Appl. Math. J. Chinese Univ.
2020, 35(1): 113-126

Induced generalized exact boundary synchronizations for a coupled system of wave equations

WANG Yan-yan

Abstract. Taking a coupled system of wave equations with Dirichlet boundary controls as an example, by splitting and merging some synchronization groups of the state variables corresponding to a given generalized synchronization matrix, this paper introduces two kinds of induced generalized exact boundary synchronizations to better determine its generalized exactly synchronizable states.

§1 Introduction

For a coupled system of wave equations with Dirichlet boundary controls, based on the exact boundary synchronization (by groups) of Li and Rao [2–6], the generalized exact boundary synchronization and the corresponding generalized exactly synchronizable states were studied in [8,9], and the results show that for the generalized exact boundary synchronization with respect to a given generalized synchronization matrix, when the coupled system satisfies some conditions (the strong compatibility for the coupling matrix and a suitable collocation for the boundary controls), the corresponding generalized exactly synchronizable state can be determined only by its initial data and is independent of applied boundary controls. When the coupled system does not possess these conditions for the given generalized synchronization matrix, in this paper we introduce some induced generalized synchronization matrices to replace the original generalized synchronization matrix, such that they meet the strong compatibility condition, then, for suitably collocated boundary controls, the corresponding induced generalized exactly synchronizable states will be independent of applied boundary controls.

Consider the following coupled system of wave equations with Dirichlet boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \tag{1.1}$$

Received: 2019-09-23.
MR Subject Classification: 93B05, 93C20.
Keywords: Coupled system of wave equations, generalized exact boundary synchronization, determination of generalized exactly synchronizable states.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-020-3913-9.

with the initial condition

$$t = 0: \ (U, U') = (\hat{U}_0, \hat{U}_1) \ \text{ in } \Omega, \tag{1.2}$$

where $U = (u^{(1)}, \dots, u^{(N)})^\intercal$ represents the state variable, $A = (a_{ij}) \in \mathbb{M}^{N\times N}(\mathbb{R})$ is a given coupling matrix with constant elements; $H = (h^{(1)}, \dots, h^{(M)})^\intercal$ $(M \le N)$ denotes the boundary control, and $D \in \mathbb{M}^{N\times M}(\mathbb{R})$ is the boundary control matrix with constant elements, standing for the collocation of boundary controls.

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and mes $\Gamma_1 \neq 0$, and satisfies the usual multiplier geometrical condition [7]: there exists $x_0 \in \mathbb{R}^n$, such that for $m = x - x_0$ we have

$$(m, \nu) > 0, \ \forall x \in \Gamma_1; \ (m, \nu) \le 0, \ \forall x \in \Gamma_0, \tag{1.3}$$

where ν is the unit outward normal vector on the boundary and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$.

Definition 1.1 (cf. [8]). *For a $(N-p)\times N(0 \le p < N)$ full row-rank matrix Θ_p, called the generalized synchronization matrix, system (1.1) is generalized exactly synchronizable with respect to Θ_p, if there exists a time $T > 0$, such that for any given initial data $(\hat{U}_0, \hat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a boundary control $H \in L^2_{\text{loc}}(0, +\infty; (L^2(\Gamma_1))^M)$ with compact support in $[0, T]$, such that the corresponding solution $U = U(t, x) \in C^0_{\text{loc}}(0, +\infty; (L^2(\Omega))^N)$ $\cap C^1_{\text{loc}}(0, +\infty; (H^{-1}(\Omega))^N)$ to problem (1.1)-(1.2) satisfies*

$$t \ge T: \ \Theta_p U \equiv 0. \tag{1.4}$$

Remark 1.2 (cf. [8]). *The generalized exactly boundary synchronization (1.4) can be written as*

$$t \ge T: \ U \in \text{Ker}(\Theta_p). \tag{1.5}$$

Taking a basis $\{\epsilon_1, \dots, \epsilon_p\}$ of $\text{Ker}(\Theta_p)$ *as a synchronization basis, condition (1.4) (i.e. (1.5)) is equivalent to that: there exists a vector function $u = (u_1, \dots, u_p)^\intercal$ of t and x, such that*

$$t \ge T: \ U = u_1\epsilon_1 + \cdots + u_p\epsilon_p = (\epsilon_1, \dots, \epsilon_p)u, \tag{1.6}$$

where the function u, being a priori unknown, is called the corresponding generalized exactly synchronizable state, and p is the group number. □

In Section 2, we will first give some basic results on the generalized exact boundary synchronization for system (1.1). Then in Section 3 and Section 4, for system (1.1) and the given generalized synchronization matrix Θ_p, we will introduce its weakly and strongly induced generalized synchronization matrices $\hat{\Theta}_q$ and $\tilde{\Theta}_r$, respectively, satisfying the following relations:

$$\text{Ker}(\tilde{\Theta}_r) \subseteq \text{Ker}(\Theta_p) \subseteq \text{Ker}(\hat{\Theta}_q) \tag{1.7}$$

with $r \le p \le q$. When we have the original generalized exact boundary synchronization with respect to Θ_p, although the corresponding generalized exactly synchronizable state usually depends on applied boundary controls, we always have the generalized exact boundary synchronization with respect to its weakly induced generalized synchronization matrix $\hat{\Theta}_q$, and the corresponding generalized exactly synchronizable state is independent of applied $(N-q)$ suitably collocated boundary controls (see Theorem 3.7). Furthermore, using more boundary controls, we can achieve a better result — the generalized exact boundary synchronization with respect to the strongly induced generalized synchronization matrix $\tilde{\Theta}_r$, which implies the original generalized exact boundary synchronization with respect to Θ_p, and through $(N-r)$ suitably collocated boundary controls the corresponding generalized exactly synchronizable state is independent of applied boundary controls (see Theorem 4.7). In Section 4, we will give examples to illustrate the relations between the original generalized exact boundary synchronization and the corresponding induced generalized exact boundary synchronizations.

