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Growth, Zeros and Fixed points of Differences of Meromorphic Solutions of Difference Equations

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Abstract. In this paper, we study the difference equation

$$a_1(z)f(z+1)+a_0(z)f(z)=0,$$

where $a_1(z)$ and $a_0(z)$ are entire functions of finite order. Under some conditions, we obtain some properties, such as fixed points, zeros etc., of the differences and forward differences of meromorphic solutions of the above equation.

§1 Introduction

In this paper, we use the basic notions of Nevanlinna's theory (see [10,15]). In addition, we use the notations $\sigma(f),\mu(f)$ to denote the order and the lower order of growth of a meromorphic function $f(z)$ respectively, and $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$. The quantity $\delta(a,f)$ is called the deficiency of the value a to $f(z)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$ that is defined as

$$\tau(f(z))=\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{f(z)-z}\right)}{\log r}.$$

Furthermore, we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o\left(T(r,f)\right)$ for all r outside of a set with finite logarithmic measure. For $n\in\mathbb{N}^+$, the forward differences $\Delta^n f(z)$ are defined in the standard way [14] by

$$\Delta f(z)=f(z+1)-f(z),\ \Delta^{n+1}f(z)=\Delta^n f(z+1)-\Delta^n f(z).$$

For convenience, we denote $\Delta^0 f(z)=f(z)$. Furthermore, a meromorphic solution $f(z)$ of a difference (or differential) equation is called *admissible* if all coefficients of the equation are small functions with respect to $f(z)$.

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Recently, many results of complex differences and difference equations are obtained, such as [1, 2, 4–7, 9, 11–13]. Chiang and Feng [7] studied the growth of meromorphic solutions of homogeneous linear difference equation, and obtained the following result.

Theorem A Let $A_0(z), \ldots, A_n(z)$ be entire functions such that there exists an integer $l, 0 \le l \le n$, such that

$$\sigma(A_l) > \max_{\substack{0 \le j \le n \\ j \ne l}} \{\sigma(A_j)\}. \tag{1.1}$$

If $f(z)(\not\equiv 0)$ is a meromorphic solution to

$$A_n(z)y(z+n) + \cdots + A_1(z)y(z+1) + A_0(z)y(z) = 0, \tag{1.2}$$

then we have $\sigma(f) \ge \sigma(A_l) + 1$.

In Theorem A, the coefficients of (1.2) should satisfy the condition (1.1). If the condition (1.1) was replaced by $\sigma(A_l) = \max_{0 \le j \le n} \{\sigma(A_j)\}$, what will be the results? Regarding this, Laine and Yang [13] obtained the following theorem.

Theorem B Let $A_0(z), \ldots, A_n(z)$ be entire functions of finite order such that among those coefficients having the maximal order $\sigma = \max\{\sigma(A_k), 0 \le k \le n\}$, exactly one has its type strictly greater than the others. If $f(z) \not\equiv 0$ is a meromorphic solution of equation (1.2), then $\sigma(f) \ge \sigma + 1$.

Laine and Yang [13] raised a question that

Question: Whether all meromorphic solutions $f(z)(\not\equiv 0)$ of equation (1.2) satisfy $\sigma(f) \ge 1 + \max_{0 \le j \le n} \{\sigma(A_j)\}$, even if there is no dominating coefficient.

For the difference equation

$$a_1(z)f(z+1) + a_0(z)f(z) = 0, \tag{1.3}$$

where $a_1(z)$ and $a_0(z)$ are entire functions, we answer the question above, and obtain the following result.

Theorem 1.1 Let $a_1(z)$ and $a_0(z)$ be entire functions of finite lower order such that $\mu(a_1) \ne \mu(a_0)$ or $\delta(0, a_1) \ne \delta(0, a_0)$. If $f(z)(\not\equiv 0)$ is a meromorphic solution of equation (1.3), then $\sigma(f) \ge \max\{\mu(a_1), \mu(a_0)\} + 1$.

The fixed points of meromorphic functions and their derivatives is a very important problem in theory of meromorphic functions. Bergweiler and Pang [3] considered fixed points of derivative and proved the following theorem.

Theorem C Let $f(z)$ be a transcendental meromorphic function and let $R(z)$ be a rational function, $R \not\equiv 0$. Suppose that all zeros and poles of $f(z)$ are multiple, except possibly finitely many. Then $f'(z) - R(z)$ has infinitely many zeros.

When $R(z) \equiv z$, Theorem C shows that $f'(z)$ has infinitely many fixed points under condition of Theorem C.

Some author considered fixed points of a meromorphic function $f(z)$, its shift $f(z+c)$ and difference $\Delta f(z)$. In general, $\tau(f(z)) \ne \tau(f(z+n)), n \in \mathbb{N}^+$ and $\tau(f(z)) \ne \tau(\Delta f(z))$. For example $f(z) = e^{i2\pi z} + z - 1$ has infinitely many fixed points and satisfy $\tau(f(z)) = \sigma(f(z)) = 1$, but its shift $f(z+1) = e^{i2\pi z} + z$ has no fixed points, and $\Delta f(z) = 1$ has only one fixed point. Chen and Shon [6] gave the following construction theorem to show that even for a meromorphic function of small growth, $\Delta f(z)$ may have only finitely many fixed points.

Theorem D Let $\phi(r)$ be a positive non-decreasing function on $[1,\infty)$ which satisfies $\lim_{r\to\infty}\phi(r)=\infty$. Then there exists a function f transcendental and meromorphic in the plane with

$$\limsup_{r\to\infty}\frac{T(r,f)}{r}<\infty,\quad \text{and}\quad \liminf_{r\to\infty}\frac{T(r,f)}{\phi(r)\log r}<\infty$$

such that $g(z)=\Delta f(z)$ has only one fixed point. Moreover, the function g satisfies

$$\limsup_{r\to\infty}\frac{T(r,g)}{\phi(r)\log r}<\infty.$$

Chen [5] discovered that Gamma function $\Gamma(z)$ satisfies

$$\tau(\Delta\Gamma(z))=\tau(\Gamma(z+n))=\sigma(\Gamma(z))=1, n\in\mathbb{N}.$$

In fact, he proved the following theorem.

