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Local times of linear multifractional stable sheets

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Abstract. Let $X^{H(u)}(u) = \{X^{H(u)}(u), u \in \mathbb{R}_+^N\}$ be linear multifractional stable sheets with index functional $H(u)$, where $H(u) = (H_1(u), \cdots, H_N(u))$ is a function with values in $(0,1)^N$. Based on some assumptions of $H(u)$, we obtain the existence of the local times of $X^{H(u)}(u)$ and establish its joint continuity and the Hölder regularity. These results generalize the corresponding results about fractional stable sheets to multifractional stable sheets.

§1 Introduction

As a suitable generalization of the Brownian motion, the fractional Brownian motion B_H (Mandelbrot and Van Ness [15]) has been applied in some scientific areas such as finance and image processing. Its Hurst index H controls almost all sample path properties. However, many data coming from applications are heavy-tailed, the fractional Brownian motion has extremely light tails, so its applications are limited. Moreover, since the pointwise Hölder exponent of B_H is almost surely constant, the fractional Brownian motion can not be used to model some phenomena for which regularity varies in space. Thus, many authors have introduced some extensions of the fractional Brownian motion, such as fractional stable process which can overcome the light tails problem and multifractional process which can solve the homogeneity (see, for example, Samorodnitsky and Taqqu [18]).

Several authors have considered some sample path properties, local time of the fractional stable fields and multifractional stable process (see, for example, Lin and Cheng [14], Nolan [17], Kôno and Shieh [13], Shieh [20], Dai and Li [8], Shevchenko [19], Chen, Wu and Xiao [7]). Recently, Ayache and Xiao [5] studied the fractional stable fields and established the joint continuity of its local time. When $\alpha \in (0, 1)$, this solves an open problem that was raised in

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Nolan [17]. Xiao [24] applied the strong local nondeterminism to studied the local times of some stable random fields.

In this paper, we consider the local times of linear multifractional stable sheets (LMFSS in short), the processes have properties of heavy tails and multifractionality. They can be regarded both as a multifractional generalization of a fractional stable sheets (Xiao [24]) and a stable generalization of multifractional Brownian sheets (Meerschaert, Wu and Xiao [16]).

This paper is organized as follows. Section 2 contains detail information on LMFSS and some assumptions on the index functional $H(u)$. In Section 3, we obtain the existence of L^2-local times and establish the joint continuity and the Hölder regularity of the (N, d) LMFSS.

Throughout this paper, an unspecified positive and finite constant will be denoted by C and $C_i, i = 1, 2, 3...$, which may not be the same in each occurrence.

§2 Preliminaries

In this section, firstly, we mainly recall the information on LMFSS. Then we introduce the local times and the definition of the one-sided sectorial local nondeterminism.

For any $0 < \alpha < 2$, $H = (H_1, \cdots, H_N) \in (0, 1)^N$. Recall that linear fractional stable sheets $X^H = \{X^H(u), u \in \mathbb{R}^N_+\}$ with values in $\mathbb{R}$ have the representation:

$$X^H(u) = \int_{\mathbb{R}^N} g(u, v) M_\alpha(dv), \quad \forall u \in \mathbb{R}^N_+, \tag{2.1}$$

where M_α is a symmetric α-stable random measure on $\mathbb{R}^N$ with Lebesgue control measure and

$$g(u, v) = c \prod_{k=1}^{N} [(u_k - v_k)_+^{H_k - 1/\alpha} - (-v_k)_+^{H_k - 1/\alpha}], \tag{2.2}$$

where constant $c > 0$ and $(\cdot)_+ = \max\{\cdot, 0\}$. For the stochastic process $X^H(u)$, some authors have considered its local time, such as Ayache, Roueff and Xiao [1] considered the local and asymptotic properties, Ayache, Roueff and Xiao [2] proved the joint continuity of local times, Xiao [24] provided some sufficient conditions for the sectorial local nondeterminism and applied the property to study the existence of local times and fractal results.

Thanks to Stoev and Taqqu [21, 22], now, we define the real LMFSS which is the extension of linear fractional stable sheets.

Definition 2.1. *Let $\alpha \in (0, 2)$ and $H(u) = (H_1(u), \cdots, H_N(u))$ be a function in $u \in \mathbb{R}^N_+$ with values in $(0, 1)^N$. A real LMFSS with index functional $H(u)$ is defined as*

$$X^{H(u)}(u) = c \int_{\mathbb{R}^N} \prod_{k=1}^{N} [(u_k - v_k)_+^{H_k(u) - 1/\alpha} - (-v_k)_+^{H_k(u) - 1/\alpha}] M_\alpha(dv), \quad \forall u \in \mathbb{R}^N_+. \tag{2.3}$$

Clearly, if $N = 1$ the stochastic process $X^{H(u)}(u)$ is linear multifractional stable motion (see Stoev and Taqqu [21, 22]), if $H(u) = H$, the process $X^{H(u)}(u)$ is linear fractional stable sheet (see Ayache, Roueff and Xiao [1, 2]).

