad \lambda = \lambda, \quad \mu = \mu,$$

$$\rho = \rho, \quad \sigma = h_2\lambda\left(d\lambda^2 + f\mu^2 + f\rho^2\right).$$

In this case, the solutions of (a),(b),(c) and (d) are in the following form:

(1): When $h_2 > 0$ and $h_4 > 0$, we get

$$q_{2,1}(\phi) = b_{-1}\sqrt{\frac{2h_4}{h_2}}\cot\left(\sqrt{\frac{h_2}{2}}\phi\right) + b_0 + b_1\sqrt{\frac{h_2}{2h_4}}\tan\left(\sqrt{\frac{h_2}{2}}\phi\right). \tag{12}$$

(2): When $h_2 < 0$ and $h_4 > 0$, we get solution of Eq.(1) as

$$q_{2,2}(\phi) = b_{-1}\sqrt{\frac{-2h_4}{h_2}}\coth\left(\sqrt{\frac{h_2}{2}}\phi\right) + b_0 + b_1\sqrt{\frac{-h_2}{2h_4}}\tanh\left(\sqrt{\frac{h_2}{2}}\phi\right). \tag{13}$$

Case 3: $h_3 = h_4 = h_5 = h_6 = 0$,

$$b_{-1} = \pm\frac{i\sqrt{6h_0(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}}, \qquad b_0 = \pm\frac{ih_1\sqrt{\frac{3}{2}(d\lambda^2+f\mu^2+f\rho^2)}}{2\sqrt{eh_0}}, \qquad b_1 = 0,$$

$$\lambda = \lambda, \qquad \mu = \mu, \qquad \rho = \rho, \qquad \sigma = \frac{\lambda\left(8h_0h_2 - 3h_1^2\right)\left(d\lambda^2+f\mu^2+f\rho^2\right)}{8h_0}.$$

This case has solution in the following form, where $h_0 \neq 0, h_1 \neq 0$ in addition to $h_2 > 0$

$$q_{3,1}(\phi) = \pm\frac{ih_1\sqrt{\frac{3}{2}(d\lambda^2+f\mu^2+f\rho^2)}}{2\sqrt{eh_0}} \pm \frac{4ih_2\sqrt{6h_0(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}\left(\exp(\sqrt{h_2}\phi) - 2h_1 + (h_1^2 - 4h_0h_2)\exp(-\sqrt{h_2}\phi)\right)}. \tag{14}$$

Case 4: $h_0 = h_1 = h_2 = h_5 = h_6 = 0$,

$$b_{-1} = 0, \qquad b_0 = \pm\frac{ih_3\sqrt{\frac{3}{2}(d\lambda^2+f\mu^2+f\rho^2)}}{2\sqrt{eh_4}}, \qquad b_1 = \pm\frac{i\sqrt{6h_4(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}},$$

$$\lambda = \lambda, \qquad \mu = \mu, \qquad \rho = \rho, \qquad \sigma = -\frac{3h_3^2\lambda\left(d\lambda^2+f\mu^2+f\rho^2\right)}{8h_4}.$$

The solution of this case is in the form

$$q_{4,1}(\phi) = \pm\frac{ih_3\sqrt{\frac{3}{2}(d\lambda^2+f\mu^2+f\rho^2)}}{2\sqrt{eh_4}} \pm \frac{4ih_3\sqrt{6h_4(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}(h_3^2\phi^2 - 4h_4)}, \ \ h_4 > 0. \tag{15}$$

Case 5: $h_1 = h_3 = h_5 = h_6 = 0$,

(a): $$b_{-1} = \pm\frac{i\sqrt{6h_0(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}}, \qquad b_0 = 0, \qquad b_1 = 0, \qquad \lambda = \lambda, \qquad \mu = \mu,$$

$$\rho = \rho, \qquad \sigma = h_2\lambda\left(d\lambda^2+f\mu^2+f\rho^2\right).$$

(b): $$b_{-1} = \pm\frac{i\sqrt{6h_0(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}}, \qquad b_0 = 0, \qquad b_1 = \pm\frac{i\sqrt{6h_4(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}},$$

$$\lambda = \lambda, \qquad \mu = \mu, \qquad \rho = \rho, \qquad \sigma = \lambda\left(h_2 - 6\sqrt{h_0h_4}\right)\left(d\lambda^2 + f\left(\mu^2+\rho^2\right)\right).$$

(c): $$b_{-1} = \pm\frac{i\sqrt{6h_0(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}}, \qquad b_0 = 0, \qquad b_1 = \pm\frac{i\sqrt{6h_4(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}},$$

$$\lambda = \lambda, \qquad \mu = \mu, \qquad \rho = \rho, \qquad \sigma = \lambda\left(6\sqrt{h_0}\sqrt{h_4} + h_2\right)\left(d\lambda^2 + f\left(\mu^2+\rho^2\right)\right).$$

(d): $$b_{-1} = 0, \qquad b_0 = 0, \qquad b_1 = \pm\frac{i\sqrt{6h_4(d\lambda^2+f\mu^2+f\rho^2)}}{\sqrt{e}}, \qquad \lambda = \lambda, \qquad \mu = \mu,$$

$$\rho = \rho, \qquad \sigma = h_2\lambda\left(d\lambda^2+f\mu^2+f\rho^2\right).$$

The solutions of Eq.(1) in the above case are the Jacobi elliptic function [37, 38], which can be constructed with different Ψ expressions.