Theorem E Let $a_1(z)$ and $a_0(z)$ be nonzero polynomials such that

$$\deg(a_1(z)+a_0(z))=\max\{\deg a_1(z),\deg a_0(z)\}\geq 1. \tag{1.4}$$

Suppose $f(z)(\not\equiv 0)$ is a finite order meromorphic solution of equation (1.3), then the following statements hold.

(i) For $j=0,1,\ldots,\tau(\Delta f(z))=\tau(f(z+j))=\sigma(f(z))\geq 1$.

(ii) For every $n\in\mathbb{N}^+,\Delta^n f(z)$ and $f(z)$ has same zeros, except possibly finitely many, and $\lambda(\Delta^n f(z))=\lambda(f(z))$.

In particular, for an entire solution $f(z)$, we have that if $\sigma(f)>1$, then

$$\lambda(\Delta^n f(z))=\lambda(f(z))\geq\sigma(f(z))-1;$$

if $\sigma(f(z))=1$, then either for all $n\in\mathbb{N}^+,\lambda(\Delta^n f(z))=1$, or for all $n\in\mathbb{N}^+,\Delta^n f(z)$ has only finitely many zeros.

We find an interesting example. For the meromorphic function $f(z)=ze^{i2\pi z}$, its shifts $f(z+n)=(z+n)e^{i2\pi z}, n\in\mathbb{N}^+$ and difference $\Delta f(z)=e^{i2\pi z}$ satisfy $\tau(\Delta f(z))=\tau(f(z+n))=\sigma(f(z))$, but its forward differences $\Delta^n f(z)\equiv 0(n\geq 2)$ have only one fixed point. That is to say, even if a meromorphic function $f(z)$ satisfies $\tau(\Delta f(z))=\tau(f(z+n))=\sigma(f(z))$, its forward differences $\Delta^n f(z)$ ($n\geq 2$ is some integer) may have finitely many fixed points. Thus, it is natural to ask that whether the forward differences $\Delta^n f(z)(n\in\mathbb{N}^+)$ satisfy $\tau(\Delta^n f(z))=\tau(f(z))$, under conditions of Theorem E, If the coefficient $a_j(z)(j\in\{0,1\})$ is transcendental, what will be the fixed points of the differences and forward differences of meromorphic solutions of equation (1.3)? We consider these questions, and obtain the following results.

Theorem 1.2 Let $a_1(z)$ and $a_0(z)$ be finite order entire functions such that $\mu(a_1)\neq\mu(a_0)$. Suppose $f(z)$ is an admissible meromorphic solution of equation (1.3), then for every $n\in\mathbb{N}$,

(i) $f(z+n)$ has no nonzero finite Nevanlinna exceptional value;

(ii) $\Delta^n f(z)$ has no nonzero finite Nevanlinna exceptional value;

(iii) $\tau(\Delta^n f(z))=\tau(f(z+n))=\sigma(f(z))$.

Theorem 1.3 Let $a_1(z)$ and $a_0(z)$ be finite order entire functions such that $\sigma(a_1) \neq \sigma(a_0)$. Suppose that $f(z)(\not\equiv 0)$ is a meromorphic solution of equation (1.3), then for every $n \in \mathbb{N}$, we have

$$\tau(\Delta^n f(z)) = \tau(f(z+n)) = \sigma(f(z)) \geq \max\{\sigma(a_1), \sigma(a_0)\} + 1.$$

Corollary 1.1 Let $a_1(z)$ and $a_0(z)$ be nonzero polynomials satisfying (1.4). Suppose that $f(z)(\not\equiv 0)$ is a finite order meromorphic solution of equation (1.3), then for every $n \in \mathbb{N}$,

(i) $\tau(\Delta^n f(z)) = \tau(f(z+n)) = \sigma(f(z)) \geq 1$;

(ii) if $f(z)$ is a finite order entire solution, then
$$\lambda(\Delta^n f(z)) = \lambda(f(z)) = \sigma(f(z)),$$
or $\Delta^n f(z)$ has only finitely many zeros.

Remark 1.1 Under conditions of Corollary 1.1, $f(z)$ and $\Delta^n f(z)$ have same zeros, except finitely many, and $\Delta^n f(z)$ has finitely many zeros possibly occurs when $\sigma(f) = 1$.

Remark 1.2 It sees that Gamma function $\Gamma(z)$ satisfies the difference equation
$$f(z+1) - zf(z) = 0.$$
We deduce from Corollary 1.1 that Gamma function $\Gamma(z)$ satisfies
$$\tau(\Delta^n \Gamma(z)) = \tau(\Gamma(z+n)) = \sigma(\Gamma(z)) = 1, n \in \mathbb{N}.$$

The following Example 1.1 shows that in Corollary 1.1, condition (1.4) cannot be weakened.

Example 1.1 The difference equation
$$-zf(z+1) + (z+1)f(z) = 0$$
has an entire solution $f(z) = ze^{i2\pi z}$. Thus, $\Delta^n f(z) \equiv 0$ $(n \geq 2)$ have only one fixed point.

The following Example 1.2 satisfies the conditions and results of Corollary 1.1.

Example 1.2 The difference equation
$$zf(z+1) + (z+1)f(z) = 0$$
has entire solutions $f_1(z) = ze^{i\pi z}$ and $f_2(z) = z\sin(\pi z)$. For every $j \in \mathbb{N}$, we have
$$\Delta^{2j} f_1(z) = 4^j(z+j)e^{i\pi z}, \qquad \Delta^{2j+1} f_1(z) = -4^j(2z+2j+1)e^{i\pi z},$$
$$\Delta^{2j} f_2(z) = 4^j(z+j)\sin(\pi z), \qquad \Delta^{2j+1} f_2(z) = -4^j(2z+2j+1)\sin(\pi z).$$
Thus, for every $n \in \mathbb{N}^+, f_1(z)$ and $\Delta^n f_1(z)$ has only one zero, but $f_2(z)$ and $\Delta^n f_2(z)$ have the same zeros except finitely many, and satisfy $\lambda(\Delta^n f_2(z)) = \lambda(f_2(z)) = \sigma(f_2(z)) = 1$.

