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New Analytical and Numerical Results For Fractional Bogoyavlensky-Konopelchenko Equation Arising in Fluid Dynamics

Ali Kurt

Abstract. In this article, $(2+1)$-dimensional time fractional Bogoyavlensky-Konopelchenko (BK) equation is studied, which describes the interaction of wave propagating along the x axis and y axis. To acquire the exact solutions of BK equation we employed sub equation method that is predicated on Riccati equation, and for numerical solutions the residual power series method is implemented. Some graphical results that compares the numerical and analytical solutions are given for different values of μ. Also comparative table for the obtained solutions is presented.

§1 Introduction

Fractional calculus has been a desirable area of research for several years and is growing rapidly and ongoing development. As a branch of infinitesimal calculus, fractional calculus specially studies the possibility of performing fractional differentiation operator on functions, such as a derivative of order 1/2. In general, fractional calculus involves the following main advantages:

1. Fractional calculus can easily express the historical dependence of the evolution of system analysis by taking the global correlation into consideration, but integer calculus is not convenient to represent this process for its locality characteristics.

2. Fractional calculus overcomes the critical defect of integer calculus that the theoretical model results often fail to coincide with the experimental results. On the contrary, it could get a good coincidence by using a few parameters.

3. When describing complicated physical mechanics problems, fractional calculus has a clearer physical significance and a simpler expression compared with the nonlinear model.

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To benefit these advantages scientists expressed different definitions of fractional derivative and integrals. During the past four decades or so, various operators of fractional calculus are declared such as those named after Riemann-Liouville, Weyl, Hadamard, Grunwald-Letnikov, Riesz, Erdelyi-Kober, Liouville-Caputo, Caputo and so on [1–4]. Almost all of these operators involve the integral representation in their definition. This situation makes the calculations harder and the exact solution of the considered mathematical model can not be obtained. Also some of these definitions do not satisfy the basic properties to be a fractional derivative definition [5]. For instance the derivative of a arbitrary constant is not zero when the Riemann-Liouville is used as derivative operator. In addition Caputo and Riemann-Liouville definitions do not satisfy the rules for derivative of quotient of two functions, derivative of product of two functions, chain rule and etc. Recently a new fractional operator called "conformable fractional derivative and integral" is introduced by Khalil et al. [6] which overcomes the above mentioned deficiencies.

Definition Let $f : [0, \infty) \to \mathbb{R}$ be a function. The μ^{th} order *"conformable fractional derivative"* (CFD) [6] of f is defined as,

$$D_\mu(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\mu}) - f(t)}{\varepsilon},$$

for all $t > 0, \mu \in (0, 1)$. Let f be a μ-differentiable function in some $(0, a), a > 0$ and $\lim_{t \to 0^+} f^{(\mu)}(t)$ exists then $f^{(\mu)}(0) = \lim_{t \to 0^+} f^{(\mu)}(t)$.

The *"conformable fractional integral"* of f starting from $a \geq 0$ is defined as:

$$I_\mu^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\mu}} dx,$$

where the integral indicates the usual Riemann improper integral, and $\mu \in (0, 1]$. The CFD satisfies the properties that are given in the following theorem [6].

Theorem 1.1. *Let $\mu \in (0, 1]$ and suppose f, g are μ-differentiable at point $t > 0$. Then*

1. $D_\mu(cf + dg) = cD_\mu(f) + cD_\mu(g)$ *where Let* $c, f \in \mathbb{R}$.
2. $D_\mu(t^p) = pt^{p-\mu}$ *where* $p \in \mathbb{R}$.
3. *Let λ be a constant then* $D_\mu(\lambda) = 0$.
4. $D_\mu(fg) = fD_\mu(g) + gD_\mu(f)$.
5. $D_\mu\left(\frac{f}{g}\right) = \frac{gD_\mu(f) - fD_\mu(g)}{g^2}$.

After this definition was expressed, which can be described as a milestone for fractional calculus, huge amount of articles has been done that involves conformable fractional derivative and integral [7–10, 26, 30, 31]. For instance Javeed et al. [11] constructed the solution of conformable fractional Burger-Poisson equation by using homotopy perturbation method. Rosales et al. [12] employed conformable fractional derivative to analyze the classical Drude model. Alharbi et al. [28] investigated the projectile motion with the aid of conformable fractional derivative.