(1): When $h_0 = 1, \ \ h_2 = -(m^2+1)$ in addition to $h_4 = m^2$, the solutions obtained are

$$q_{5,1(1)}(\phi) = b_{-1}dc(\phi) + b_0 + b_1cd(\phi), \tag{16}$$

$$q_{5,1(2)}(\phi) = b_{-1}ns(\phi) + b_0 + b_1sn(\phi). \tag{17}$$

(2): When $h_0 = m^2, \;\; h_2 = -m^2 - 1$ in addition to $h_4 = 1$, the solutions obtained are

$$q_{5,2(1)}(\phi) = b_{-1}sn(\phi) + b_0 + b_1 ns(\phi), \tag{18}$$

$$q_{5,2(2)}(\phi) = b_{-1}cd(\phi) + b_0 + b_1 dc(\phi). \tag{19}$$

(3): When $h_0 = m^2 - 1, \;\; h_2 = 2 - m^2$ in addition to $h_4 = -1$, the solution is

$$q_{5,3}(\phi) = b_{-1}nd(\phi) + b_0 + b_1 dn(\phi). \tag{20}$$

(4): When $h_0 = 1 - m^2, \;\; h_2 = 2m^2 - 1$ in addition to $h_4 = -m^2$, the solution is

$$q_{5,4}(\phi) = b_{-1}nc(\phi) + b_0 + b_1 cn(\phi). \tag{21}$$

(5): When $h_0 = -m^2, \;\; h_2 = -1 + 2m^2$ in addition to $h_4 = 1 - m^2$, the solution is

$$q_{5,5}(\phi) = b_{-1}cn(\phi) + b_0 + b_1 nc(\phi). \tag{22}$$

(6): When $h_0 = -1, \;\; h_2 = 2 - m^2$ in addition to $h_4 = m^2 - 1$, the solution is

$$q_{5,6}(\phi) = b_{-1}dn(\phi) + b_0 + b_1 nd(\phi). \tag{23}$$

(7): When $h_0 = 1 - m^2, \;\; h_2 = 2 - m^2$ in addition to $h_4 = 1$, the solution is

$$q_{5,7}(\phi) = b_{-1}sc(\phi) + b_0 + b_1 cs(\phi). \tag{24}$$

(8): When $h_0 = 1, \;\; h_2 = 2 - m^2$ in addition to $h_4 = 1 - m^2$, the solution is

$$q_{5,8}(\phi) = b_{-1}cs(\phi) + b_0 + b_1 sc(\phi). \tag{25}$$

(9): When $h_0 = 1, \;\; h_2 = 2m^2 - 1$ and $h_4 = m^2(-1 + m^2)$, the solution is

$$q_{5,9}(\phi) = b_{-1}ds(\phi) + b_0 + b_1 sd(\phi). \tag{26}$$

(10): When $h_0 = m^2(-1 + m^2), \;\; h_2 = 2m^2 - 1$ in addition to $h_4 = 1$, we get the solution

$$q_{5,10}(\phi) = b_{-1}sd(\phi) + b_0 + b_1 ds(\phi). \tag{27}$$

(11): When $h_0 = \frac{1}{4}, \;\; h_2 = \frac{(1 - m^2)}{2}$ in addition to $h_4 = \frac{1}{4}$, we get the solution

$$q_{5,11}(\phi) = \frac{b_{-1}}{(ns(\phi) \pm cs(\phi))} + b_0 + b_1(ns(\phi) \pm cs(\phi)). \tag{28}$$

(12): When $h_0 = \frac{(1 - m^2)}{4}, \;\; h_2 = \frac{(1 + m^2)}{2}$ in addition to $h_4 = \frac{1}{4}$, we get the solution

$$q_{5,12}(\phi) = \frac{b_{-1}}{(nc(\phi) \pm sc(\phi))} + b_0 + b_1(nc(\phi) \pm sc(\phi)). \tag{29}$$

(13): When $h_0 = \frac{m^2}{4}, \;\; h_2 = \frac{(m^2 - 2)}{2}$ in addition to $h_4 = \frac{1}{4}$, we get the solution

$$q_{5,13}(\phi) = \frac{b_{-1}}{(ns(\phi) + ds(\phi))} + b_0 + b_1(ns(\phi) + ds(\phi)). \tag{30}$$

(14): When $h_0 = \frac{m^2}{4}, \;\; h_2 = \frac{(m^2 - 2)}{2}$in addition to $h_4 = \frac{m^2}{4}$, we get the solutions

$$q_{5,14(1)}(\phi) = \frac{b_{-1}}{(sn(\phi) \pm icn(\phi))} + b_0 + b_1(sn(\phi) \pm icn(\phi)), \tag{31}$$

$$q_{5,14(2)}(\phi) = b_{-1}\left(i\sqrt{1 - m^2}sn(\phi) \pm cn(\phi)\right) nd(\phi) + b_0 + b_1 \frac{dn(\phi)}{\left(i\sqrt{1 - m^2}sn(\phi) \pm cn(\phi)\right)}. \tag{32}$$

(15): When $h_0 = 1$, $h_2 = 2 - 4m^2$ in addition to $h_4 = 1$, we get the solution

$$q_{5,15}(\phi) = b_{-1}cn(\phi)ns(\phi)nd(\phi) + b_0 + b_1 sn(\phi)dn(\phi)nc(\phi). \tag{33}$$

(16): When $h_0 = \dfrac{(m-1)^2}{4A_1^2}$, $h_2 = \dfrac{(1+m^2+6m)}{2}$ and $h_4 = \dfrac{A_1^2(-1+m)^2}{4}$, we get the solution

$$q_{5,16}(\phi) = b_{-1}A_1(1+sn(\phi))(1+msn(\phi))nd(\phi)nc(\phi) + b_0 + \frac{b_1 dn(\phi)cn(\phi)}{A_1(1+sn(\phi))(1+msn(\phi))}. \tag{34}$$