§2 Lemmas for the Proof of Theorems and Corollary

Lemma 2.1 [7] Let $f(z)$ be a meromorphic function of finite order σ and let η be a nonzero complex number. Then for each $\varepsilon > 0$, we have
$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.2 [9] Let $c \in \mathbb{C}$ and $f(z)$ be a nonconstant meromorphic function of finite order. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f).$$

Lemma 2.3 [7] Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be a complex number. Then

$$T(r, f(z+c)) = T(r, f(z)) + S(r,f).$$

Lemma 2.2 and Lemma 2.3 show that

Lemma 2.4 Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be a complex number. Then

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r,f).$$

Lemma 2.5 [8,16] Let $T_1(r)$ and $T_2(r)$ be real valued, nonnegative and nondecreasing functions on $[r_0, \infty), (r_0 > 0$ is a real constant), and satisfying

$$T_1(r) = O(T_2(r)), \quad (r \to \infty, r \notin E)$$

where E is a set with finite linear measure. Then

$$\liminf_{r\to\infty} \frac{\log^+ T_1(r)}{\log r} \leq \liminf_{r\to\infty} \frac{\log^+ T_2(r)}{\log r},$$

$$\limsup_{r\to\infty} \frac{\log^+ T_1(r)}{\log r} \leq \limsup_{r\to\infty} \frac{\log^+ T_2(r)}{\log r}.$$

Remark 2.1 If E is a set with finite logarithmic measure, then Lemma 2.5 still holds.

Lemma 2.6 [4] Let $f(z)$ be a transcendental merormophic function with $\sigma(f) < 1$, and let $g_1(z)$ and $g_2(z) \not\equiv 0$ be polynomials, $c_1, c_2 (\neq c_1)$ be constants. Then

$$h(z) = g_2(z) f(z+c_2) + g_1(z) f(z+c_1)$$

is transcendental.

Lemma 2.7 [9,13] Let $w(z)$ be a nonconstant finite order meromorphic solution of

$$P(z,w) = 0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a) \not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r,a) = S(r,w)$, then

$$m\left(r, \frac{1}{w-a}\right) = S(r,w)$$

outside of a possible exceptional set of finite logarithmic measure.

From the proof of Lemma 2.7, we easily obtain

Remark 2.2 Let $w(z)$ be a nonconstant finite order meromorphic solution of

$$P(z,w) = 0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$, so that the coefficients of $P(z,w)$ are meromorphic functions $a_j(z) (j = 1, \ldots, s)$ satisfying $\sigma(a_j) < \sigma(w)$. If $P(z,a) \not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r,a) = S(r,w)$, then

$$m\left(r, \frac{1}{w-a}\right) \leq \sum_{j=1}^{s} m(r, a_j) + S(r,w)$$

outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.8 Let $a_1(z)$ and $a_0(z)$ be entire functions such that $\mu(a_1) \neq \mu(a_0)$ or $\delta(0, a_1) \neq \delta(0, a_0)$. Then equation (1.3) has no nonzero rational solution.

Proof. It is easy to see that at least one of $a_1(z)$ and $a_0(z)$ is transcendental. Without loss of generality, assume $a_0(z)$ is transcendental. Suppose that $R(z)$ is a nonzero rational solution of equation (1.3). So

$$a_1(z)R(z+1) + a_0(z)R(z) = 0.$$

Thus,

$$a_1(z) = -\frac{R(z)}{R(z+1)} a_0(z).$$

Since $R(z)$ is a rational function and $a_0(z)$ is transcendental, then

$$T(r, a_1) = T(r, a_0) + O(\log r) = T(r, a_0) + o\{T(r, a_0)\}, \tag{2.1}$$

and

$$N\left(r, \frac{1}{a_1}\right) = N\left(r, \frac{1}{a_0}\right) + O(\log r) = N\left(r, \frac{1}{a_0}\right) + o\{T(r, a_0)\}. \tag{2.2}$$

If $\mu(a_1) \neq \mu(a_0)$, but by (2.1), we have $\mu(a_1) = \mu(a_0)$. A contradiction.

If $\delta(0, a_1) \neq \delta(0, a_0)$, but by (2.1) and (2.2), we obtain

$$\delta(0, a_1) = 1 - \overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{a_1}\right)}{T(r, a_1)} = 1 - \overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{a_0}\right) + o\{T(r, a_0)\}}{T(r, a_0) + o\{T(r, a_0)\}} = \delta(0, a_0).$$

A contradiction.

Lemma 2.9 Let $a_1(z)$ and $a_0(z)$ be finite order entire functions such that $\mu(a_1) \neq \mu(a_0)$ or $\sigma(a_1) \neq \sigma(a_0)$. Then

$$P_n(z) := \binom{n}{n} \prod_{j=0}^{n-1} a_0(z+j) + \sum_{k=1}^{n-1} \binom{n}{k} \prod_{j=0}^{k-1} a_0(z+j) \prod_{s=k}^{n-1} a_1(z+s) + \binom{n}{0} \prod_{j=0}^{n-1} a_1(z+j) \not\equiv 0.$$

Proof. Without loss of generality, we assume $\mu(a_0) > \mu(a_1)$(or $\sigma(a_0) > \sigma(a_1)$). Suppose that $P_n(z) \equiv 0$. So

$$-\binom{n}{n} \prod_{j=0}^{n-1} a_0(z+j) = \sum_{k=1}^{n-1} \binom{n}{k} \prod_{j=0}^{k-1} a_0(z+j) \prod_{s=k}^{n-1} a_1(z+s) + \binom{n}{0} \prod_{j=0}^{n-1} a_1(z+j). \tag{2.3}$$

Denote

$$b_k(z) = \binom{n}{k} \prod_{j=0}^{k-1} \frac{a_0(z+j)}{a_0(z)} \prod_{s=k}^{n-1} \frac{a_1(z+s)}{a_1(z)}, \quad k = 1, \ldots, n-1. \tag{2.4}$$

It follows from Lemma 2.2 and (2.4) that

$$m(r, b_k) \leq S(r, a_0) + S(r, a_1), \quad k = 1, \ldots, n-1. \tag{2.5}$$

By (2.4), we see that (2.3) can be rewritten as the form

$$\begin{aligned} -\binom{n}{n} \prod_{j=0}^{n-1} a_0(z+j) &= \sum_{k=1}^{n-1} \binom{n}{k} \prod_{j=0}^{k-1} \frac{a_0(z+j)}{a_0(z)} \prod_{s=k}^{n-1} \frac{a_1(z+s)}{a_1(z)} a_0^k(z) a_1^{n-k}(z) + \binom{n}{0} \prod_{j=0}^{n-1} a_1(z+j) \\ &= \sum_{k=1}^{n-1} b_k(z) a_0^k(z) a_1^{n-k}(z) + \binom{n}{0} \prod_{j=0}^{n-1} a_1(z+j). \end{aligned} \tag{2.6}$$