§2 Preliminaries

According to [9], for Θ_p and the basis $\{\epsilon_1, \dots, \epsilon_p\}$ of $\mathrm{Ker}(\Theta_p)$, let the normalized transformation be given by

$$X = \begin{pmatrix} \Theta_p \\ (y_1, \dots, y_p)^\intercal \end{pmatrix}, \tag{2.1}$$

where $\{y_1, \dots, y_p\}$ is bi-orthonormal to $\{\epsilon_1, \dots, \epsilon_p\}$: $(y_1, \dots, y_p)^\intercal(\epsilon_1, \dots, \epsilon_p) = I_p$ (identity matrix of order p), which guarantees the reversibility of X. Under this transformation, the state variable U turns into

$$\tilde{U} = XU = \begin{pmatrix} W_p \\ V_p \end{pmatrix}, \tag{2.2}$$

where

$$W_p = \Theta_p U,\ V_p = (y_1, \dots, y_p)^\intercal U, \tag{2.3}$$

and the generalized exact boundary synchronization (1.6) is equivalent to that

$$t \geq T: \ W_p = 0,\ V_p = u. \tag{2.4}$$

Denoting

$$\tilde{A} = XAX^{-1} = \begin{pmatrix} \bar{A}_p & Z_1 \\ Z_2^\intercal & \tilde{A}_p \end{pmatrix}, \tag{2.5}$$

where $\bar{A}_p$ and $\tilde{A}_p$ are square matrices of order $(N-p)$ and p, respectively, the null controllable part W_p satisfies

$$\begin{cases} W_p'' - \Delta W_p + \bar{A}_p W_p = -Z_1 V_p & \text{in } (0, +\infty) \times \Omega, \\ W_p = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ W_p = \Theta_p DH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: (W_p, W_p') = \Theta_p(\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{2.6}$$

while, the synchronizable state part V_p satisfies

$$\begin{cases} V_p'' - \Delta V_p + \tilde{A}_p V_p = -Z_2^\intercal W_p & \text{in } (0, +\infty) \times \Omega, \\ V_p = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ V_p = (y_1, \dots, y_p)^\intercal DH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: (V_p, V_p') = (y_1, \dots, y_p)^\intercal(\hat{U}_0, \hat{U}_1) & \text{in } \Omega. \end{cases} \tag{2.7}$$

Lemma 2.1 (cf. [9]). *Assume A satisfies the condition of Θ_p-compatibility*

$$A\,\mathrm{Ker}(\Theta_p) \subseteq \mathrm{Ker}(\Theta_p), \tag{2.8}$$

then $Z_1 = 0$ in (2.5), hence the generalized exact boundary synchronization for system (1.1) with respect to Θ_p is equivalent to the exact boundary null controllability for system (2.6), and thus equivalent to

$$\mathrm{rank}(\Theta_p D) = N - p. \tag{2.9}$$

We say that A satisfies the condition of Θ_p-strong compatibility if

$$\begin{cases} \mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \dots, \epsilon_p\} \text{ is an invariant subspace of } A, \\ A^\intercal \text{ admits an invariant subspace } \mathrm{Span}\{y_1, \dots, y_p\} \text{ which is} \\ \text{bi-orthonormal to } \mathrm{Span}\{\epsilon_1, \dots, \epsilon_p\}, \end{cases} \tag{2.10}$$

which is equivalent to $Z_1 = Z_2 = 0$ in (2.5), then $\bar{A}_p$ and $\tilde{A}_p$ are called the generalized reduced matrix and the generalized row-sum matrix, respectively.

Lemma 2.2 (cf. [9]). *(i) If system (1.1) is generalized exactly synchronizable with respect to Θ_p, and the synchronizable state part V_p is independent of applied boundary controls H, then A necessarily satisfies the condition of Θ_p-strong compatibility (2.10), and D satisfies* $\mathrm{Ker}(D^\intercal) = \mathrm{Span}\{y_1, \dots, y_p\}$.

(ii) If A satisfies the condition of Θ_p-strong compatibility (2.10), then there exists a boundary control matrix D with $\mathrm{Ker}(D^\intercal) = \mathrm{Span}\{y_1, \dots, y_p\}$, such that system (1.1) is generalized exactly synchronizable with respect to Θ_p, and system (2.7) of the synchronizable state part becomes

$$\begin{cases} V_p'' - \Delta V_p + \tilde{A}_p V_p = 0 & \text{in } (0, +\infty) \times \Omega, \\ V_p = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0 : (V_p, V_p') = (y_1, \dots, y_p)^\intercal (\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{2.11}$$

whose solution is determined only by the initial data and independent of applied boundary controls H, therefore the generalized exactly synchronizable state $u = V_p$ is independent of applied boundary controls H.

§3 Weakly induced generalized exact boundary synchronization (more groups and fewer controls)

According to [8], $\mathrm{Ker}(\Theta_p)$ can be extended to $\mathrm{Span}\{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}$ (not unique) by means of a Jordan basis $\{\tilde{\xi}_1, \dots, \tilde{\xi}_q, \dots, \tilde{\xi}_N\}$ $(q \geq p)$ of $\mathbb{R}^N$, under which A can be represented by its real Jordan form [1]:

$$A(\tilde{\xi}_1, \dots, \tilde{\xi}_q, \tilde{\xi}_{q+1}, \dots, \tilde{\xi}_N) = (\tilde{\xi}_1, \dots, \tilde{\xi}_q, \tilde{\xi}_{q+1}, \dots, \tilde{\xi}_N) \begin{pmatrix} J_q & 0 \\ 0 & J_{N-q} \end{pmatrix}, \tag{3.1}$$

in which q is the minimal value satisfying $\mathrm{Span}\{\tilde{\xi}_1, \dots, \tilde{\xi}_q\} \supseteq \mathrm{Ker}(\Theta_p)$. Then $\mathrm{Ker}(\hat{\Theta}_q) \overset{\text{def.}}{=} \mathrm{Span}\{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}$ is a minimal extension of $\mathrm{Ker}(\Theta_p)$, which realizes the corresponding condition of strong compatibility (see (3.3)). Thus, for the given generalized synchronization matrix Θ_p, we can introduce a generalized synchronization matrix $\hat{\Theta}_q$ by extending $\mathrm{Ker}(\Theta_p)$ through the coupling matrix A as follows.