Recall that, the more general method to prove the joint continuity of local times is to establish local nondeterminism first, and then apply the local nondeterminism to study the local times. However, Ayache, Roueff and Xiao [2] do not prove the local nondeterminism, they only give the increments of the linear fractional stable sheet (see there in Lemma 2.1), then the existence of local times follows from Theorem 21.9 in Geman and Horowitz [11]. The joint continuity of local times follows from Lemma 2.4 in Ayache, Roueff and Xiao [2] and a multiparameter version of the Kolmogorov continuity theorem. Lemma 2.4 can be regarded as the tightness of moments of the local time L, with the method derived from the theory based on (25.5) and (25.7) in Geman and Horowitz [11].

In this paper, we will show the existence and joint continuity of local times of LMFSS theoretically, and its Hölder regularity will also be given. As a more general fractional stable process, LMFSS is multifractional generalization of a fractional stable sheets. It is not easy to obtain the increments of LMFSS, since u and $H(u)$ in (2.3) are changing at the same time. Thus by controlling a single variable, before the proof of Lemma 3.2, we show Lemma 3.1 at first. Moreover, making full use of one-sided sectorial local nondeterminism, we obtain the existence of local times of LMFSS. The conditions in (3.21) and Theorem 3.2 desire the convergence of moments in Lemmas 3.6 and 3.7, then by the multiparameter version of Kolmogorov continuity theorem and the way in Ehm [10], we can obtain joint continuity and Hölder regularity of local times of $X^{H(u)}(u)$. Furthermore, it can be expected that the existing results of local times related to linear fractional stable sheet can be extended to LMFSS.

In the following, we briefly recall the local times and the one-sided sectorial local nondeterminism. Some surveys and complete literature could be found in Geman and Horowitz [11], Dozzi [9], Xiao [24] and the references therein.

Let $Y(t)$ be a Borel vector field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. For any Borel set $I \subset \mathbb{R}^N$, the occupation measure of Y on I, is defined as

$$\mu_I(\cdot) = \lambda_N\{t \in I : Y(t) \in \cdot\},$$

which is the Borel measure on $\mathbb{R}^d$. If μ_I is almost surely absolutely continuous with respect to the Lebesgue measure λ_d, then Y is said to have local times on I and define its local time $L(\cdot, I)$ to be the Radon-Nikodým derivative of μ_I with respect to λ_d, that is,

$$L(x, I) = \frac{d\mu_I}{d\lambda_d}(x), \quad \forall x \in \mathbb{R}^d,$$

where x is the so-called the *space variable*, and I is called *time variable*. Intuitively, $L(x, I)$ measures the amount of "time" Y spent at x during I. Sometimes, we write $L(x, t)$ in place of $L(x, [0, t])$. Notice that if Y has local times on I then for any Borel set $J \subset I$, $L(x, J)$ also exists. It follows from Geman and Horowitz [11] that the local times satisfies the occupation density formula: for any Borel function $g(t, x) \geq 0$ on $I \times \mathbb{R}^d$,

$$\int_I g(t, Y(t))dt = \int_{\mathbb{R}^d} \int_I g(t, x)L(x, dt)dx. \tag{2.4}$$

Especially, we let $I = \prod_{k=1}^N [b_k, b_k + h_k]$. If the local time $L(x, \prod_{k=1}^N [b_k, b_k + t_k])$ is a continuous function of $(x, t_1, \cdots, t_N) \in \mathbb{R}^d \times \prod_{k=1}^N [0, h_k]$, then we call Y has a jointly continuous local time on I.

Now, we introduce one-sided sectorial local nondeterminism on I, which is important to the proofs of Lemma 3.3.