By Lemma 2.2, we have

$$m\left(r,\prod_{j=0}^{n-1}a_0(z+j)\right)\le m(r,a_0^n(z))+m\left(r,\prod_{j=0}^{n-1}\frac{a_0(z+j)}{a_0(z)}\right)$$
$$=nm(r,a_0)+S(r,a_0).$$

Again by Lemma 2.2, we have

$$nm(r,a_0)=m(r,a_0^n(z))\le m\left(r,\prod_{j=0}^{n-1}a_0(z+j)\right)+m\left(r,\prod_{j=0}^{n-1}\frac{a_0(z)}{a_0(z+j)}\right)$$
$$=m\left(r,\prod_{j=0}^{n-1}a_0(z+j)\right)+S(r,a_0).$$

Combining the last two inequalities, we obtain

$$m\left(r,\prod_{j=0}^{n-1}a_0(z+j)\right)=nm(r,a_0)+S(r,a_0).\tag{2.7}$$

By (2.5), we have

$$m\left(r,\sum_{k=1}^{n-1}b_ka_0^ka_1^{n-k}\right)=m\left(r,a_0\left(\sum_{k=1}^{n-1}b_ka_0^{k-1}a_1^{n-k}\right)\right)$$
$$\le m(r,a_0)+m\left(r,\sum_{k=2}^{n-1}b_ka_0^{k-1}a_1^{n-k}+b_1a_1^{n-1}\right)$$
$$\le m(r,a_0)+m\left(r,\sum_{k=2}^{n-1}b_ka_0^{k-1}a_1^{n-k}\right)+(n-1)m(r,a_1)+m(r,b_1)+O(1)$$
$$=m(r,a_0)+m\left(r,a_0\left(\sum_{k=2}^{n-1}b_ka_0^{k-2}a_1^{n-k}\right)\right)+(n-1)m(r,a_1)+m(r,b_1)+O(1)$$
$$\le 2m(r,a_0)+m\left(r,\sum_{k=3}^{n-1}b_ka_0^{k-2}a_1^{n-k}+b_2a_1^{n-2}\right)+(n-1)m(r,a_1)+m(r,b_1)+O(1)$$
$$\le 2m(r,a_0)+m\left(r,a_0\left(\sum_{k=3}^{n-1}b_ka_0^{k-3}a_1^{n-k}\right)\right)+\sum_{j=n-2}^{n-1}jm(r,a_1)+\sum_{j=1}^{2}m(r,b_j)+O(1)$$
$$\cdots$$
$$\le (n-1)m(r,a_0)+\sum_{j=1}^{n-1}jm(r,a_1)+\sum_{j=1}^{n-1}m(r,b_j)+O(1)$$
$$=(n-1)m(r,a_0)+\frac{n(n-1)}{2}m(r,a_1)+S(r,a_0)+S(r,a_1).$$

Thus,

$$m\left(r,\sum_{k=1}^{n-1}b_ka_0^ka_1^{n-k}\right)\le (n-1)m(r,a_0)+\frac{n(n-1)}{2}m(r,a_1)+S(r,a_0)+S(r,a_1).\tag{2.8}$$

By Lemma 2.2 and (2.6)–(2.8), we have

$$\begin{aligned}
nm(r,a_0) &= m\left(r,\prod_{j=0}^{n-1}a_0(z+j)\right)+S(r,a_0)\\
&\le m\left(r,\sum_{k=1}^{n-1}b_k a_0^k a_1^{n-k}\right)+m\left(r,\prod_{j=0}^{n-1}a_1(z+j)\right)\\
&\le (n-1)m(r,a_0)+\frac{n(n-1)}{2}m(r,a_1)+nm(r,a_1)+S(r,a_0)+S(r,a_1)\\
&= (n-1)m(r,a_0)+\frac{n(n+1)}{2}m(r,a_1)+S(r,a_0)+S(r,a_1).
\end{aligned}$$

Hence,

$$m(r,a_0)\le\frac{n(n+1)}{2}m(r,a_1)+S(r,a_0)+S(r,a_1).$$

Since a_0 and a_1 are entire functions, then

$$T(r,a_0)\le\frac{n(n+1)}{2}T(r,a_1)+S(r,a_0)+S(r,a_1),$$

which means $\mu(a_0)\le\mu(a_1)$ (or $\sigma(a_0)\le\sigma(a_1)$), by Lemma 2.5 and Remark 2.1. It is a contradiction.

§3 Proof of Theorems and Corollary

Proof of Theorem 1.1

Without loss of generality, assume $\mu(a_1)\ge\mu(a_0)$ and denote $\mu=\max\{\mu(a_1),\mu(a_0)\}=\mu(a_1)$. Suppose that $f(z)(\not\equiv 0)$ is a meromorphic solution of equation (1.3) satisfying $\sigma(f)=\sigma<\mu+1$. By Lemma 2.8, we know $f(z)$ is transcendental. For any given $\varepsilon>0$, choose α such that

$$\sigma-1+\varepsilon<\alpha<\mu-\varepsilon. \tag{3.1}$$

By the equation (1.3), we have

$$\frac{a_1(z)}{a_0(z)}=-\frac{f(z)}{f(z+1)}. \tag{3.2}$$

By Lemma 2.1, (3.1) and (3.2), we have

$$\begin{aligned}
T(r,a_1)=m(r,a_1)\le& m(r,a_0)+m\left(r,\frac{f(z)}{f(z+1)}\right)\\
=&T(r,a_0)+O(r^{\sigma-1+\varepsilon})\\
=&T(r,a_0)+o(r^{\alpha}).
\end{aligned}\tag{3.3}$$

Consider the following two cases.

Case 1. $\mu(a_1)\ne\mu(a_0)$. Thus, $\mu=\mu(a_1)>\mu(a_0)$.