Definition 3.1. *An $(N-q)\times N(p\le q\le N)$ full row-rank matrix $\hat{\Theta}_q$ is called the weakly induced generalized synchronization matrix corresponding to Θ_p, if* $\mathrm{Ker}(\hat{\Theta}_q)$ *as an extension of* $\mathrm{Ker}(\Theta_p)$*:*

$$\mathrm{Ker}(\hat{\Theta}_q)\supseteq \mathrm{Ker}(\Theta_p) \tag{3.2}$$

possesses the property that A satisfies the condition of $\hat{\Theta}_q$-strong compatibility:

$$\begin{cases}\mathrm{Ker}(\hat{\Theta}_q) \textit{ is an invariant subspace of } A,\\ A^{\mathsf{T}} \textit{ admits an invariant subspace which is bi-orthonormal to } \mathrm{Ker}(\hat{\Theta}_q),\end{cases} \tag{3.3}$$

and q is the minimal value satisfying (3.2) and (3.3). In particular, $\hat{\Theta}_N$ is a zero matrix when $q=N$.

Remark 3.2. *$q=p$ if and only if A satisfies the condition of Θ_p-strong compatibility (2.10). Thus we can only consider the case that the condition of Θ_p-strong compatibility (2.10) fails, namely, $q>p$.* □

In what follows we will give the generalized exact boundary synchronization with respect to the weakly induced generalized synchronization matrix $\hat{\Theta}_q$, called its weakly induced generalized exact boundary synchronization for short, including its relation to the original generalized exact boundary synchronization with respect to Θ_p, then we will show the benefits of introducing it.

Theorem 3.3. *The generalized exact boundary synchronization (1.4) with respect to Θ_p implies its weakly induced generalized exact boundary synchronization:*

$$t\ge T:\quad \hat{\Theta}_q U\equiv 0. \tag{3.4}$$

And system (1.1) possesses the weakly induced generalized exact boundary synchronization (3.4) if and only if

$$\mathrm{rank}(\hat{\Theta}_q D)=N-q. \tag{3.5}$$

Proof. It follows from (3.2) that the generalized exact boundary synchronization (1.5) implies

$$t\ge T:\ U\in \mathrm{Ker}(\hat{\Theta}_q), \tag{3.6}$$

which is just its weakly induced generalized exact boundary synchronization (3.4). Noting the condition of $\hat{\Theta}_q$-strong compatibility (3.3), by Lemma 2.1 we get the second part of the conclusion. □

Remark 3.4. *Noting that condition (3.5) can also be written as* $\mathrm{rank}(D^{\mathsf{T}}\hat{\Theta}_q^{\mathsf{T}})=N-q=\mathrm{rank}(\hat{\Theta}_q^{\mathsf{T}})$*, namely,* $\mathrm{Ker}(D^{\mathsf{T}})\cap \mathrm{Im}(\hat{\Theta}_q^{\mathsf{T}})=\{0\}$*, when the boundary control matrix D is given beforehand, for a generalized synchronization matrix Θ_p, some of its weakly induced generalized synchronization matrices $\hat{\Theta}_q$ may satisfy (3.5) and then hold the corresponding weakly induced generalized exact boundary synchronization (3.4), while, some may not. It should be selected suitably in applications(see Example 3.9).* □

Denoting

$$\mathrm{Ker}(\hat{\Theta}_q)=\mathrm{Span}\{\hat{\epsilon}_1,\dots,\hat{\epsilon}_q\}, \tag{3.7}$$

the weakly induced generalized exact boundary synchronization (3.4) can be written as: there exists a q-dimensional vector function $\hat{u} = (\hat{u}_1, \dots, \hat{u}_q)^\intercal$ such that

$$t \geq T: \; U = \hat{u}_1 \hat{\epsilon}_1 + \cdots + \hat{u}_q \hat{\epsilon}_q = (\hat{\epsilon}_1, \dots, \hat{\epsilon}_q)\hat{u}, \tag{3.8}$$

in which, the generalized synchronizable state $\hat{u}$ with respect to the weakly induced generalized synchronization matrix $\hat{\Theta}_q$ can be called as the weakly induced generalized exactly synchronizable state for short.

Theorem 3.5. *If we have the generalized exact boundary synchronization (1.6) with respect to Θ_p, then the weakly induced generalized exactly synchronizable state $\hat{u}$ can be expressed by its original generalized exactly synchronizable state u as follows:*

$$\hat{u} = \hat{Q}u, \tag{3.9}$$

where $\hat{Q}$ is a $q \times p$ matrix of full column-rank, given by

$$(\epsilon_1, \dots, \epsilon_p) = (\hat{\epsilon}_1, \dots, \hat{\epsilon}_q)\hat{Q}. \tag{3.10}$$

Proof. By $\mathrm{Ker}(\hat{\Theta}_q) \supseteq \mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \dots, \epsilon_p\}$, there exists a (unique) $q \times p$ matrix $\hat{Q}$ of full column-rank, such that (3.10) holds. Then plugging (3.10) into (1.6) and noting (3.8), we get (3.9). □

Remark 3.6. *$\hat{Q}$ in (3.9) represents the way of splitting the original synchronization groups (see Example 5.2).* □

Therefore, the generalized exact boundary synchronization with respect to Θ_p implies that with respect to $\hat{\Theta}_q$, which is easier to be realized and thus demands fewer boundary controls. We will show that when we reduce suitably the number of boundary controls, the generalized exactly synchronizable state with respect to $\hat{\Theta}_q$ can be independent of applied boundary controls.