(17): When $h_0 = \dfrac{(m+1)^2}{4A_1^2}$, $h_2 = \dfrac{(1+m^2+6m)}{2}$ and $h_4 = \dfrac{A_1^2(1+m)^2}{4}$, we get the solution

$$q_{5,17}(\phi) = b_{-1}A_1(1+sn(\phi))(1-msn(\phi))nd(\phi)nc(\phi) + b_0 + \frac{b_1 dn(\phi)cn(\phi)}{A_1(1+sn(\phi))(1-msn(\phi))}. \tag{35}$$

(18): When $h_0 = -2m^3 + m^4 + m^2$, $h_2 = \dfrac{-4}{m}$ and $h_4 = 6m - m^2 - 1$, we get the solution

$$q_{5,18}(\phi) = \frac{b_{-1}(1+msn^2(\phi))nd(\phi)nc(\phi)}{m} + b_0 + \frac{b_1 m dn(\phi)cn(\phi)}{(1+msn^2(\phi))}. \tag{36}$$

(19): When $h_0 = 2m^3 + m^4 + m^2$, $h_2 = -6m - m^2 - 1$ and $h_4 = \dfrac{-4}{m}$, we get the solution

$$q_{5,19}(\phi) = \frac{b_{-1}(msn^2(\phi)-1)nd(\phi)nc(\phi)}{m} + b_0 + \frac{b_1 m dn(\phi)cn(\phi)}{(msn^2(\phi)-1)}. \tag{37}$$

(20): When $h_0 = 2 + 2\sqrt{1-m^2} - m^2$, $h_2 = 6\sqrt{1-m^2} - m^2 + 2$ and $h_4 = 4\sqrt{1-m^2}$, we get the solution

$$q_{5,20}(\phi) = \frac{b_{-1}\left(\sqrt{1-m^2} - dn^2(\phi)\right)ns(\phi)nc(\phi)}{m^2} + b_0 + \frac{b_1 m^2 sn(\phi)cn(\phi)}{\left(\sqrt{1-m^2} - dn^2(\phi)\right)}. \tag{38}$$

(21): When $h_0 = 2 - 2\sqrt{1-m^2} - m^2$, $h_2 = 6\sqrt{1-m^2} - m^2 + 2$ and $h_4 = -4\sqrt{1-m^2}$, we get the solution

$$q_{5,21}(\phi) = -\left(\frac{b_{-1}\left(\sqrt{1-m^2} + dn^2(\phi)\right)ns(\phi)nc(\phi)}{m^2} + b_0 + \frac{b_1 m^2 sn(\phi)cn(\phi)}{\left(\sqrt{1-m^2} + dn^2(\phi)\right)}\right). \tag{39}$$

(22): When $h_0 = \dfrac{m^2-1}{4(A_3^2m^2 - A_2^2)}$, $h_2 = \dfrac{m^2+1}{2}$ and $h_4 = \dfrac{(A_3^2m^2 - A_2^2)(m^2-1)}{4}$, we get the solution

$$q_{5,22}(\phi) = \frac{b_{-1}\left(A_2cn(\phi) + A_3dn(\phi)\right)ns(\phi)}{\sqrt{\dfrac{A_2^2 - A_3^2}{A_2^2 - A_3^2m^2}}} + b_0 + \frac{b_1\sqrt{\dfrac{A_2^2 - A_3^2}{A_2^2 - A_3^2m^2}}\,sn(\phi)}{A_2cn(\phi) + A_3dn(\phi)}. \tag{40}$$

(23): When $h_0 = \dfrac{m^2}{4(A_3^2 + A_2^2)}$, $h_2 = \dfrac{m^2 - 2}{2}$ and $h_4 = \dfrac{(A_3^2 + A_2^2)}{4}$, we get the solution

$$q_{5,23}(\phi) = \frac{b_{-1}\left(A_2 sn(\phi) + A_3 cn(\phi)\right) nd(\phi)}{\sqrt{\dfrac{A_2^2 + A_3^2 - A_3^2 m^2}{A_2^2 + A_3^2}}} + b_0 + \frac{b_1 \sqrt{\dfrac{A_2^2 + A_3^2 - A_3^2 m^2}{A_2^2 + A_3^2}} dn(\phi)}{A_2 sn(\phi) + A_3 cn(\phi)}. \tag{41}$$

(24): When $h_0 = \dfrac{2m - m^2 - 1}{A_2^2}$, $h_2 = 2m^2 + 2$ and $h_4 = -A_2^2 m^2 - A_2^2 - 2A_2^2 m$, the solution is

$$q_{5,24}(\phi) = \frac{b_{-1} A_2 \left(msn^2(\phi) + 1\right)}{(msn^2(\phi) - 1)} + b_0 + \frac{b_1 \left(msn^2(\phi) - 1\right)}{A_2\left(msn^2(\phi) + 1\right)}. \tag{42}$$

(25): When $h_0 = -\dfrac{2m + m^2 + 1}{A_2^2}$, $h_2 = 2m^2 + 2$ and $h_4 = -A_2^2 m^2 + A_2^2 + 2A_2^2 m$, we get the solution

$$q_{5,25}(\phi) = \frac{b_{-1} A_2 \left(msn^2(\phi) - 1\right)}{(msn^2(\phi) + 1)} + b_0 + \frac{b_1 \left(msn^2(\phi) + 1\right)}{A_2\left(msn^2(\phi) - 1\right)}. \tag{43}$$