By (3.1) and the definition of lower order, when r is sufficiently large,

$$r^{\alpha}\le r^{\mu-\varepsilon}\le\frac{1}{2}T(r,a_1). \tag{3.4}$$

By (3.3) and (3.4), we have

$$\begin{aligned} T(r,a_1) &\le T(r,a_0)+o(r^{\alpha}) \\ &\le T(r,a_0)+\frac{1}{2}T(r,a_1), \end{aligned}$$

which yields

$$\frac{1}{2}T(r,a_1)\le T(r,a_0),$$

that is, $\mu(a_1)\le\mu(a_0)=\mu$. Thus, $\mu(a_1)=\mu(a_0)=\mu$. A contradiction.

Case 2. $\delta(0,a_1)\ne\delta(0,a_0)$. By Case 1, we may assume $\mu(a_1)=\mu(a_0)=\mu$.

By (3.1) and (3.3), we have

$$T(r,a_1)\le T(r,a_0)+o\{T(r,a_0)\}.$$

By (3.1), (3.2) and Lemma 2.1, we have

$$\begin{aligned} T(r,a_0)=m(r,a_0) \le& m(r,a_1)+m\left(r,\frac{f(z+1)}{f(z)}\right) \\ =& T(r,a_1)+O(r^{\sigma-1+\varepsilon}) \\ =& T(r,a_1)+o(r^{\alpha}). \end{aligned} \tag{3.5}$$

By (3.1), (3.3) and (3.5), we obtain

$$\begin{aligned} T(r,a_1) &= T(r,a_0)+o(r^{\alpha}) \\ &= T(r,a_0)+o\{T(r,a_0)\} \\ &= (1+o(1))T(r,a_0). \end{aligned} \tag{3.6}$$

Again by (3.2) and Lemma 2.1, we have

$$\begin{aligned} m\left(r,\frac{1}{a_1}\right) &\le m\left(r,\frac{1}{a_0}\right)+m\left(r,\frac{a_0}{a_1}\right)+O(1) \\ &= m\left(r,\frac{1}{a_0}\right)+m\left(r,\frac{f(z+1)}{f(z)}\right)+O(1) \\ &= m\left(r,\frac{1}{a_0}\right)+O(r^{\sigma-1+\varepsilon}) \\ &= m\left(r,\frac{1}{a_0}\right)+o\{T(r,a_0)\}. \end{aligned}$$

Similarly, we get

$$m\left(r,\frac{1}{a_0}\right)\le m\left(r,\frac{1}{a_1}\right)+o\{T(r,a_0)\}.$$

Combining the last two inequalities, we obtain

$$m\left(r,\frac{1}{a_1}\right)=m\left(r,\frac{1}{a_0}\right)+o\{T(r,a_0)\}. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\varliminf_{r\to\infty}\frac{m\left(r,\frac{1}{a_1}\right)}{T(r,a_1)}=\varliminf_{r\to\infty}\frac{m\left(r,\frac{1}{a_0}\right)+o\{T(r,a_0)\}}{T(r,a_0)+o\{T(r,a_0)\}}=\varliminf_{r\to\infty}\frac{m\left(r,\frac{1}{a_0}\right)}{T(r,a_0)}.$$

So

$$\delta(0,a_1)=\delta(0,a_0),$$

which is a contradiction.

Proof of Theorem 1.2

Suppose that $f(z)$ is an admissible meromorphic solution of equation (1.3).

(i) we prove that $f(z+n)$ has no nonzero finite Nevanlinna exceptional value.

Set

$$P(z,f) := a_1(z)f(z+1) + a_0(z)f(z) = 0. \tag{3.8}$$

By Lemma 2.8, we know equation (1.3) has no nonzero rational solution. Then for any $b \in \mathbb{C}\setminus\{0\}$, we have $P(z,b) \not\equiv 0$. By Lemma 2.7 and $P(z,b) \not\equiv 0$, we see that

$$m\left(r, \frac{1}{f(z)-b}\right) = S(r,f).$$

Thus,

$$N\left(r, \frac{1}{f(z)-b}\right) = T(r,f(z)) + S(r,f). \tag{3.9}$$

By Lemma 2.3, Lemma 2.4 and (3.9), we have

$$\begin{aligned} N\left(r, \frac{1}{f(z+n)-b}\right) &= N\left(r, \frac{1}{f(z)-b}\right) + S(r,f) \\ &= T(r,f(z)) + S(r,f) \\ &= T(r,f(z+n)) + S(r,f(z+n)). \end{aligned}$$

Hence, $\delta(b, f(z+n)) = 0$. That is, $f(z+n)$ has no nonzero Nevanlinna exceptional value.

(ii) we prove that $\Delta^n f(z)$ has no nonzero finite Nevanlinna exceptional value.

For any $k \in \mathbb{N}^+$, by equation (1.3), we have

$$\begin{aligned} f(z+k) &= -\frac{a_0(z+k-1)}{a_1(z+k-1)} f(z+k-1) \\ &= \frac{a_0(z+k-1)}{a_1(z+k-1)} \frac{a_0(z+k-2)}{a_1(z+k-2)} f(z+k-2) \\ &= \cdots \\ &= (-1)^k \prod_{j=0}^{k-1} \frac{a_0(z+j)}{a_1(z+j)} f(z). \end{aligned} \tag{3.10}$$

By (3.10), we obtain

$$\begin{aligned} \Delta^n f(z) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(z+k) \\ &= \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} (-1)^k \prod_{j=0}^{k-1} \frac{a_0(z+j)}{a_1(z+j)} f(z) + \binom{n}{0} (-1)^n f(z) \\ &= (-1)^n f(z) \left(\sum_{k=1}^{n} \binom{n}{k} \prod_{j=0}^{k-1} \frac{a_0(z+j)}{a_1(z+j)} + \binom{n}{0} \right) \\ &= (-1)^n Q_n(z) f(z), \end{aligned} \tag{3.11}$$

where

$$Q_n(z)=\sum_{k=1}^{n}\binom{n}{k}\prod_{j=0}^{k-1}\frac{a_0(z+j)}{a_1(z+j)}+\binom{n}{0}. \tag{3.12}$$