By the condition of $\hat{\Theta}_q$-strong compatibility (3.3),

$$\mathrm{Span}\{y_1, \dots, y_q\} \tag{3.11}$$

as an $A^\intercal$-invariant subspace is bi-orthonormal to (3.7), then there are the generalized reduced matrix $\bar{A}_q$ and the generalized row-sum matrix $\tilde{A}_q$ such that

$$XAX^{-1} = \begin{pmatrix} \bar{A}_q & 0 \\ 0 & \tilde{A}_q \end{pmatrix}, \text{ where } X = \begin{pmatrix} \hat{\Theta}_q \\ (y_1, \dots, y_q)^\intercal \end{pmatrix}. \tag{3.12}$$

Thus by Lemma 2.2, we have

Theorem 3.7. *If and only if* $\mathrm{Ker}(D^\intercal)$ *is invariant for* $A^\intercal$ *and bi-orthonormal to* $\mathrm{Ker}(\hat{\Theta}_q)$*, there exists*

$$\mathrm{Span}\{y_1, \dots, y_q\} = \mathrm{Ker}(D^\intercal) \tag{3.13}$$

such that the corresponding synchronizable part $V_q = (y_1, \dots, y_q)^\intercal U$ of system (1.1) with respect to $\hat{\Theta}_q$ satisfies a problem independent of applied boundary controls:

$$\begin{cases} V_q'' - \Delta V_q + \tilde{A}_q V_q = 0 & \text{in } (0, +\infty) \times \Omega, \\ V_q = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0: (V_q, V_q') = (y_1, \dots, y_q)^\intercal (\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{3.14}$$

then the generalized exactly synchronizable state $\hat{u} = V_q$ with respect to $\hat{\Theta}_q$ is independent of applied boundary controls.

Remark 3.8. *By (3.11) and (3.7), the projection operator discussed in [8] is*

$$P = (\hat{\epsilon}_1, \dots, \hat{\epsilon}_q)(y_1, \dots, y_q)^\intercal,$$

thus the independence of applied boundary controls for the synchronizable part $U_s = PU$ of system (1.1) with respect to Θ_p is actually that for its synchronizable state part $V_q = (y_1, \dots, y_q)^\intercal U$ with respect to the corresponding weakly induced synchronization matrix $\hat{\Theta}_q$.

Noting that $\mathrm{rank}(D) = N - q$ by Lemma 2.2 and that $\mathrm{rank}(D) \geq N - p$ under the generalized exact boundary synchronization with respect to Θ_p, it is necessary to have $q = p$, that is to say, A satisfies the condition of Θ_p-strong compatibility (2.10), in order to ensure the independence of applied boundary controls for the synchronizable part $U_s = PU$, i.e., for the synchronizable state part V_q with respect to $\hat{\Theta}_q$.

However, the weakly induced generalized exact boundary synchronization is weaker than the original one, as a result, Theorem 3.7 does not require the condition of Θ_p-strong compatibility (2.10) for the coupling matrix A. □

Example 3.9. *Let $N = 3$. Consider the coupled system of wave equations given by*

$$\begin{cases} u^{(1)''} - \Delta u^{(1)} + u^{(1)} + u^{(2)} = 0 & \text{in } (0, +\infty) \times \Omega, \\ u^{(2)''} - \Delta u^{(2)} + u^{(2)} = 0 & \text{in } (0, +\infty) \times \Omega, \\ u^{(3)''} - \Delta u^{(3)} + u^{(3)} = 0 & \text{in } (0, +\infty) \times \Omega, \end{cases} \tag{3.15}$$

in which the coupling matrix is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.16}$$

Given the generalized synchronization matrix

$$\Theta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.17}$$

it is easy to see that

$$\mathrm{Ker}(\Theta_1) = \mathrm{Span}\{\epsilon_1\}$$

and $A\epsilon_1 = \epsilon_1$, in which $\epsilon_1 = (1, 0, 0)^\intercal$, and $x = \epsilon_2 = (0, 1, \gamma)^\intercal$ meets $(A - I)x = \epsilon_1$, where γ is an arbitrarily given real constant. Then the minimal extension of $\mathrm{Ker}(\Theta_1)$ satisfying the condition of strong compatibility (3.3) is

$$\mathrm{Ker}(\hat{\Theta}_2) = \mathrm{Span}\{\epsilon_1, \epsilon_2\},$$

and the corresponding weakly induced generalized synchronization matrix is

$$\hat{\Theta}_2 = \begin{pmatrix} 0 & \gamma & -1 \end{pmatrix}. \tag{3.18}$$

Thus there are infinitely many weakly induced generalized synchronization matrices $\hat{\Theta}_2$ as γ varies. The generalized exact boundary synchronization with respect to $\hat{\Theta}_2$ means that

$$t \geq T: \ \hat{\Theta}_2 U = 0, \ \text{namely, } U = (\epsilon_1, \epsilon_2)\hat{u} = (\hat{u}_1, \hat{u}_2, \gamma\hat{u}_2)^\intercal, \tag{3.19}$$

where $\hat{u} = (\hat{u}_1, \hat{u}_2)^\intercal$ is the corresponding weakly induced generalized exactly synchronizable state.