(26): When $f_0 = f_4 = \dfrac{1}{4}$ and $h_2 = \dfrac{1 - 2m^2}{2}$,the solutions are

$$q_{5,26(1)}(\phi) = b_{-1}\left(mcn(\phi) \pm i\sqrt{1 - m^2}\right) nd(\phi) + b_0 + \frac{b_1 dn(\phi)}{\left(mcn(\phi) \pm i\sqrt{1 - m^2}\right)}, \tag{44}$$

$$q_{5,26(2)}(\phi) = \frac{b_{-1}}{msn(\phi) \pm idn(\phi)} + b_0 + b_1\left(msn(\phi) \pm idn(\phi)\right), \tag{45}$$

$$q_{5,26(3)}(\phi) = \frac{b_{-1}}{mns(\phi) \pm cs(\phi)} + b_0 + b_1\left(mns(\phi) \pm cs(\phi)\right), \tag{46}$$

$$q_{5,26(4)}(\phi) = b_{-1}\left(1 \pm cn(\phi)\right) ns(\phi) + b_0 + \frac{b_1 sn(\phi)}{(1 \pm cn(\phi))}. \tag{47}$$

(27): When $h_0 = h_4 = \dfrac{m^2 - 1}{4}$ and $h_2 = \dfrac{1 + m^2}{2}$, the solutions are

$$q_{5,27(1)}(\phi) = b_{-1}\left(1 \pm msn(\phi)\right) nd(\phi) + b_0 + \frac{b_1 dn(\phi)}{(1 \pm msn(\phi))}, \tag{48}$$

$$q_{5,27(2)}(\phi) = \frac{b_{-1}}{msd(\phi) \pm nd(\phi)} + b_0 + b_1\left(msd(\phi) \pm nd(\phi)\right). \tag{49}$$

(28): When $h_0 = h_4 = \dfrac{1 - m^2}{4}$ and $h_2 = \dfrac{1 + m^2}{2}$, the solutions are

$$q_{5,28(1)}(\phi) = b_{-1}\left(1 \pm sn(\phi)\right) nc(\phi) + b_0 + \frac{b_1 cn(\phi)}{(1 \pm sn(\phi))}, \tag{50}$$

$$q_{5,28(2)}(\phi) = \frac{b_{-1}}{(nc(\phi) \pm sc(\phi))} + b_0 + b_1\left(nc(\phi) \pm sc(\phi)\right). \tag{51}$$

(29): When $h_0 = -\dfrac{(1 - m^2)^2}{4}$, $h_2 = \dfrac{1 + m^2}{2}$ in addition to $h_4 = \dfrac{1}{4}$, the solution is

$$q_{5,29}(\phi) = \frac{b_{-1}}{(mcn(\phi) \pm dn(\phi))} + b_0 + b_1\left(mcn(\phi) \pm dn(\phi)\right). \tag{52}$$

(30): When $h_0 = \frac{1}{4}$, $h_2 = \frac{1+m^2}{2}$ and $h_4 = \frac{(1-m^2)^2}{4}$, the solution is

$$q_{5,30}(\phi) = b_{-1}\left(dn(\phi) \pm cn(\phi)\right) ns(\phi) + b_0 + \frac{b_1 sn(\phi)}{(dn(\phi) \pm cn(\phi))}. \tag{53}$$

(31): When $h_0 = \frac{1}{4}$, $h_2 = \frac{m^2-2}{2}$ and $h_4 = \frac{m^4}{4}$, the solution is

$$q_{5,31(1)}(\phi) = b_{-1}\left(\sqrt{1-m^2} \pm dn(\phi)\right) nc(\phi) + b_0 + \frac{b_1 cn(\phi)}{\left(\sqrt{1-m^2} \pm dn(\phi)\right)}, \tag{54}$$

$$q_{5,31(2)}(\phi) = b_{-1}\left(1 \pm dn(\phi)\right) ns(\phi) + b_0 + \frac{b_1 sn(\phi)}{(1 \pm dn(\phi))}. \tag{55}$$

Where $\phi = \lambda\iota + \mu\tau + \rho\eta - \sigma\zeta$ fulfills Eq.(9)-(55), m is a modulus in the open interval $(0,1)$, $A_1, A_2, A_3 (A_1A_2A_3 \neq 0)$ and A_4 are arbitrary constants. The Jacobi elliptic functions are bi-periodic and have the subsequent characteristics of trigonometric functions

$$sn^2(\phi) = 1 - cn^2(\phi),\ \ dn^2(\phi) = 1 - m^2 sn^2(\phi),\ \ sn'(\phi) = cn(\phi)dn(\phi),$$
$$cn'(\phi) = -sn(\phi)dn(\phi),\ \ dn'(\phi) = -m^2 sn(\phi)cn(\phi).$$

Once $m \longrightarrow 1$, the functions in Jacobi?s form reprobate to functions of hyperbolic form, thus

$$sn(\phi) \longrightarrow \tanh(\phi),\ \ cn(\phi) \longrightarrow \mathrm{sech}(\phi).$$

Once $m \longrightarrow 0$, the functions in Jacobi?s form reprobate to functions of trigonometric form, thus

$$sn(\phi) \longrightarrow \sin(\phi),\ \ cos(\phi) \longrightarrow \mathrm{sech}(\phi).$$

Case 6: $h_0 = h_1 = h_5 = h_6 = 0$,

$$b_{-1} = 0, \qquad b_0 = \pm\frac{ih_3\sqrt{\frac{3}{2}(d\lambda^2 + f\mu^2 + f\rho^2)}}{2\sqrt{eh_4}}, \qquad b_1 = \frac{i\sqrt{6h_4(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}},$$
$$\lambda = \lambda, \qquad \mu = \mu, \qquad \rho = \rho, \qquad \sigma = \frac{\lambda\left(8h_2h_4 - 3h_3^2\right)\left(d\lambda^2 + f\mu^2 + f\rho^2\right)}{8h_4}.$$