Yaslan and Girgin [14] implemented G'/G^2 expansion method to get the exact solutions of conformable time fractional Caudrey-Dodd-Gibbon equation, Calogero-Bogoyavlenskii-Schiff (CBS) equation and Ablowitz-Kaup-Newell-Segur equations. Cenesiz et al. [15] established the exact and numerical solutions of fractional Hirota-Satsuma Coupled KdV System where the fractional terms are in conformable sense. Also many researches including conformable derivative can be cited therein [27–29].

In this article the new analytical and numerical solutions of $(2+1)$ dimensional time fractional BK Equation

$$D_t^{\mu} D_x u + \alpha D_x^4 u + \beta D_x^3 D_y u + 6\alpha D_x^2 u D_x u + 4\beta D_x D_y u D_x u + 4\beta D_x^2 u D_y u = 0, \tag{1}$$

that identifies the interaction of a Riemann wave propagating along the y-axis and a long wave propagating along the x-axis in a fluid where u is a function of the space coordinates x and y and time coordinate t and the parameters α and β are constant. To the best of our knowledge all the obtained results are seen for the first time in the literature and will benefit the scientists that study on wave propagation.

§2 Brief Description of Implemented Methods

2.1 Sub Equation Method

Lets describe the considered method called sub equation method [16] which established on the Riccati equation

$$\varphi'(\xi) = \sigma + (\varphi(\xi))^2 . \tag{2}$$

The general form of nonlinear time fractional partial differential equation is

$$P\left(u, D_t^{\mu} u, D_x u, D_y u, D_x^2 u, D_y^2 u, D_x D_y u ...\right) = 0, \tag{3}$$

where $D_t^{\mu} u$ denotes conformable derivative operator with fractional order. Regarding the fractional wave transformation [17]

$$u(x, y, t) = U(\xi), \ \xi = kx + ly + w\frac{t^{\mu}}{\mu}, \tag{4}$$

where k, w, l are any constants to be evaluated later and the chain rule [18], Eq. (3) becomes an integer order nonlinear ordinary differential equation

$$G(U(\xi), U'(\xi), U''(\xi), ...) = 0. \tag{5}$$

Suppose that the solution of the reduced Eq. (5) in the following form

$$U(\xi) = \sum_{i=0}^{N} a_i \varphi^i(\xi), \ a_N \neq 0, \tag{6}$$

where a_i $(0 \leq i \leq N)$ are stable coefficients to be examined and N is can be acquired by balancing principle [19] in Eq. (5) and $\varphi(\xi)$ is the solution of Riccati Eq. (2). A set of the solutions which verify the Eq. (2) is given below.

$$\varphi(\xi)=\begin{cases} -\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right), & \sigma<0\\ -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right), & \sigma<0\\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right), & \sigma>0\\ -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right), & \sigma>0\\ -\frac{1}{\xi+\varpi},\ \varpi\ is\ a\ cons., & \sigma=0 \end{cases} \tag{7}$$

Collecting the all the obtained data sets we attain a polynomial including $\varphi(\xi)$. Vanishing the coefficients of $\varphi^i(\xi)$ $(i=0,1,...,N)$ produces an algebraic system in k,l,w,a_i $(i=0,1,...,N)$. Calculating the solution of these nonlinear algebraic equation we obtain the values for l,w,k,a_i $(i=0,1,...,N)$. Substituting all the obtained values in the formulas (7) we get the exact solutions for Eq. (3).

2.2 Residual Power Series Method

To illustrate the idea of RPSM [20–23], one can regard the following equation [24]:

$$T_\mu u(x,y,t)+N[x,y]u(x,y,t)+L[x,y]u(x,y,t)=c(x,y,t), \tag{8}$$

where $x\in\mathbb{R}$, $n-1<n\mu\le n$, $t>0$ and given with the initial condition

$$f_0(x,y)=u(x,y,0)=f(x,y). \tag{9}$$

where the operator $L[x,y]$ is linear, the operator $N[x,y]$ is nonlinear and $c(x,y,t)$ are arbitrary function.