Since we only need one boundary control to realize the weakly induced generalized exact boundary synchronization (3.19), we suppose that the boundary control matrix $D \in \mathbb{M}^{3\times 1}(\mathbb{R})$. *Then the weakly induced generalized exact boundary synchronization (3.19) holds if and only if* γ *satisfies* $\text{rank}(\hat{\Theta}_2 D) = 1$, *namely,*

$$D = (d_1, d_2, d_3)^{\intercal}, \text{ with } \gamma d_2 - d_3 \neq 0. \tag{3.20}$$

Besides, it is easy to see that $\text{Span}\{y_1, y_2\}$ *is an* $A^{\intercal}$*-invariant subspace which is bi-orthonormal to* $\text{Span}\{\epsilon_1, \epsilon_2\}$, *where* $y_1 = (1, k\gamma, -k)^{\intercal}$ *and* $y_2 = (0,1,0)^{\intercal}$ *with* k *being a real constant. Thus, only when* $\text{Ker}(D^{\intercal}) = \text{Span}\{y_1, y_2\}$, *namely, when*

$$D = (kd_3, 0, d_3)^{\intercal}, \text{ where } d_3 \neq 0 \text{ and } k \text{ is a real constant}, \tag{3.21}$$

the corresponding synchronizable state part $V_2 = (v_1, v_2)^{\intercal} = (y_1, y_2)^{\intercal} U = (u^{(1)} + k\gamma u^{(2)} - ku^{(3)}, u^{(2)})^{\intercal}$ *satisfies*

$$\begin{cases} v_1'' - \Delta v_1 + v_1 + v_2 = 0 & \text{in } (0,+\infty)\times\Omega, \\ v_2'' - \Delta v_2 + v_2 = 0 & \text{in } (0,+\infty)\times\Omega, \\ v_1 = v_2 = 0 & \text{on } (0,+\infty)\times\Gamma, \\ t=0: (V_2, V_2') = (y_1, y_2)^{\intercal}(\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{3.22}$$

which depends only on the initial data $(\hat{U}_0, \hat{U}_1)$, k*(i.e., the boundary control matrix* D*) and* γ*(i.e., the weakly induced generalized synchronization matrix* $\hat{\Theta}_2$*), but not on applied boundary controls, therefore, the corresponding generalized exactly synchronizable state* $\hat{u} = V_2$ *is independent of applied boundary controls.* □

§4 Strongly induced generalized exact boundary synchronization (fewer groups and more controls)

Similarly, for the given generalized synchronization matrix Θ_p, we can introduce a generalized synchronization matrix by extending $\text{Im}(\Theta_p^{\intercal})$ through $A^{\intercal}$ as follows.

Definition 4.1. *An* $(N-r)\times N (0 \le r \le p)$ *full row-rank matrix* $\tilde{\Theta}_r$ *is called the strongly induced generalized synchronization matrix corresponding to* Θ_p, *if* $\text{Im}(\tilde{\Theta}_r^{\intercal})$ *as an extension of* $\text{Im}(\Theta_p^{\intercal})$:

$$\text{Im}(\tilde{\Theta}_r^{\intercal}) \supseteq \text{Im}(\Theta_p^{\intercal}) \tag{4.1}$$

possesses the property that $A^{\intercal}$ *satisfies*

$$\begin{cases} \text{Im}(\tilde{\Theta}_r^{\intercal}) \text{ is an invariant subspace of } A^{\intercal}, \\ A \text{ admits an invariant subspace which is bi-orthonormal to } \text{Im}(\tilde{\Theta}_r^{\intercal}), \end{cases} \tag{4.2}$$

and $(N-r)$ *is the minimal value satisfying (4.1) and (4.2).*

Remark 4.2. *Condition (4.2) is actually the condition of* $\tilde{\Theta}_r$*-strong compatibility:*

$$\begin{cases} \text{Ker}(\tilde{\Theta}_r) \text{ is an invariant subspace of } A, \\ A^{\intercal} \text{ admits an invariant subspace which is bi-orthonormal to } \text{Ker}(\tilde{\Theta}_r). \end{cases} \tag{4.3}$$

Evidently, $r = p$ *if and only if* A *satisfies the condition of* Θ_p*-strong compatibility (2.10). Thus we can only consider the case that the condition of* Θ_p*-strong compatibility (2.10) fails, namely,* $0 \le r < p$. □

Now we look at the generalized exact boundary synchronization with respect to the strongly induced generalized synchronization matrix $\tilde{\Theta}_r$, called its strongly induced generalized exact boundary synchronization for short, including its relation to the original generalized exact boundary synchronization with respect to Θ_p, and show the benefits of introducing it.

Theorem 4.3. *The generalized exact boundary synchronization (1.4) with respect to* Θ_p *can be implied by its strongly induced generalized exact boundary synchronization:*

$$t \ge T: \quad \tilde{\Theta}_r U \equiv 0. \tag{4.4}$$

And system (1.1) possesses the strongly induced generalized exact boundary synchronization (4.4) if and only if

$$\mathrm{rank}(\tilde{\Theta}_r D) = N - r. \tag{4.5}$$

Proof. It follows from (4.1) that $\mathrm{Ker}(\tilde{\Theta}_r) \subseteq \mathrm{Ker}(\Theta_p)$, thus the generalized exact boundary synchronization (1.5) can be implied by

$$t \ge T: \ U \in \mathrm{Ker}(\tilde{\Theta}_r), \tag{4.6}$$

which is just the strongly induced generalized exact boundary synchronization (4.4). Noting the condition of strong $\tilde{\Theta}_r$-compatibility (4.3), by Lemma 2.1 we have the second part of the conclusion. □

Remark 4.4. *Similar to Remark 3.4, when the boundary control matrix* D *is given, for a given generalized synchronization matrix* Θ_p*, some of its strongly induced generalized synchronization matrices* $\tilde{\Theta}_r$ *may satisfy (4.5) and then have the corresponding strongly induced generalized exact boundary synchronization (4.4), while, some may not. This asks us to take a suitable choice in practices (see Example 4.8).* □

Denoting $\mathrm{Ker}(\tilde{\Theta}_r) = \mathrm{Span}\{\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_r\}$, the strongly induced generalized exact boundary synchronization (4.4) is equivalent to that: there exists an r-dimensional vector function $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_r)^\intercal$ such that

$$t \ge T: \ U = \tilde{u}_1\tilde{\epsilon}_1 + \cdots + \tilde{u}_r\tilde{\epsilon}_r = (\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_r)\tilde{u}, \tag{4.7}$$

in which, the generalized synchronizable state $\tilde{u}$ with respect to the strongly induced generalized synchronization matrix $\tilde{\Theta}_r$ can be called as the strongly induced generalized exactly synchronizable state for short.