Observe that

$$\begin{aligned}Q_n(z)&=\frac{\binom{n}{n}\prod\limits_{j=0}^{n-1}a_0(z+j)+\sum\limits_{k=1}^{n-1}\binom{n}{k}\prod\limits_{j=0}^{k-1}a_0(z+j)\prod\limits_{s=k}^{n-1}a_1(z+s)+\binom{n}{0}\prod\limits_{j=0}^{n-1}a_1(z+j)}{\prod\limits_{j=0}^{n-1}a_1(z+j)}\\&=\frac{P_n(z)}{\prod\limits_{j=0}^{n-1}a_1(z+j)},\end{aligned} \tag{3.13}$$

where

$$P_n(z)=\binom{n}{n}\prod_{j=0}^{n-1}a_0(z+j)+\sum_{k=1}^{n-1}\binom{n}{k}\prod_{j=0}^{k-1}a_0(z+j)\prod_{s=k}^{n-1}a_1(z+s)+\binom{n}{0}\prod_{j=0}^{n-1}a_1(z+j). \tag{3.14}$$

Since $\mu(a_1)\neq\mu(a_0)$, by Lemma 2.9, we know $P_n(z)\not\equiv 0$. So $\Delta^n f(z)\not\equiv 0$ by (3.11) and (3.13).

Since $a_1(z)$ and $a_0(z)$ are small with respect to $f(z)$, by (3.11), (3.12) and $\Delta^n f(z)\not\equiv 0$, we have

$$T(r,\Delta^n f)=T(r,f)+S(r,f). \tag{3.15}$$

For any given $b\in\mathbb{C}\setminus\{0\}$, suppose that $P\left(z,\frac{b}{(-1)^nQ_n(z)}\right)\equiv 0$. By (3.8), we have

$$a_1(z)\frac{b}{(-1)^nQ_n(z+1)}+a_0(z)\frac{b}{(-1)^nQ_n(z)}=0.$$

Thus,

$$a_1(z)Q_n(z)+a_0(z)Q_n(z+1)=0. \tag{3.16}$$

By (3.16) and Lemma 2.2, we have

$$m(r,a_1)=m\left(r,a_0\frac{Q_n(z+1)}{Q_n(z)}\right)\leq m(r,a_0)+m\left(r,\frac{Q_n(z+1)}{Q_n(z)}\right)=m(r,a_0)+S(r,Q_n).$$

Again by (3.16) and Lemma 2.2, we have

$$m(r,a_0)=m\left(r,a_1\frac{Q_n(z)}{Q_n(z+1)}\right)\leq m(r,a_1)+m\left(r,\frac{Q_n(z)}{Q_n(z+1)}\right)=m(r,a_1)+S(r,Q_n).$$

Thus,

$$T(r,a_0)=m(r,a_0)=m(r,a_1)+S(r,Q_n)=T(r,a_1)+S(r,Q_n). \tag{3.17}$$

By (3.12), (3.17) and Lemma 2.3, we know

$$S(r,Q_n)\leq S(r,a_1)+S(r,a_0)=S(r,a_1)=S(r,a_0). \tag{3.18}$$

By (3.17) and (3.18), we obtain

$$T(r,a_0)=T(r,a_1)+S(r,a_1),$$

together with Lemma 2.5 and Remark 2.1, we see that $\mu(a_1)=\mu(a_0)$ holds. A contradiction. So,

$$P\left(z,\frac{b}{(-1)^nQ_n(z)}\right)\not\equiv 0.$$

By this and Lemma 2.7, we have

$$m\left(r, \frac{1}{f(z)-\frac{b}{(-1)^n Q_n(z)}}\right) = S(r,f)$$

which leads to

$$N\left(r, \frac{1}{f(z)-\frac{b}{(-1)^n Q_n(z)}}\right) = T(r,f)+S(r,f). \tag{3.19}$$

By (3.11), we have

$$\begin{aligned}\Delta^n f(z) - b &= (-1)^n Q_n(z) f(z) - b \\ &= (-1)^n Q_n(z)\left(f(z) - \frac{b}{(-1)^n Q_n(z)}\right),\end{aligned}$$

together with (3.15) and (3.19), we have

$$\begin{aligned}N\left(r, \frac{1}{\Delta^n f(z)-b}\right) &\geq N\left(r, \frac{1}{f(z)-\frac{b}{(-1)^n Q_n(z)}}\right) - N(r, Q_n(z)) \\ &\geq T(r,f) - T(r, Q_n(z)) + S(r,f) \\ &= T(r,f) + S(r,f) \\ &= T(r, \Delta^n f) + S(r, \Delta^n f).\end{aligned}$$

Thus,

$$N\left(r, \frac{1}{\Delta^n f(z)-b}\right) = T(r,\Delta^n f) + S(r,\Delta^n f).$$

Hence, $\delta(b, \Delta^n f) = 0$. That is, $\Delta^n f(z)$ has no nonzero Nevanlinna exceptional value.

(iii) we prove $\tau(f(z+n)) = \sigma(f(z)), \quad \tau(\Delta^n f(z)) = \sigma(f(z))$.

By Lemma 2.8, equation (1.3) has no nonzero rational solution, so $P(z,z) \not\equiv 0$. By Lemma 2.7 and $P(z,z) \not\equiv 0$, we see that

$$m\left(r, \frac{1}{f(z)-z}\right) = S(r,f).$$

Thus,

$$N\left(r, \frac{1}{f(z)-z}\right) = T(r,f(z)) + S(r,f).$$

Hence, $\tau(f(z)) = \sigma(f(z))$.

Now we prove that for every $n \in \mathbb{N}^+, \tau(f(z+n)) = \sigma(f(z))$. By (1.3), we have

$$a_1(z+n)g(z+1) + a_0(z+n)g(z) = 0, \tag{3.20}$$

where $g(z) = f(z+n)$.

By Lemma 2.3, Lemma 2.5 and Remark 2.1, we know $\mu(a_j(z+n)) = \mu(a_j(z)) (j = 0,1)$. Since $\mu(a_1(z)) \neq \mu(a_0(z))$, we get

$$\mu(a_1(z+n)) \neq \mu(a_0(z+n)). \tag{3.21}$$

By (3.20), (3.21) and the above result, we obtain $\tau(g(z)) = \sigma(g(z))$. That is, $\tau(f(z+n)) = \sigma(f(z))$.