The RPSM established on the indicating the solution of the Eq. (8) according to (9) with fractional power series expansion around $t=0$.

$$f_{(n-1)}(x,y)=T_t^{(n-1)\mu}u(x,y,0)=h(x,y). \tag{10}$$

The expanded version of the solution can be regarded as

$$u(x,y,t)=f(x,y)+\sum_{n=1}^{\infty}f_n(x,y)\frac{t^{n\mu}}{\mu^n n!}. \tag{11}$$

Then, $u_k(x,y,t)$ can be written as:

$$u_k(x,y,t)=f(x,y)+\sum_{n=1}^{k}f_n(x,y)\frac{t^{n\mu}}{\mu^n n!}. \tag{12}$$

Since the $1st$ approximate solution $u_1(x,y,t)$ is

$$u_1(x,y,t)=f(x,y)+f_1(x,y)\frac{t^\mu}{\mu}, \tag{13}$$

so $u_k(x,y,t)$ can be rearranged as follows

$$u_k(x,y,t)=f(x,y)+f_1(x,y)\frac{t^\mu}{\mu}+\sum_{n=2}^{k}f_n(x,y)\frac{t^{n\mu}}{\mu^n n!},\ k=2,3,4,... \tag{14}$$

for $0<\mu\le 1$, $0\le t<\mathbb{R}^{\frac{1}{v}}$, $x\in I$.

Initially we express the residual function and the $k-th$ residual function

$$Res(x,y,t) = T_\mu u(x,y,t) + N[x,y]u(x,y,t) + L[x,y]u(x,y,t) - c(x,y,t), \tag{15}$$

$$Res_k(x,t) = T_\mu u_k(x,y,t) + N[x,y]u_k(x,y,t) + L[x,y]u_k(x,y,t) - c(x,y,t),\ k = 1,2,3,... \tag{16}$$

respectively. Obviously, $Res(x,y,t) = 0$ and $\lim_{k\to\infty} Res_k(x,y,t) = Res(x,y,t)$ for each $x \in I$ and $t \geq 0$. Indeed this bring about $\frac{\partial^{(n-1)\mu}}{\partial t^{(n-1)\mu}} Res_k(x,y,t) = 0$ for $n = 1,2,3,...,k$ [22, 23, 25]. Evaluating the solution of the equation $\frac{\partial^{(n-1)\mu}}{\partial t^{(n-1)\mu}} Res_k(x,y,0) = 0$ concludes the necessary $f_n(x,y)$ coefficients. Thus the numerical solutions can be acquired respectively.

§3 Analytical Solutions of the Bogoyavlensky-Konopelchenko Equation

Consider the time fractional BK equation

$$D_t^\mu D_x u + \alpha D_x^4 u + \beta D_x^3 D_y u + 6\alpha D_x^2 u D_x u + 4\beta D_x D_y u D_x u + 4\beta D_x^2 u D_y u = 0, \tag{17}$$

where $D_t^\mu u$ is conformable derivative of function $u(x,y,t)$ with fractional order. Implementing the chain [18] and wave transform (4) to Eq. (17), the reduced form can be expressed as follows

$$kwU' + (\alpha k^4 + \beta k^3 l)U''' + (3\alpha k^3 + 4\beta k^2 l)(U')^2 = 0. \tag{18}$$

Regard that the solution of Eq. (18) is in terms of $\varphi(\xi)$ where is the exact solutions of Eq. (2) as follows.

$$U(\xi) = \sum_{i=0}^{N} a_i \varphi^i(\xi),\ a_N \neq 0. \tag{19}$$

With use of the balancing principle [19], we have $N = 1$. Gathering all the obtained data in Eq. (18), an algebraic equation system come to exist with respect to k, l, w, a_0, a_1. Solving the obtain system yields following solution set

$$a_1 = \frac{3kw}{2\left(\alpha k^3\sigma - w\right)}, l = \frac{w - 4\alpha k^3\sigma}{4\beta k^2\sigma}, \tag{20}$$

and k, w, a_0 are free constants. When $\sigma < 0$, using (7) and (4) the traveling wave solutions of Eq. (17) can be deducted

$$u_1(x,y,t) = a_0 - \frac{3k\sqrt{-\sigma}w\tanh\left(\sqrt{-\sigma}\xi\right)}{2\left(\alpha k^3\sigma - w\right)},$$