Definition 2.2. *(Xiao [24]). Let $Z = \{Z(u), u \in \mathbb{R}^N\}$ be an α-stable random field with following representation,*

$$Z(u) = \int_{\wedge} g(u,x) M_\alpha(dx), \tag{2.5}$$

where M_α is a symmetric α-stable ($S\alpha S$) random measure on a measurable space $(\wedge, \mathcal{F})$ with control measure m and $g(u,\cdot) : \wedge \to \mathbb{R}$ ($u \in \mathbb{R}^N$) is a family of measurable functions on $\wedge$ satisfying

$$Z(u) = \int_{\wedge} |g(u,x)|^\alpha m(dx) < \infty, \quad \forall u \in \mathbb{R}^N. \tag{2.6}$$

Then Z is said to be one-sided sectorial locally nondeterminism on I if for every $u, v \in I$ with $|u - v|$ sufficiently small

$$||Z(u)||_\alpha > 0, \ \ \forall \ u \in I, \qquad ||Z(u) - Z(v)||_\alpha > 0, \tag{2.7}$$

and there exists positive constant C such that for every $n \geq 2$ and $u^1, \cdots, u^n \in I$ with $u_l^k \leq u_l^n$ for all $1 \leq k \leq n-1$ and some $1 \leq l \leq N$, we have

$$||Z(u^n)|Z(u^1), \cdots, Z(u^{n-1})||_\alpha \geq C \min_{1 \leq k \leq n-1} (u_l^n - u_l^k)^{H_l}. \tag{2.8}$$

Here Denote

$$||\sum_{k=1}^n a_k Z(u^k)||_\alpha := ||\sum_{k=1}^n a_k g(u^k, \cdot)||_{L^\alpha}, \tag{2.9}$$

for the scale parameter of $\sum_{k=1}^n a_k Z(u^k)$.

Thanks to Meerschaert, Wu and Xiao [16], in this paper, we will use the following metric $\rho_M(u,v)$ in $\mathbb{R}^N$

$$\rho_M(u,v) = \sum_{k=1}^N |u_k - v_k|^{M_k}, \quad \forall u, v \in \mathbb{R}^N, \tag{2.10}$$

where $M = (M_1, \cdots, M_N) \in (0,1)^N$ is a fixed vector.

In the following, we will assume that $H(u) = (H_1(u), \cdots, H_N(u))$ is a function in $u \in \mathbb{R}_+^N$ with values in $(0,1)^N$ which satisfies conditions $C1$ and $C2$:

C1. There are constants $a \in (0,1)$, $b = \max\{M_1, \cdots, M_N\}$ such that for any $k \in \{1, 2, \cdots, N\}$, $a \leq H_k(u) \leq M_k(u) \leq b$, $u \in \mathbb{R}_+^N$.

C2. There are constants $c_k = c_k(I) > 0$ and $\delta > 0$ satisfying

$$|H_k(u) - H_k(v)| \leq c_k \rho_M(u,v), \quad \forall u, v \in I \text{ with } |u - v| < \delta.$$

In this paper, $s = (s_1, s_2, \cdots, s_N) \in \mathbb{R}^N$, $\langle c \rangle = (c, \cdots, c)$. For every $u, v \in \mathbb{R}^N$, if $u_i < v_i$ ($i = 1, 2, \cdots, N$), define closed interval (or rectangle) $[u,v] = \prod_{i=1}^N [u_i, v_i]$. $\mathcal{A} = \{[u,v], u, v \in [\varepsilon, T]^N\}$ denotes the collection of $[u,v]$.

§3 Local times of LMFSS

In this section, we will consider the local times of LMFSS $X^{H(u)}(u)$. We obtain the existence of the local times of $X^{H(u)}(u)$ and establish its joint continuity and Hölder regularity. Firstly, we give some lemmas that are useful for the proofs of the main results.

Lemma 3.1. *Suppose* $0 < \epsilon < T$ *and* $0 < a < b < 1$. $\{X^k(u),(u,k) \in \mathbb{R}_+^N \times [a,b]^N\}$ *be a real-valued LMFSS defined by* (2.3) *with* $H(u) = k$. *Then for any* $u \in [\epsilon, T]^N$ *and* $H_1 = (h_1, \cdots, h_N), H_2 = (h_1^{'}, \cdots, h_N^{'}) \in [a,b]^N$, *we have*

$$||X^{H_1}(u) - X^{H_2}(u)||_\alpha^\alpha \le C|H_1 - H_2|^\alpha, \tag{3.1}$$

where the constant $C > 0$ *depending on* a, b, ϵ, T, N.