Theorem 4.5. *If we have the strongly induced generalized exact boundary synchronization (4.7), then the original generalized exactly synchronizable state* u*, defined by (1.6), can be expressed through its strongly induced exactly synchronizable state* $\tilde{u}$ *by*

$$u = \tilde{Q}\tilde{u}, \tag{4.8}$$

where $\tilde{Q}$ *is a* $p \times r$ *matrix of full column-rank, given by*

$$(\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_r) = (\epsilon_1, \dots, \epsilon_p)\tilde{Q}. \tag{4.9}$$

Remark 4.6. *$\tilde{Q}$ in (4.8) represents the way of merging synchronization groups (see Example 5.2).* □

Therefore, the generalized exact boundary synchronization of system (1.1) with respect to $\tilde{\Theta}_r$ implies that with respect to Θ_p. The strongly induced generalized exact boundary synchronization (4.6) possesses more requirements and fewer synchronization groups ($r \le p$), hence more boundary controls are needed. When the boundary control matrix D is suitably set, the strongly induced generalized exactly synchronizable state $\tilde{u}$ can be independent of applied boundary controls.

Due to the condition of $\tilde{\Theta}_r$-strong compatibility (4.3), $\text{Span}\{y_1, \dots, y_r\}$ as an invariant subspace of $A^\intercal$ is bi-orthonormal to $\text{Ker}(\tilde{\Theta}_r) = \text{Span}\{\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_r\}$, then there exist the generalized reduced matrix $\bar{A}_r$ and the row-sum matrix $\tilde{A}_r$ such that

$$XAX^{-1} = \begin{pmatrix} \bar{A}_r & 0 \\ 0 & \tilde{A}_r \end{pmatrix}, \text{ where } X = \begin{pmatrix} \tilde{\Theta}_r \\ (y_1, \dots, y_r)^\intercal \end{pmatrix}. \tag{4.10}$$

Thus by Lemma 2.2 we have

Theorem 4.7. *If and only if $\text{Ker}(D^\intercal)$ is invariant for $A^\intercal$ and bi-orthonormal to $\text{Ker}(\tilde{\Theta}_r)$, there exists*

$$\text{Span}\{y_1, \dots, y_r\} = \text{Ker}(D^\intercal) \tag{4.11}$$

such that the corresponding synchronizable state part $V_r = (y_1, \dots, y_r)^\intercal U$ of system (1.1) with respect to $\tilde{\Theta}_r$ satisfies the following problem independent of boundary controls:

$$\begin{cases} V_r'' - \Delta V_r + \tilde{A}_r V_r = 0 & \text{in } (0, +\infty) \times \Omega, \\ V_r = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0 : (V_r, V_r') = (y_1, \dots, y_r)^\intercal (\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{4.12}$$

then the generalized exactly synchronizable state $\tilde{u} = V_r$ with respect to $\tilde{\Theta}_r$ is independent of applied boundary controls.

Example 4.8. *Let $N = 3$. For the coupled system of wave equations given by (3.15) in which the coupling matrix is (3.16), consider the strongly induced generalized exact boundary synchronizations for*

$$\Theta_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}. \tag{4.13}$$

Noting that

$$\text{Im}(\Theta_2^\intercal) = \text{Span}\{e_2\}$$

and $A^\intercal e_2 = e_2$, where $e_2 = (0, 1, 0)^\intercal$, there is $e_1 = (1, 0, -\alpha)^\intercal$ such that $(A^\intercal - I)e_1 = e_2$, where α is an arbitrarily given real number. Thus the minimal extension of $\text{Im}(\Theta_2^\intercal)$ satisfying the condition of strong compatibility (4.2) is

$$\text{Im}(\tilde{\Theta}_1^\intercal) = \text{Span}\{e_1, e_2\},$$

then

$$\text{Ker}(\tilde{\Theta}_1) = \text{Span}\{\epsilon\},$$

where $\epsilon = (\alpha, 0, 1)^\intercal$, and the corresponding strongly induced generalized exact synchronization *matrix is*

$$\tilde{\Theta}_1 = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.14}$$

correspondingly there are many strongly induced generalized exact boundary synchronizations (4.6) as α varies. The generalized exact boundary synchronization with respect to $\tilde{\Theta}_1$ is written as

$$t \geq T: \ \tilde{\Theta}_1 U = 0, \ \text{namely}, \ U = \epsilon \tilde{u} = (\alpha \tilde{u}, 0, \tilde{u})^{\intercal}, \tag{4.15}$$

where $\tilde{u}$ is the corresponding strongly induced generalized exactly synchronizable state.