$$u_2(x,y,t) = a_0 - \frac{3k\sqrt{-\sigma}w\coth\left(\sqrt{-\sigma}\xi\right)}{2\left(\alpha k^3\sigma - w\right)},$$

where $\xi = kx + \frac{\left(w-4\alpha k^3\sigma\right)}{4\beta k^2\sigma}y + \frac{wt^\mu}{\mu}$. In a similar way, for $\sigma > 0$, we obtain

$$u_3(x,y,t) = a_0 + \frac{3k\sqrt{\sigma}w\tan\left(\sqrt{\sigma}\xi\right)}{2\left(\alpha k^3\sigma - w\right)},$$

$$u_4(x,y,t) = a_0 - \frac{3k\sqrt{\sigma}w\cot\left(\sqrt{\sigma}\xi\right)}{2\left(\alpha k^3\sigma - w\right)},$$

where $\xi = kx + \frac{\left(w-4\alpha k^3\sigma\right)}{4\beta k^2\sigma}y + \frac{wt^\mu}{\mu}$.

§4 Approximate Solutions of the Bogoyavlensky-Konopelchenko Equation

Consider the nonlinear time-fractional BK Eq. (17). The initial condition obtained from the exact solution is

$$u(x,y,0)=a_0-\frac{3k\sqrt{-\sigma}w\tanh\left(\sqrt{-\sigma}\left(\frac{y\left(w-4\alpha k^3\sigma\right)}{4\beta k^2\sigma}+kx\right)\right)}{2\left(\alpha k^3\sigma-w\right)}.$$

For residual power series

$$u(x,y,t)=f(x,y)+\sum_{n=1}^{\infty}f_n(x,y)\frac{t^{n\mu}}{\mu^n n!},$$

and k.truncated series of $u(x,t)$

$$u_k(x,y,t)=f(x,y)+\sum_{n=1}^{k}f_n(x,y)\frac{t^{n\mu}}{\mu^n n!},\ k=1,2,3,...$$

Therefore, the k-th residual functions of time-fractional B-K equation is:

$$Resu_k(x,y,t)=(\partial_t^{\mu}u_k)_x+\alpha\left(u_k\right)_{xxxx}+\beta\left(u_k\right)_{xxxy}+6\alpha\left(u_k\right)_{xx}\left(u_k\right)_x+4\beta\left(u_k\right)_{xx}\left(u_k\right)_y$$
$$+4\beta\left(u_k\right)_{xy}\left(u_k\right)_x.$$

To determine the coefficients $f_1(x,y)$, in $u_1(x,y,t)$, we should replace the $1st$ truncated series $u_1(x,y,t)=f(x,y)+f_1(x,y)\frac{t^{\mu}}{\mu}$ into the $1st$ truncated residual function as

$$\begin{aligned}Resu_1(x,y,t)=&\,6\alpha\left(f_x+\frac{t^{\mu}\left(f_1\right)_x}{\mu}\right)\left(f_{xx}+\frac{t^{\mu}\left(f_1\right)_{xx}}{\mu}\right)\\&+\alpha\left(f_{xxxx}+\frac{t^{\mu}\left(f_1\right)_{xxxx}}{\mu}\right)\\&+4\beta\left(f_x(x,y)+\frac{t^{\mu}\left(f_1\right)_x}{\mu}\right)\left(f_{xy}(x,y)+\frac{t^{\mu}\left(f_1\right)_{xy}}{\mu}\right)\\&+4\beta\left(f_y+\frac{t^{\mu}(f_1)_y}{\mu}\right)\left(f_{xx}+\frac{t^{\mu}\left(f_1\right)_{xx}}{\mu}\right)\\&+\beta\left(f_{xxxy}+\frac{t^{\mu}\left(f_1\right)_{xxxy}}{\mu}\right)+\left(f_1\right)_x,\end{aligned}$$

for $f=f(x,y)$. Now for the substitution of $t=0$ through the equation $Resu_1(x,y,t)$ to obtain

$$\left(f_1\right)_x=-6\alpha f_xf_{xx}-\alpha f_{xxxx}-4\beta f_xf_{xy}-4\beta f_yf_{xx}-\beta f_{xxxy}.$$