Now, the solutions of Eq.(1) for this case are as follows [37]:

(32): When $h_3^2 - 4h_2h_4 = 0$ and $2\alpha^2 - 9\beta < 0$, we get the solutions

$$q_{6,1}(\phi) = -\frac{b_{-1}h_3}{h_2\left(1 + \epsilon\tanh\left(\sqrt{\frac{h_2}{2}}\phi\right)\right)} + b_0 - b_1\frac{h_2}{h_3}\left(1 + \epsilon\tanh\left(\sqrt{\frac{h_2}{2}}\phi\right)\right), \tag{56}$$

$$q_{6,2}(\phi) = -\frac{b_{-1}h_3}{h_2\left(1 + \epsilon\coth\left(\sqrt{\frac{h_2}{2}}\phi\right)\right)} + b_0 - b_1\frac{h_2}{h_3}\left(1 + \epsilon\coth\left(\sqrt{\frac{h_2}{2}}\phi\right)\right). \tag{57}$$

(33): When $h_3^2 - 4h_2h_4 > 0$ and $h_2 > 0$, we get the solution

$$q_{6,3}(\phi) = \frac{b_{-1}\left(\epsilon\sqrt{\Theta} - h_3\mathrm{sech}(\sqrt{h_2}\phi)\right)}{2h_2\mathrm{sech}(\sqrt{h_2}\phi)} + b_0 + \frac{2b_1h_2\mathrm{sech}(\sqrt{h_2}\phi)}{\epsilon\sqrt{\Theta} - h_3\mathrm{sech}(\sqrt{h_2}\phi)}. \tag{58}$$

(34): When $h_3^2 - 4h_2h_4 > 0$ and $h_2 < 0$, we get the solutions

$$q_{6,4}(\phi) = \frac{b_{-1}\left(\epsilon\sqrt{\Theta} - h_3\mathrm{sech}(\sqrt{-h_2}\phi)\right)}{2h_2\mathrm{sech}(\sqrt{-h_2}\phi)} + b_0 + \frac{2b_1h_2\mathrm{sech}(\sqrt{-h_2}\phi)}{\epsilon\sqrt{\Theta} - h_3\mathrm{sech}(\sqrt{-h_2}\phi)}, \tag{59}$$

$$q_{6,5}(\phi) = \frac{b_{-1}\left(\epsilon\sqrt{-\Theta} - h_3\mathrm{sech}(\sqrt{-h_2}\phi)\right)}{2h_2\mathrm{sech}(\sqrt{-h_2}\phi)} + b_0 + \frac{2b_1h_2\mathrm{sech}(\sqrt{-h_2}\phi)}{\epsilon\sqrt{-\Theta} - h_3\mathrm{sech}(\sqrt{-h_2}\phi)}. \tag{60}$$

(35): When $h_3^2 - 4h_2h_4 < 0$ and $h_2 > 0$, we get the solution

$$q_{6,6}(\phi) = \frac{b_{-1}\left(\epsilon\sqrt{-\Theta} - h_3\mathrm{sech}(\sqrt{h_2}\phi)\right)}{2h_2\mathrm{sech}(\sqrt{h_2}\phi)} + b_0 + \frac{2b_1h_2\mathrm{sech}(\sqrt{h_2}\phi)}{\epsilon\sqrt{-\Theta} - h_3\mathrm{sech}(\sqrt{h_2}\phi)}. \tag{61}$$

Where $\phi = \lambda\iota + \mu\tau + \rho\eta - \sigma\zeta$ and $\Theta = h_3^2 - 4h_2h_4$.

Case 7: $h_2 = h_4 = h_5 = h_6 = 0$,

$$b_{-1} = \pm\frac{i\sqrt{6h_0(d\lambda^2 + f\mu^2 + f\rho^2)}}{\sqrt{e}}, \qquad b_0 = \frac{ih_1\sqrt{\frac{3}{2}(d\lambda^2 + f\mu^2 + f\rho^2)}}{2\sqrt{eh_0}}, \qquad b_1 = 0,$$

$$\lambda = \lambda, \qquad \mu = \mu, \qquad \rho = \rho, \qquad \sigma = -\frac{3h_1^2\lambda\left(d\lambda^2 + f\mu^2 + f\rho^2\right)}{8h_0}.$$

In this case, the solution is in the Weierstrass elliptic function as show in the form

$$q_{7,1}(\phi) = b_{-1}\wp\left(\frac{\sqrt{h_3}}{2}\phi; g_2, g_3\right) + b_0 + b_1\wp\left(\frac{\sqrt{h_3}}{2}\phi; g_2, g_3\right), \qquad h_3 > 0, \tag{62}$$

where $g_2 = -\frac{4h_1}{h_3}$, $g_3 = -\frac{4h_0}{h_3}$ and $\wp$ are called invariants of Weierstrass elliptic function.

2.1 Physical explanation of the results

Here, we give the physical explanation of the excess solutions regained and stated in this work. The discussions are conducted on a case-by-case basis:

For **Case 1:** The Eq.(9) is a bright soliton solution to the Eq.(1), but under the inverse limitation situations, the singular periodic solution is restored and they are enumerated in Eqs.(10) and (11).

For **Case 2:** The Eq.(12) recorded indicates singular periodic solution, but when the limitation is reversed, a dark singular combo optical soliton is attained, which is given by Eq.(13).