Next we prove that $\tau(\Delta^n f) = \sigma(f(z))$. Let

$$R_n(z) = \frac{z}{(-1)^n Q_n(z)}. \tag{3.22}$$

Using a same method as in the proof of (ii), we have $P(z, R_n) \not\equiv 0$. By $P(z, R_n) \not\equiv 0$ and Lemma 2.7, we have

$$m\left(r, \frac{1}{f - R_n}\right) = S(r, f).$$

Hence,

$$N\left(r, \frac{1}{f - R_n}\right) = T(r, f) + S(r, f). \tag{3.23}$$

By (3.11) and (3.22), we have

$$\begin{aligned} \Delta^n f(z) - z &= (-1)^n Q_n(z) f(z) - z \\ &= (-1)^n Q_n(z) \left(f(z) - \frac{z}{(-1)^n Q_n(z)} \right) \\ &= (-1)^n Q_n(z)(f(z) - R_n(z)). \end{aligned} \tag{3.24}$$

By (3.23) and (3.24), we have

$$\begin{aligned} N\left(r, \frac{1}{\Delta^n f - z}\right) &\geq N\left(r, \frac{1}{f - R_n}\right) - N(r, Q_n) \\ &\geq T(r, f) - T(r, Q_n) + S(r, f) \\ &= T(r, f) + S(r, f), \end{aligned}$$

which yields $\tau(\Delta^n f(z)) \geq \sigma(f(z))$. Since $\tau(\Delta^n f(z)) \leq \sigma(f(z))$, so $\tau(\Delta^n f(z)) = \sigma(f(z))$.

Proof of Theorem 1.3

Suppose $f(z) \not\equiv 0$ is a meromorphic solution of equation (1.3). Then

$$\sigma(f(z)) \geq \max\{\sigma(a_1), \sigma(a_0)\} + 1 \tag{3.25}$$

by $\sigma(a_1) \neq \sigma(a_0)$ and Theorem A.

We proceed to prove that $\tau(f(z+n)) = \sigma(f(z)) \geq \max\{\sigma(a_0), \sigma(a_1)\} + 1$. Set

$$P(z, f) := a_1(z) f(z+1) + a_0(z) f(z) = 0. \tag{3.26}$$

Since $\sigma(a_1) \neq \sigma(a_0)$, then

$$\sigma((z+1)a_1(z) + z a_0(z)) = \max\{\sigma(a_1(z)), \sigma(a_0(z))\}.$$

Thus, $P(z, z) = (z+1)a_1(z) + z a_0(z) \not\equiv 0$. By $P(z, z) \not\equiv 0$ and Remark 2.2, we have

$$m\left(r, \frac{1}{f(z) - z}\right) \leq m(r, a_0(z)) + m(r, a_1(z)) + S(r, f),$$

which implies that

$$N\left(r, \frac{1}{f(z) - z}\right) \geq T(r, f(z)) - m(r, a_0(z)) - m(r, a_1(z)) + S(r, f). \tag{3.27}$$

By (3.25) and (3.27), $\tau(f(z)) \geq \sigma(f(z))$ holds, together with the fact $\tau(f(z)) \leq \sigma(f(z))$, we know $\tau(f(z)) = \sigma(f(z))$.

By (1.3), we have

$$a_1(z+n) g(z+1) + a_0(z+n) g(z) = 0, \tag{3.28}$$

where $g(z) = f(z+n)$.

By Lemma 2.3, Lemma 2.5 and Remark 2.1, we know $\sigma(a_j(z+n)) = \sigma(a_j(z)) (j = 0, 1)$.

Since $\sigma(a_1(z)) \neq \sigma(a_0(z))$, we get

$$\sigma(a_1(z+n)) \neq \sigma(a_0(z+n)). \tag{3.29}$$

By (3.28), (3.29) and the above result, we obtain $\tau(g(z)) = \sigma(g(z))$. That is, $\tau(f(z+n)) = \sigma(f(z))$.

Next we prove that $\tau(\Delta^n f(z)) = \sigma(f(z))$. We also obtain (3.10)–(3.14).

Since $\sigma(a_1) \neq \sigma(a_0)$, by Lemma 2.9, we know $P_n(z) \not\equiv 0$. So $\Delta^n f(z) \not\equiv 0$ by (3.11) and (3.13).

Let

$$R_n(z) = \frac{z}{(-1)^n Q_n(z)}. \tag{3.30}$$

Assert that $P(z, R_n) \not\equiv 0$. Otherwise, by (3.26),

$$a_1(z)R_n(z+1) + a_0(z)R_n(z) = 0.$$

So $\sigma(R_n) \geq \max\{\sigma(a_1), \sigma(a_0)\}+1$ by Theorem A. On the other hand, $\sigma(R_n) \leq \max\{\sigma(a_1), \sigma(a_0)\}$ by (3.12) and (3.30). A contradiction. By $P(z, R_n) \not\equiv 0$ and Remark 2.2, we have

$$m\left(r, \frac{1}{f - R_n}\right) \leq m(r, a_1) + m(r, a_0) + S(r, f)$$

together with (3.12), (3.30) and Lemma 2.3, we obtain

$$\begin{aligned} N\left(r, \frac{1}{f - R_n}\right) &\geq T(r, f - R_n) - m(r, a_1) - m(r, a_0) + S(r, f) \\ &\geq T(r, f) - T(r, R_n) - m(r, a_1) - m(r, a_0) + S(r, f) \\ &= T(r, f) - T(r, Q_n) - m(r, a_1) - m(r, a_0) + S(r, f) \\ &= T(r, f) + O(T(r, a_1)) + O(T(r, a_0)). \end{aligned} \tag{3.31}$$

By (3.11) and (3.30), we have

$$\begin{aligned} \Delta^n f(z) - z &= (-1)^n Q_n(z) f(z) - z \\ &= (-1)^n Q_n(z) \left(f(z) - \frac{z}{(-1)^n Q_n(z)}\right) \\ &= (-1)^n Q_n(z)(f(z) - R_n(z)). \end{aligned} \tag{3.32}$$

It follows from (3.12),(3.31),(3.32) and Lemma 2.3 that

$$\begin{aligned} N\left(r, \frac{1}{\Delta^n f - z}\right) &\geq N\left(r, \frac{1}{f - R_n}\right) - N(r, Q_n) \\ &\geq T(r, f) + O(T(r, a_1)) + O(T(r, a_0)) - T(r, Q_n) + S(r, f) \\ &= T(r, f) + O(T(r, a_1)) + O(T(r, a_0)) + S(r, f), \end{aligned}$$

together with (3.25), we obtain $\tau(\Delta^n f(z)) \geq \sigma(f(z))$. Combining the fact that $\tau(\Delta^n f(z)) \leq \sigma(f(z))$, so $\tau(\Delta^n f(z)) = \sigma(f(z))$.