Solving this differential equation gives the first unknown parameter as

$$f_1=-3\alpha f_x^2-\alpha f-4\beta f_xf_y-\beta f_{xxy}.$$

Thus, we obtain the $1st$ RPS approximate solutions of time-fractional B-K equation as

$$u_1(x,y,t)=f+\frac{t^{\mu}\left(-3\alpha f_x^2-\alpha f_{xxx}-4\beta f_xf_y-\beta f_{xxy}\right)}{\mu}.$$

Again, to determine the second unknown coefficient $f_2(x,y)$, we replace the $2nd$ truncated series solution $u_2(x,y,t)=f(x,y)+f_1(x,y)\frac{t^{\mu}}{\mu}+f_2(x,y)\frac{t^{2\mu}}{2\mu^2}$ into the $2nd$ truncated residual function and obtain

$$
\begin{aligned}
Resu_2(x,y,t) = {} & 6\alpha\left(f_x + \frac{t^{\mu}(f_1)_x}{\mu} + \frac{t^{2\mu}(f_2)_x}{2\mu^2}\right)\left(f_{xx} + \frac{t^{2\mu}(f_2)_{xx}}{2\mu^2} + \frac{t^{\mu}(f_1)_{xx}}{\mu}\right) \\
& + \alpha\left(f_{xxxx} + \frac{t^{\mu}(f_1)_{xxxx}}{\mu} + \frac{t^{2\mu}(f_2)_{xxxx}}{2\mu^2}\right) \\
& + 4\beta\left(f_x + \frac{t^{\mu}(f_1)_x}{\mu} + \frac{t^{2\mu}(f_2)_x}{2\mu^2}\right) \\
& \times\left(f_{xy} + \frac{t^{\mu}(f_1)_{xy}}{\mu} + \frac{t^{2\mu}(f_2)_{xy}}{2\mu^2}\right) + 4\beta\left(f_y + \frac{t^{\mu}(f_1)_y}{\mu} + \frac{t^{2\mu}(f_2)_y}{2\mu^2}\right) \\
& \times\left(f_{xx} + \frac{t^{\mu}(f_1)_{xx}}{\mu} + \frac{t^{2\mu}(f_2)_{xx}}{2\mu^2}\right) \\
& + \beta\left(f_{xxxy} + \frac{t^{\mu}(f_1)_{xxxy}}{\mu} + \frac{t^{2\mu}(f_2)_{xxxy}}{2\mu^2}\right) \\
& + \frac{t^{\mu}(f_2)_x}{\mu} + (f_1)_x .
\end{aligned}
$$

Now, applying T_μ on both sides of $Resu_2(x,y,t)$ and equating to 0 for $t=0$ gives:

$$
\begin{aligned}
(f_2)_x = {} & -6\alpha(f_1)_x f_{xx} - 6\alpha f_x (f_1)_{xx} - 4\beta(f_1)_x f_{xy} - 4\beta f_x \\
& \times(f_1)_{xy} - 4\beta(f_1)_y f_{xx} - 4\beta f_y (f_1)_{xx} - \alpha(f_1)_{xxxx} - \beta(f_1)_{xxxy} .
\end{aligned}
$$

Solving this differential equation gives

$$
f_2(x,y) = -2(f_1)_{xy}(3\alpha f_x + 2\beta f_y) - 4\beta(f_1)_x (f_1)_y - \alpha(f_1)_{xxx} - \beta(f_1)_{xxy} .
$$

The 2^{nd} RPS approximate solutions of time-fractional B-K equation is:

$$
u_2(x,y,t) = f + \frac{t^{\mu} f_1}{\mu} + \frac{t^{2\mu}\left(-2(f_1)_x(3\alpha f_x + 2\beta f_y) - 4\beta f_x (f_1)_y - \alpha(f_1)_{xxx} - \beta(f_1)_{xxy}\right)}{2\mu^2}.
$$

In a similar way, operating the same calculation for $n=3$ and 4 to get the following results.