For **Case 3:** The Eq.(14) is also another form of soliton-like solution attained from the addition procedure.

For **Case 4:** The Eq.(15) indicates a plane wave solution.

For **Case 5:** Here, the solutions are recorded according to the Jacobi's elliptic function, where m denotes the elliptic modulus. Therefore, the limit values for these recorded solutions as m draw closer to 0 or 1 are listed below:

Firstly, when $m \longrightarrow 0$, Eq.(16)-(55) correspondingly tend to:

$$q_{5,1(1)}(\phi) = b_{-1}\text{sec}(\phi) + b_0 + b_1\text{cos}(\phi), \tag{63}$$

$$q_{5,1(2)}(\phi) = b_{-1}\text{csc}(\phi) + b_0 + b_1\text{sin}(\phi), \tag{64}$$

$$q_{5,2(1)}(\phi) = b_{-1}\text{sin}(\phi) + b_0 + b_1\text{csc}(\phi), \tag{65}$$

$$q_{5,2(2)}(\phi) = b_{-1}\text{cos}(\phi) + b_0 + b_1\text{sec}(\phi), \tag{66}$$

$$q_{5,3}(\phi) = b_{-1}(\phi) + b_0 + b_1(\phi), \tag{67}$$

$$q_{5,4}(\phi) = b_{-1}\text{sec}(\phi) + b_0 + b_1\text{cos}(\phi), \tag{68}$$

$$q_{5,5}(\phi) = b_{-1}\text{cos}(\phi) + b_0 + b_1\text{sec}(\phi), \tag{69}$$

$$q_{5,6}(\phi) = b_{-1}(\phi) + b_0 + b_1(\phi), \tag{70}$$

$$q_{5,7}(\phi) = b_{-1}\text{tan}(\phi) + b_0 + b_1\text{cot}(\phi), \tag{71}$$

$$q_{5,8}(\phi) = b_{-1}\text{cot}(\phi) + b_0 + b_1\text{tan}(\phi), \tag{72}$$

$$q_{5,9}(\phi) = b_{-1}\text{csc}(\phi) + b_0 + b_1\text{sin}(\phi), \tag{73}$$

$$q_{5,10}(\phi) = b_{-1}\text{sin}(\phi) + b_0 + b_1\text{csc}(\phi), \tag{74}$$

$$q_{5,11}(\phi) = \frac{b_{-1}}{(\text{csc}(\phi) \pm \text{cot}(\phi))} + b_0 + b_1(\text{csc}(\phi) \pm \text{cot}(\phi)), \tag{75}$$

$$q_{5,12}(\phi) = \frac{b_{-1}}{(\text{sec}(\phi) \pm \text{tan}(\phi))} + b_0 + b_1(\text{sec}(\phi) \pm \text{tan}(\phi)), \tag{76}$$

$$q_{5,13}(\phi) = \frac{b_{-1}}{2\text{csc}(\phi)} + b_0 + 2b_1\text{csc}(\phi), \tag{77}$$

$$q_{5,14(1)}(\phi) = \frac{b_{-1}}{(\text{sin}(\phi) \pm i\text{cos}(\phi))} + b_0 + b_1(\text{sin}(\phi) \pm i\text{cos}(\phi)), \tag{78}$$

$$q_{5,14(2)}(\phi) = b_{-1}\left(i\text{sin}(\phi) \pm \text{cos}(\phi)\right) + b_0 + \frac{b_1}{(i\text{sin}(\phi) \pm \text{cos}(\phi))}, \tag{79}$$

$$q_{5,15}(\phi) = b_{-1}\text{cot}(\phi) + b_0 + b_1\text{tan}(\phi), \tag{80}$$

$$q_{5,16}(\phi) = b_{-1}A_1(1 + \text{sin}(\phi))\text{sec}(\phi) + b_0 + \frac{b_1\text{cos}}{A_1(1 + \text{sin}(\phi))}, \tag{81}$$

$$q_{5,17}(\phi) = b_{-1}A_1(1 + \text{sin}(\phi))\text{sec}(\phi) + b_0 + \frac{b_1\text{cos}}{A_1(1 + \text{sin}(\phi))}, \tag{82}$$

$$q_{5,22}(\phi) = \frac{b_{-1}\left(A_2\cos(\phi) + A_3\right)\csc(\phi)}{\sqrt{\frac{A_2^2 - A_3^2}{A_2^2}}} + b_0 + \frac{b_1\sqrt{\frac{A_2^2 - A_3^2}{A_2^2}}\sin(\phi)}{A_2\cos(\phi) + A_3}, \tag{83}$$

$$q_{5,23}(\phi) = b_{-1}\left(A_2\sin(\phi) + A_3\cos\right) + b_0 + \frac{b_1}{A_2\sin(\phi) + A_3\cos(\phi)}, \tag{84}$$

$$q_{5,24}(\phi) = -b_1A_2 + b_0 - \frac{b_1}{A_2}, \tag{85}$$

$$q_{5,25}(\phi) = -b_1A_2 + b_0 - \frac{b_1}{A_2}, \tag{86}$$

$$q_{5,26(1)}(\phi) = \pm ib_{-1} + b_0 \mp ib_1, \tag{87}$$

$$q_{5,26(2)}(\phi) = \mp ib_{-1} + b_0 \pm ib_1, \tag{88}$$

$$q_{5,26(3)}(\phi) = \pm b_{-1}\tan(\phi) + b_0 \pm b_1\cot(\phi), \tag{89}$$

$$q_{5,26(4)}(\phi) = b_{-1}\left(1 \pm \cos(\phi)\right)\csc(\phi) + b_0 + \frac{b_1\sin(\phi)}{\left(1 \pm \cos(\phi)\right)}, \tag{90}$$