Suppose that the boundary control matrix is $D \in \mathbb{M}^{3\times 2}(\mathbb{R})$, then the strongly induced generalized exact boundary synchronization (4.15) holds if and only if α satisfies $\operatorname{rank}(\tilde{\Theta}_1 D) = 2$, *namely,*

$$D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_{3\times 2}, \ \text{where } d_1 - \alpha d_3 \text{ and } d_2 \text{ are linearly independent.} \tag{4.16}$$

As $y = (0, -k, 1)^{\intercal}$ is an eigenvector of $A^{\intercal}$ and bi-orthonormal to $\epsilon = (\alpha, 0, 1)^{\intercal}$, when D satisfies $\operatorname{Ker}(D^{\intercal}) = \operatorname{Span}\{y\}$, *namely,*

$$D = \begin{pmatrix} d_1 \\ d_2 \\ kd_2 \end{pmatrix}_{3\times 2}, \ \text{where } d_1 \text{ and } d_2 \text{ are linearly independent, and } k \text{ is a real number,} \tag{4.17}$$

the corresponding synchronizable state part $v = y^{\intercal} U = u^{(3)} - ku^{(2)}$ satisfies

$$\begin{cases} v'' - \Delta v + v = 0 & \text{in } (0, +\infty) \times \Omega, \\ v = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0: (v, v') = y^{\intercal}(\hat{U}_0, \hat{U}_1) & \text{in } \Omega, \end{cases} \tag{4.18}$$

which depends only on the initial data $(\hat{U}_0, \hat{U}_1)$ and k (i.e., the boundary control matrix D), but not on applied boundary controls, therefore the generalized exactly synchronizable state $\tilde{u} = v$ is independent of applied boundary controls. □

§5 Examples of induced generalized exact boundary synchronizations

At last we offer some examples to explain the relations between the original generalized exact boundary synchronization and the corresponding induced generalized exact boundary synchronizations.

Example 5.1. *Assume the coupling matrix A of system (1.1) is similar to a Jordan block. Then, for any given generalized synchronization matrix $\Theta_p (0 < p < N)$,*

- *its weakly induced generalized synchronization matrix is a zero matrix 0, hence the corresponding weakly induced generalized exact boundary synchronization always holds without any demands on boundary controls as well as on the boundary control matrix;*
- *its strongly induced generalized synchronization matrix is an invertible matrix $\tilde{\Theta}_0$, there-*

fore the corresponding strongly induced generalized exact boundary synchronization is actually exact boundary null controllability of system (1.1), and requires N boundary controls, i.e., the boundary control matrix D should be invertible and the corresponding generalized exactly synchronizable state is $u \equiv 0$.

These two induced generalized exact boundary synchronizations are both trivial. □

Example 5.2. *Let $N = 5$. Consider the coupled system of wave equations*

$$\begin{cases} u^{(1)''} - \Delta u^{(1)} + u^{(1)} = 0 & \text{in } (0, +\infty) \times \Omega, \\ u^{(2)''} - \Delta u^{(2)} - u^{(1)} + 2u^{(2)} = 0 & \text{in } (0, +\infty) \times \Omega, \\ u^{(3)''} - \Delta u^{(3)} - 2u^{(1)} + u^{(2)} + 2u^{(3)} = 0 & \text{in } (0, +\infty) \times \Omega, \\ u^{(4)''} - \Delta u^{(4)} - 2u^{(1)} + u^{(2)} - u^{(3)} + 3u^{(4)} = 0 & \text{in } (0, +\infty) \times \Omega, \\ u^{(5)''} - \Delta u^{(5)} + u^{(5)} = 0 & \text{in } (0, +\infty) \times \Omega \end{cases} \tag{5.1}$$

with the boundary conditions

$$\begin{cases} u^{(1)} = u^{(2)} = u^{(3)} = u^{(4)} = u^{(5)} = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ u^{(1)} = u^{(3)} = u^{(5)} = 0, \ u^{(2)} = h^{(1)}, \ u^{(4)} = h^{(2)} & \text{on } (0, +\infty) \times \Gamma_1, \end{cases} \tag{5.2}$$

where the coupling matrix and the boundary control matrix are

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 \\ -2 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5.3}$$

respectively. Provide a generalized synchronization matrix

$$\Theta_3 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \tag{5.4}$$

and a basis of its kernel space:

$$\epsilon_1 = (1, 1, 0, 0, 0)^\intercal, \ \epsilon_2 = (0, 0, 1, 1, 0)^\intercal, \ \epsilon_3 = (0, 0, 0, 0, 1)^\intercal. \tag{5.5}$$

Since $A\epsilon_1 = \epsilon_1 - \epsilon_2$, $A\epsilon_2 = 2\epsilon_2$ and $A\epsilon_3 = 4\epsilon_3$, A possesses the condition of Θ_3-compatibility. According to Lemma 2.1, it follows from $\text{rank}(\Theta_3 D) = N - 3 = 2$ *that the system is generalized exactly synchronizable with respect to Θ_3:*

$$t \geq T: \ \Theta_3 U = 0, \ \text{namely, } U = (\epsilon_1, \epsilon_2, \epsilon_3)u = (u^{(1)}, u^{(1)}, u^{(3)}, u^{(3)}, u^{(5)})^\intercal, \tag{5.6}$$

which is just the usual exact boundary synchronization by 3 groups. It is easy to see that the generalized exactly synchronizable state $u = (u^{(1)}, u^{(3)}, u^{(5)})^\intercal$ is governed by the equations of $u^{(1)}, u^{(3)}$ and $u^{(5)}$, where the equation of $u^{(3)}$ relies on $u^{(2)}$ then on applied boundary controls.

Now we consider the weakly and strongly induced generalized exact boundary synchronizations for Θ_3. Denoting

$$\xi_1 = (1, 1, 1, 1, 0)^\intercal, \ \xi_2 = (0, 0, 1, 1, 0)^\intercal, \ \xi_3 = (0, 1, 0, 0, 0)^\intercal, \ \xi_4 = (0, 0, 0, 1, 0)^\intercal,$$
$$\xi_5 = (0, 0, 0, 0, 1)^\intercal,$$

we have

$$A(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = (\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \tag{5.7}$$

and $\mathrm{Ker}(\Theta_3) = \mathrm{Span}\{\epsilon_1, \epsilon_2, \epsilon_3\} = \mathrm{Span}\{\xi_1, \xi_2, \xi_5\}$. *Therefore,*