$$
\begin{aligned}
f_3(x,y) = {} & -2(f_2)_x(3\alpha f_y + 2\beta f_y) - 4\beta(f_2)_y f_x - 6\alpha\left(f_1^2\right)_x \\
& -\alpha(f_2)_{xxx} - 8\beta(f_1)_x(f_1)_y - \beta(f_2)_{xxy}),
\end{aligned}
$$

$$
\begin{aligned}
u_3(x,y,t) = {} & f + \frac{t^{\mu} f_1}{\mu} + \frac{t^{2\mu} f_2}{2\mu^2} \\
& + \frac{t^{3\mu}\left(-2(f_2)_x(3\alpha f_x + 2\beta f_y) - 4\beta(f_2)_y f_x - 6\alpha\left(f_1^2\right)_x\right)}{6\mu^3} \\
& + \frac{t^{3\mu}\left(-\alpha(f_2)_{xxx} - 8\beta(f_1)_x(f_1)_y - \beta(f_2)_{xxy}\right)}{6\mu^3},
\end{aligned}
$$

$$
\begin{aligned}
f_4(x,y) = {} & -2(f_3)_x(3\alpha f_x + 2\beta f_y) - 4\beta(f_3)_y f_x - 6(f_2)_x\left(3\alpha(f_1)_x + 2\beta(f_1)_y\right) - \alpha(f_3)_{xxx} \\
& -12\beta(f_2)_y(f_1)_x - \beta(f_3)_{xxy} ,
\end{aligned}
$$

$$\begin{aligned} u_4(x,y,t) &= f+\frac{t^{\mu}f_1}{\mu}+\frac{t^{2\mu}f_2}{2\mu^2}+\frac{t^{3\mu}f_3}{6\mu^3}+\frac{t^{4\mu}}{24\mu^4}\Big(-2\left(f_3\right)_x\left(3\alpha f_x+2\beta f_y\right)-4\beta\left(f_3\right)_y f_x \\ &\quad -6\left(f_2\right)_x\left(3\alpha\left(f_1\right)_x+2\beta\left(f_1\right)_y\right)\Big) \\ &\quad +\frac{t^{4\mu}\left(-\alpha\left(f_3\right)_{xxx}-12\beta\left(f_2\right)_y\left(f_1\right)_x-\beta\left(f_3\right)_{xxy}\right)}{24\mu^4}. \end{aligned}$$

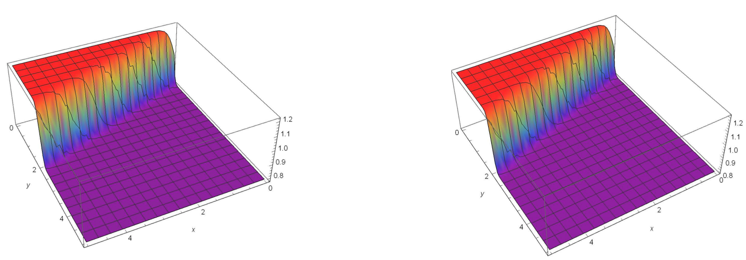
In Table 1, the fourth-order approximate RPSM solutions of time-fractional B-K equation are compared numerically with the exact solution

$$u(x,y,t)=a0-\frac{3k\sqrt{-\sigma}w\tanh\left(\sqrt{-\sigma}\left(\frac{y\left(w-4\alpha k^3\sigma\right)}{4\beta k^2\sigma}+kx+\frac{wt^{\mu}}{\mu}\right)\right)}{2\left(\alpha k^3\sigma-w\right)},$$

The graphical representations indicates that obtained exact solutions and numerical results are compatible for different values of μ. Also the table that given below shows that the absolute error values are in acceptable limits.

Table 1: RPSM approximate results and comparison with the exact solutions by absolute errors for $a_0=1$, $k=2$, $w=3$, $\sigma=-4$, $\alpha=2.5$, $\beta=1$, and $t=0.1$.