$$q_{5,27(1)}(\phi) = b_{-1} + b_0 + b_1, \tag{91}$$

$$q_{5,27(2)}(\phi) = \pm b_{-1} + b_0 \pm b_1, \tag{92}$$

$$q_{5,28(1)}(\phi) = b_{-1}\left(1 \pm \sin(\phi)\right)\sec(\phi) + b_0 + \frac{b_1\cos(\phi)}{\left(1 \pm \sin(\phi)\right)}, \tag{93}$$

$$q_{5,28(2)}(\phi) = \frac{b_{-1}}{\left(\sec(\phi) \pm \tan(\phi)\right)} + b_0 + b_1\left(\sec(\phi) \pm \tan(\phi)\right), \tag{94}$$

$$q_{5,29}(\phi) = \pm b_{-1} + b_0 \pm b_1, \tag{95}$$

$$q_{5,30}(\phi) = b_{-1}\left(1 \pm \cos(\phi)\right)\csc(\phi) + b_0 + \frac{b_1\sin(\phi)}{\left(1 \pm \cos(\phi)\right)}, \tag{96}$$

$$q_{5,31(1)}(\phi) = 2b_{-1}\sec(\phi) + b_0 + \frac{b_1\cos(\phi)}{2}, \tag{97}$$

$$q_{5,31(2)}(\phi) = 2b_{-1}\csc(\phi) + b_0 + \frac{b_1\sin(\phi)}{2}. \tag{98}$$

Conversely, when $m \longrightarrow 1$, Eq.(16)-(55) correspondingly tend to:

$$q_{5,1(1)}(\phi) = b_{-1} + b_0 + b_1, \tag{99}$$

$$q_{5,1(2)}(\phi) = b_{-1}\coth(\phi) + b_0 + b_1\tanh(\phi), \tag{100}$$

$$q_{5,2(1)}(\phi) = b_{-1}\tanh(\phi) + b_0 + b_1\coth(\phi), \tag{101}$$

$$q_{5,2(2)}(\phi) = b_{-1} + b_0 + b_1, \tag{102}$$

$$q_{5,3}(\phi) = b_{-1}\cosh(\phi) + b_0 + b_1\mathrm{sech}(\phi), \tag{103}$$

$$q_{5,4}(\phi) = b_{-1}\cosh(\phi) + b_0 + b_1\mathrm{sech}(\phi), \tag{104}$$

$$q_{5,5}(\phi) = b_{-1}\mathrm{sech}(\phi) + b_0 + b_1\cosh(\phi), \tag{105}$$

$$q_{5,6}(\phi) = b_{-1}\mathrm{sech}(\phi) + b_0 + b_1\cosh(\phi), \tag{106}$$

$$q_{5,7}(\phi) = b_{-1}\sinh(\phi) + b_0 + b_1\mathrm{csch}(\phi), \tag{107}$$

$$q_{5,8}(\phi) = b_{-1}\mathrm{csch}(\phi) + b_0 + b_1\sinh(\phi), \tag{108}$$

$$q_{5,9}(\phi) = b_{-1}\mathrm{csch}(\phi) + b_0 + b_1\sinh(\phi), \tag{109}$$

$$q_{5,10}(\phi) = b_{-1}\sinh(\phi) + b_0 + b_1\mathrm{csch}(\phi), \tag{110}$$

$$q_{5,11}(\phi) = \frac{b_{-1}}{(\coth(\phi) \pm \mathrm{csch}(\phi))} + b_0 + b_1(\coth(\phi) \pm \mathrm{csch}(\phi)), \tag{111}$$

$$q_{5,12}(\phi) = \frac{b_{-1}}{(\cosh(\phi) \pm \sinh(\phi))} + b_0 + b_1(\cosh(\phi) \pm \sinh(\phi)), \tag{112}$$

$$q_{5,13}(\phi) = \frac{b_{-1}}{\coth(\phi) + \mathrm{csch}(\phi)} + b_0 + b_1\left(\coth(\phi) + \mathrm{csch}(\phi)\right), \tag{113}$$

$$q_{5,14(1)}(\phi) = \frac{b_{-1}}{(\tanh(\phi) \pm i\mathrm{sech}(\phi))} + b_0 + b_1(\tanh(\phi) \pm i\mathrm{sech}(\phi)), \tag{114}$$

$$q_{5,14(2)}(\phi) = \pm b_{-1} + b_0 \pm b_1, \tag{115}$$

$$q_{5,15}(\phi) = b_{-1}\coth(\phi) + b_0 + b_1\tanh(\phi), \tag{116}$$

$$q_{5,16}(\phi) = b_{-1}A_1(1 + \tanh(\phi))^2\cosh^2(\phi) + b_0 + \frac{b_1\mathrm{sech}^2(\phi)}{A_1(1 + \tanh(\phi))^2}, \tag{117}$$

$$q_{5,17}(\phi) = b_{-1}A_1 + b_0 + \frac{b_1}{A_1}, \tag{118}$$

$$q_{5,22}(\phi) = b_{-1}(A_2 + A_3)\mathrm{csch}(\phi) + b_0 + \frac{b_1\sinh(\phi)}{A_2 + A_3}, \tag{119}$$

$$q_{5,23}(\phi) = \frac{b_{-1}\left(A_2\tanh(\phi) + A_3\mathrm{sech}(\phi)\right)\cosh(\phi)}{\sqrt{\dfrac{A_2^2}{A_2^2 + A_3^2}}} + b_0 + \frac{b_1\sqrt{\dfrac{A_2^2}{A_2^2 + A_3^2}}\mathrm{sech}(\phi)}{A_2\tanh(\phi) + A_3\mathrm{sech}(\phi)}, \tag{120}$$