- *the weakly induced generalized synchronization matrix is*

$$\hat{\Theta}_4 = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \end{pmatrix}, \tag{5.8}$$

which is the last row of the original generalized synchronization matrix Θ_3*, and* $\mathrm{Ker}(\hat{\Theta}_4) = \mathrm{Span}\{\xi_1, \xi_2, \xi_3, \xi_5\}$. *For simplicity, we take the synchronization basis of* $\mathrm{Ker}(\hat{\Theta}_4)$ *as*

$$\hat{\epsilon}_1 = (1,0,0,0,0)^{\intercal},\ \hat{\epsilon}_2 = (0,1,0,0,0)^{\intercal},\ \hat{\epsilon}_3 = (0,0,1,1,0)^{\intercal},\ \hat{\epsilon}_4 = (0,0,0,0,1)^{\intercal}. \tag{5.9}$$

Since $\mathrm{rank}(\hat{\Theta}_4 D) = 1$*, the system possesses the generalized exact boundary synchronization with respect to* $\hat{\Theta}_4$*:*

$$t \geq T:\ \hat{\Theta}_4 U = 0,\ \text{namely, } U = (\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3, \hat{\epsilon}_4)\hat{u} = (u^{(1)}, u^{(2)}, u^{(3)}, u^{(3)}, u^{(5)})^{\intercal}, \tag{5.10}$$

which is the usual exact boundary synchronization by 4 *groups, and the corresponding generalized exactly synchronizable state is* $\hat{u} = (u^{(1)}, u^{(2)}, u^{(3)}, u^{(5)})^{\intercal}$*, where* $u^{(2)}$ *depends on applied boundary controls. However, if we diminish the number of boundary controls and take the boundary control matrix to be*

$$\hat{D} = (0,0,0,1,0)^{\intercal}, \tag{5.11}$$

then the system realizes the generalized exact boundary synchronization with respect to $\hat{\Theta}_4$*, and the corresponding generalized exactly synchronizable state* $\hat{u}$ *is independent of applied boundary controls.*

- *the strongly induced synchronization matrix is*

$$\tilde{\Theta}_2 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}, \tag{5.12}$$

which adds an additional row to the original generalized synchronization matrix Θ_3*, and* $\mathrm{Ker}(\tilde{\Theta}_2) = \mathrm{Span}\{\xi_1, \xi_5\}$. *Choose naturally a synchronization basis of* $\mathrm{Ker}(\tilde{\Theta}_2)$ *to be*

$$\tilde{\epsilon}_1 = \xi_1 = (1,1,1,1,0)^{\intercal},\ \tilde{\epsilon}_2 = \xi_5 = (0,0,0,0,1)^{\intercal}. \tag{5.13}$$

Since $\mathrm{rank}(\tilde{\Theta}_2 D) \leq \mathrm{rank}(D) < 3$*, we have to use more boundary controls to realize the generalized exact boundary synchronization with respect to* $\tilde{\Theta}_2$*:*

$$t \geq T:\ \tilde{\Theta}_2 U = 0,\ \text{namely, } U = (\tilde{\epsilon}_1, \tilde{\epsilon}_2)\tilde{u} = (u^{(1)}, u^{(1)}, u^{(1)}, u^{(1)}, u^{(5)})^{\intercal}, \tag{5.14}$$

which is just the usual exact boundary synchronization by 2 *groups, where the generalized exactly synchronizable state is* $\tilde{u} = (u^{(1)}, u^{(5)})^{\intercal}$. *By increasing the rank of boundary control matrix,* *suitably reset it as*

$$\tilde{D} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.15}$$

correspondingly, the system is generalized exactly synchronizable with respect to $\tilde{\Theta}_2$, *and the generalized exactly synchronizable state* $\tilde{u}$ *is independent of applied boundary controls.*

Comparing to the original generalized synchronization matrix and the original boundary control matrix, we can see the contraction or extension of induced generalized synchronization matrices as well as boundary control matrices. In fact, (5.6) shows that the original generalized exact boundary synchronization is in three groups: $u^{(1)} = u^{(2)}$, $u^{(3)} = u^{(4)}$ *and* $u^{(5)}$; *and the weakly induced generalized exact boundary synchronization (5.10) splits one of its groups into two groups* $u^{(1)}$ *and* $u^{(2)}$, *and the rest does not change; while the strongly induced generalized exact boundary synchronization (5.14) merges two of its groups into one group* $u^{(1)} = u^{(2)} = u^{(3)} = u^{(4)}$, *and the rest remains the same.* □

References

[1] I Gohberg, P Lancaster, L Rodman. *Invariant Subspaces of Matrices with Applications*, John Wiley & Sons, Inc, New York, 1986.

[2] T T Li, B P Rao. *Synchronisation exacte d'un système couplé d'équations des ondes par des contrôles frontières de Dirichlet*, C R Math Acad Sci Paris, 2012, 350(15-16): 767-772.

[3] T T Li, B P Rao. *Exact synchronization for a coupled system of wave equations with Dirichlet boundary controls*, Chin Ann Math Ser B, 2013, 34(1): 139-160.

[4] T T Li, B P Rao. *A note on the exact synchronization by groups for a coupled system of wave equations*, Math Methods Appl Sci, 2015, 38(2): 241-246.

[5] T T Li, B P Rao. *On the exactly synchronizable state to a coupled system of wave equations*, Port Math, 2015, 72(2-3): 83-100.

[6] T T Li, B P Rao. *Exact synchronization by groups for a coupled system of wave equations with Dirichlet boundary controls*, J Math Pures Appl, 2016, 105(1): 86-101.

[7] J L Lions. *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués Tome 1*, Masson, Paris, 1988.

[8] Y Y Wang. *Generalized exact boundary synchronization for a coupled system of wave equations with Dirichlet boundary controls*, to appear in Chin Ann Math, Ser B.

[9] Y Y Wang. *On the generalized exact boundary synchronization for a coupled system of wave equations*, Math Methods Appl Sci, 2019, 42(18): 7011–7129.

School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
Email: yanyanwang@fudan.edu.cn