Proof of Corollary 1.1

(i) By Theorem E, we know for every $n \in \mathbb{N}, \tau(f(z+n)) = \sigma(f(z)) \geq 1$.

Now we prove that $\tau(\Delta^n f(z)) = \sigma(f(z))$. Using a same method as in the proof of Theorem 1.2, we obtain (3.10)–(3.14).

Assert that $P_n(z) \not\equiv 0$.

If $\deg a_1(z) < \deg a_0(z)$, we see that there is only one term $\binom{n}{n} \prod_{j=0}^{n-1} a_0(z+j)$ whose degree is the highest one. So that $P_n(z) \not\equiv 0$.

If $\deg a_0(z) < \deg a_1(z)$, we see that there is only one term $\binom{n}{0} \prod_{j=0}^{n-1} a_1(z+j)$ whose degree is the highest one. So that $P_n(z) \not\equiv 0$.

Now suppose that $\deg a_0(z) = \deg a_1(z)$. Let b_0 and b_1 be the leading coefficient of $a_0(z)$ and $a_1(z)$ respectively. By (1.4), we know $b_0 + b_1 \neq 0$. By (3.14), the leading coefficient of $P_n(z)$ is

$$\binom{n}{n} b_0^n + \sum_{k=1}^{n-1} \binom{n}{k} b_0^k b_1^{n-k} + \binom{n}{0} b_1^n = \sum_{k=0}^{n} \binom{n}{k} b_0^k b_1^{n-k} = (b_0+b_1)^n \neq 0.$$

So that $P_n(z) \not\equiv 0$. Thus, $\Delta^n f(z) \not\equiv 0$ by (3.11) and (3.13).

Let

$$R_n(z) = \frac{z}{(-1)^n Q_n(z)}, \tag{3.33}$$

and

$$P(z,f) := a_1(z) f(z+1) + a_0(z) f(z) = 0.$$

Suppose $P(z, R_n(z)) \equiv 0$. That is,

$$a_1(z) R_n(z+1) + a_0(z) R_n(z) = 0.$$

Since $R_n(z)$ is a nonzero rational function, then

$$-\frac{a_0(z)}{a_1(z)} = \frac{R_n(z+1)}{R_n(z)} \to 1 \quad (z \to \infty).$$

Hence,

$$-\frac{a_0(z)+a_1(z)}{a_1(z)} = -\frac{a_0(z)}{a_1(z)} - 1 \to 0 \quad (z \to \infty),$$

which means $\deg(a_1(z) + a_0(z)) < \deg a_1(z)$. It contradicts with (1.4). So, $P(z, R_n(z)) \not\equiv 0$.

By $P(z, R_n(z)) \not\equiv 0$ and Lemma 2.7, we have

$$m\left(r, \frac{1}{f(z) - R_n(z)}\right) = S(r,f),$$

which yields

$$N\left(r, \frac{1}{f(z) - R_n(z)}\right) = T(r, f(z)) + S(r,f).$$

By (3.11) and (3.33), we get

$$\begin{aligned}
\Delta^n f(z) - z &= (-1)^n Q_n(z) f(z) - z \\
&= (-1)^n Q_n(z) \left(f(z) - \frac{z}{(-1)^n Q_n(z)} \right) \\
&= (-1)^n Q_n(z) (f(z) - R_n(z)).
\end{aligned}$$

Combining with the last two equalities, we obtain

$$N\left(r, \frac{1}{\Delta^n f(z) - z}\right) = N\left(r, \frac{1}{f(z) - R_n(z)}\right) + O(\log r) = T(r,f) + S(r,f).$$

Hence, $\tau(\Delta^n f(z)) = \sigma(f(z))$.

(ii) Suppose $f(z)$ is a finite order entire solution of equation (1.3) satisfying $\lambda(f) < \sigma(f)$.

By Hadamard's factorization theorem, $f(z)$ assumes the form

$$f(z) = P(z)e^{h(z)}, \tag{3.34}$$

where $P(z)$ is an entire function satisfying

$$\sigma(P) = \lambda(P) = \lambda(f) < \sigma(f), \tag{3.35}$$

and $h(z)$ is a polynomial satisfying

$$\deg h = \sigma(f). \tag{3.36}$$

Substituting (3.34) into equation (1.3), we have

$$a_1(z)P(z+1)e^{h(z+1)} + a_0(z)P(z)e^{h(z)} = 0.$$

So

$$\left(a_1(z)e^{h(z+1)-h(z)}\right) P(z+1) + a_0(z)P(z) = 0. \tag{3.37}$$

If $\deg h \geq 2$, by calculation, we know $\deg(h(z+1) - h(z)) = \deg h - 1 \geq 1$. Thus,

$$\sigma\left(a_1(z)e^{h(z+1)-h(z)}\right) = \deg h - 1 > 0 = \sigma(a_0).$$

Applying Theorem A to equation (3.37), we have $\sigma(P) \geq (\deg h - 1) + 1 = \deg h$, which contradicts with (3.35) and (3.36).

If $\deg h = 1$, then $h(z+1) - h(z)$ is a constant. If $P(z)$ is transcendental with $\sigma(P) < 1$, by Lemma 2.6, we know $\left(a_1(z)e^{h(z+1)-h(z)}\right) P(z+1) + a_0(z)P(z)$ is transcendental, which contradicts with (3.37). So, $P(z)$ must be a polynomial. Thus, $f(z)$ has finitely many zeros.

Hence, either $\lambda(f) = \sigma(f)$ or $f(z)$ has finitely many zeros.

Since $a_1(z)$ and $a_0(z)$ are polynomials, by (3.11) and (3.12), we know $\Delta^n f(z)$ and $f(z)$ have the same zeros, except possibly finitely many. Thus, $\lambda(\Delta^n f(z)) = \lambda(f(z)) = \sigma(f(z))$ or $\Delta^n f(z)$ has finitely many zeros.

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