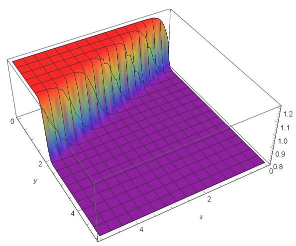
	$\mu=0.50$			$\mu=0.75$			$\mu=0.95$		
x	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*	*RPSM*	*Exact*	*Abs. Error*
0.0	0.783133	0.783134	1.28127E-6	0.783133	0.783133	2.03316E-9	0.783133	0.783133	4.5705E-11
0.1	0.783133	0.783136	2.85150E-6	0.783133	0.783133	4.52488E-9	0.783133	0.783133	1.0171E-10
0.2	0.783133	0.783140	6.34607E-6	0.783133	0.783133	1.00703E-8	0.783133	0.783133	2.2638E-10
0.3	0.783135	0.783149	1.41231E-5	0.783133	0.783133	2.24119E-8	0.783133	0.783133	5.0381E-10
0.4	0.783137	0.783168	3.14299E-5	0.783133	0.783133	4.98785E-8	0.783133	0.783133	1.12127E-9
0.5	0.783143	0.783213	6.99404E-5	0.783133	0.783133	1.11006E-7	0.783133	0.783133	2.49542E-9
0.6	0.783155	0.783311	1.55615E-4	0.783134	0.783134	2.47047E-7	0.783133	0.783133	5.55363E-9
0.7	0.783182	0.783528	3.46129E-4	0.783135	0.783136	5.49802E-7	0.783133	0.783133	1.23597E-8
0.8	0.783243	0.784013	7.69339E-4	0.783139	0.783140	1.22356E-6	0.783134	0.783134	2.75063E-8
0.9	0.783379	0.785087	1.70734E-3	0.783147	0.783150	2.72283E-6	0.783137	0.783137	6.12131E-8
1.0	0.783681	0.787457	3.77590E-3	0.783164	0.783171	6.05853E-6	0.783142	0.783142	1.36216E-7



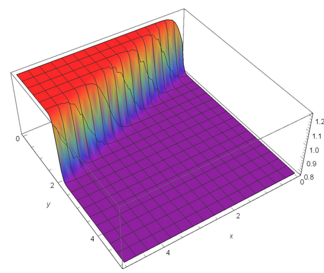
(a) RPSM solution

(b) Exact solution

Figure 1: RPSM solution and exact solution for $a_0=1$, $k=2$, $w=3$, $\sigma=-4$, $\alpha=2.5$, $\beta=1$, $t=0$ and $\mu=0.50$.

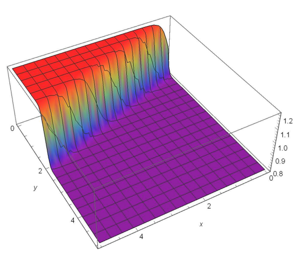


(a) RPSM solution

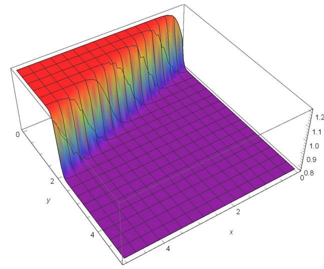


(b) Exact solution

Figure 2: RPSM solution and exact solution for $a_0 = 1$, $k = 2$, $w = 3$, $\sigma = -4$, $\alpha = 2.5$, $\beta = 1$, $t = 0$ and $\mu = 0.75$.



(a) RPSM solution



(b) Exact solution

Figure 3: RPSM solution and exact solution for $a_0 = 1$, $k = 2$, $w = 3$, $\sigma = -4$, $\alpha = 2.5$, $\beta = 1$, $t = 0$ and $\mu = 0.95$.

§5 Conclusion

In this article the author obtained the numerical and analytical solutions of conformable time fractional BK equation that arises in interaction of the wave propagation with the aid of sub equation method and RPSM. The 3D graphical representation for comparisons of the obtained results are given for different values of μ. Also comparative table is represented to express the effectiveness, reliableness and accuracy of the implemented methods. All the exact, numerical solutions and also, graphical representations, tables are obtained with the aid of computer software called Mathematica. This work reported here is a first step towards understanding structural and physical behaviour of the models arising in fluid dynamics. We hope that this work will be very useful in better understanding the physical structure occurring at real life events that correspond to the models of fluid dynamics. We believe our manuscript is very timely and will interest a broad range of scientists that work on mathematical models arising in fluid dynamics.

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Department of Mathematics, Pamukkale University, Denizli 20180, Turkey.
Email: pau.dr.alikurt@gmail.com, akurt@pau.edu.tr