$$q_{5,24}(\phi) = -b_1A_2\cosh2(\phi) + b_0 - \frac{b_1\mathrm{sech}2(\phi)}{A_2}, \tag{121}$$

$$q_{5,25}(\phi) = -b_1A_2\mathrm{sech}2(\phi) + b_0 - \frac{b_1\cosh2(\phi)}{A_2}, \tag{122}$$

$$q_{5,26(1)}(\phi) = b_1 + b_0 + b_1, \tag{123}$$

$$q_{5,26(2)}(\phi) = \frac{b_{-1}}{\tanh(\phi) \pm i\mathrm{sech}(\phi)} + b_0 + b_1\left(\tanh(\phi) \pm i\mathrm{sech}(\phi)\right), \tag{124}$$

$$\tag{125}$$

$$q_{5,26(3)}(\phi) = \frac{b_{-1}}{\coth(\phi) \pm \text{csch}(\phi)} + b_0 + b_1 (\coth(\phi) \pm \text{csch}(\phi)), \tag{126}$$

$$q_{5,26(4)}(\phi) = b_{-1} (1 \pm \text{sech}(\phi)) \coth(\phi) + b_0 + \frac{b_1 \tanh(\phi)}{(1 \pm \text{sech}(\phi))}, \tag{127}$$

$$q_{5,27(1)}(\phi) = b_{-1} (1 \pm \tanh(\phi)) \cosh(\phi) + b_0 + \frac{b_1 \text{sech}(\phi)}{(1 \pm \tanh(\phi))}, \tag{128}$$

$$q_{5,27(2)}(\phi) = \frac{b_{-1}}{\sinh(\phi) \pm \cosh(\phi)} + b_0 + b_1 (\sinh(\phi) \pm \cosh(\phi)), \tag{129}$$

$$q_{5,27(1)}(\phi) = b_{-1} (1 \pm \tanh(\phi)) \cosh(\phi) + b_0 + \frac{b_1 \text{sech}(\phi)}{(1 \pm \tanh(\phi))}, \tag{130}$$

$$q_{5,27(2)}(\phi) = \frac{b_{-1}}{\sinh(\phi) \pm \cosh(\phi)} + b_0 + b_1 (\sinh(\phi) \pm \cosh(\phi)), \tag{131}$$

$$q_{5,28(1)}(\phi) = b_{-1} (1 \pm \tanh(\phi)) \cosh(\phi) + b_0 + \frac{b_1 \text{sech}(\phi)}{(1 \pm \tanh(\phi))}, \tag{132}$$

$$q_{5,28(2)}(\phi) = \frac{b_{-1}}{(\cosh(\phi) \pm \sinh(\phi))} + b_0 + b_1 (\cosh(\phi) \pm \sinh(\phi)), \tag{133}$$

$$q_{5,29}(\phi) = \frac{b_{-1}}{2\text{sech}(\phi)} + b_0 + 2b_1 \text{sech}(\phi), \tag{134}$$

$$q_{5,30}(\phi) = 2b_{-1} \text{csch}(\phi) + b_0 + \frac{b_1 \sinh(\phi)}{2}, \tag{135}$$

$$q_{5,31(1)}(\phi) = \pm b_{-1} + b_0 \pm b_1, \tag{136}$$

$$q_{5,31(2)}(\phi) = b_{-1} (1 \pm \text{sech}(\phi)) \coth(\phi) + b_0 + \frac{b_1 \tanh(\phi)}{(1 \pm \text{sech}(\phi))}. \tag{137}$$

For the case where m tends to 0, the solutions listed are periodic waves, periodic singular waves or plane waves. Conversely, for the case where m tends to 1, the solutions suggest dark singular combo solitons, bright singular combo solitons, singular solitons and some solutions are not categorized solitons.

For **Case 6:** For this case, solutions (56)-(57) suggest dark singular combo solitons whereas both solutions (58) and (61) suggest singular solitons. Eqs. (59)-(60) indicates a composition of singular soliton in addition to periodic singular wave.

For **Case 7:** Here, the Eq.(62) is a Weierstrass elliptic function solutions.

§3 Conclusion

In this work, we efficiently use the modified extended direct algebraic method to attain the exact traveling wave solutions for the nonlinear 3-D mKdV-ZK equation. With this method, we reduce the mKdV-ZK equation to ODE, making it easy to be solved. Implementing this method and the solutions of the auxiliary first order nonlinear ODE (ordinary differential equation), new diversity of traveling wave solutions of the nonlinear 3-D mKdV-ZK equation are obtained under certain parameter constraints, comprising bright solitons, dark solitons, singular periodic solutions, and pairs of bright, dark and singular solitons. Furthermore, we retrieve new solutions of the mKdV-ZK equation for the first time, such as constant wave, combo solitons and some uncategorized solitons. The method also recovers solutions from the Jacobi?s elliptic and the Weierstrass elliptic functions.

By comparing our solutions in this work with the solutions in [39], we obtain more and new exact solutions with solitons properties. In addition, Lu D et al [40], applied the same extended direct algebraic method on the mKdV-ZK equation but most of the solutions we attain are different by comparing with our solutions due to the auxiliary equation. Our solutions recovered new solutions for the mKdV-ZK equation for the first time like constant wave, combo solitons and uncategorized solitons which are not in [40]. This method can be extended to several NEEs due to its efficient, simplicity and ability to execute tedious and complex algebraic calculations.

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