Proof. For $H_1, H_2 \in [a,b]^N$, define $h^l = (h_1^{'}, \cdots, h_l^{'}, h_{l+1}, \cdots, h_N)$ for $l = 1, 2, \cdots, N$ and $h^0 = H_1 = (h_1, \cdots, h_N)$. Using the triangle-type inequality, we obtain

$$\begin{aligned} &||X^{H_1}(u) - X^{H_2}(u)||_\alpha^\alpha \\ &= ||X^{H_1}(u) - X^{h^1}(u) + X^{h^1}(u) - X^{h^2}(u) + \cdots + X^{h^{N-1}}(u) - X^{h^N}(u)||_\alpha^\alpha \\ &\le C_1 \sum_{l=1}^N ||X^{h^{l-1}}(u) - X^{h^l}(u)||_\alpha^\alpha. \end{aligned} \tag{3.2}$$

For fixed $l = 1, 2, \cdots, N$, we have

$$\begin{aligned} &\big|\big|X^{h^{l-1}}(u) - X^{h^l}(u)\big|\big|_\alpha^\alpha \\ &= \int_{\mathbb{R}^{N-1}} \prod_{j=1}^{l-1} \left|(u_j - v_j)_+^{h_j^{\prime}-1/\alpha} - (-v_j)_+^{h_j^{\prime}-1/\alpha}\right|^\alpha \cdot \prod_{j=l+1}^{N} \left|(u_j - v_j)_+^{h_j-1/\alpha} - (-v_j)_+^{h_j-1/\alpha}\right|^\alpha d\widetilde{v}_l \\ &\times \int_{\mathbb{R}} \left|(u_l - v_l)_+^{h_l-1/\alpha} - (-v_l)_+^{h_l-1/\alpha} - \Big((u_l - v_l)_+^{h_l^{\prime}-1/\alpha} - (-v_l)_+^{h_l^{\prime}-1/\alpha}\Big)\right|^\alpha dv_l \\ &=: I_1 \times I_2, \end{aligned} \tag{3.3}$$

where $\widetilde{v}_l = (v_1, \cdots, v_{l-1}, v_{l+1}, \cdots, v_N)$.

It is clear that

$$\begin{aligned} I_1 = &\prod_{j=1}^{l-1} \int_{\mathbb{R}} \left|(u_j - v_j)_+^{h_j^{\prime}-1/\alpha} - (-v_j)_+^{h_j^{\prime}-1/\alpha}\right|^\alpha dv_j \\ &\cdot \prod_{j=l+1}^{N} \int_{\mathbb{R}} \left|(u_j - v_j)_+^{h_j-1/\alpha} - (-v_j)_+^{h_j-1/\alpha}\right|^\alpha dv_j, \end{aligned} \tag{3.4}$$

which is bounded by a constant independent of l, since $\int_{\mathbb{R}} |(u_j - v_j)_+^{h_j-1/\alpha} - (-v_j)_+^{h_j-1/\alpha}|^\alpha dv_j$ is finite.

By the mean value theorem, we have

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} \left|(u_l - v_l)_+^{h_l^{''}-1/\alpha} \log(u_l - v_l)_+ - (-v_l)_+^{h_l^{''}-1/\alpha} \log(-v_l)_+\right|^\alpha |h_l - h_l^{'}|^\alpha dv_l \\ &\le C_2 |h_l - h_l^{'}|^\alpha, \end{aligned} \tag{3.5}$$

for all $H_1, H_2 \in [a,b]^N$, $u \in [\epsilon, T]^N$, where $h_l^{''} \in (h_l \wedge h_l^{'}, h_l \vee h_l^{'})$. Combining (3.2)–(3.5) with Hölder inequality, we obtain that

$$||X^{H_1}(u) - X^{H_2}(u)||_\alpha^\alpha \le C_3 \sum_{l=1}^N |h_l - h_l^{'}|^\alpha \le C_4 \left(\sum_{l=1}^N |h_l - h_l^{'}|^2\right)^{\alpha/2} = C_5 |H_1 - H_2|^\alpha.$$

This completes the proof. □

Lemma 3.2. *Let $\{X^{H(u)}(u)\}$ be a LMFSS in $\mathbb{R}$. Then, there exist constants $\delta > 0$, C' and C'' satisfying for any $u, v \in [\epsilon, T]^N$ with $|u-v| < \delta$, we have*

$$C' \sum_{k=1}^{N} |u_k - v_k|^{\alpha H_k(\widehat{u})} \le ||X^{H(u)}(u) - X^{H(v)}(v)||_\alpha^\alpha \le C'' \sum_{k=1}^{N} |u_k - v_k|^{\alpha H_k(\widehat{u})}, \tag{3.6}$$

where $\widehat{u} \in \prod_{k=1}^N [v_k \wedge u_k, v_k \vee u_k]$.

Proof. Using the triangle-type inequality,

$$|u+v+w|^\alpha \le 3(|u|^\alpha + |v|^\alpha + |w|^\alpha)$$

and

$$|u+v+w-v-w|^\alpha \le 3(|u+v+w|^\alpha + |v|^\alpha + |w|^\alpha).$$

We have

$$\begin{aligned} &\frac{1}{3}||X^{H(\widehat{u})}(u) - X^{H(\widehat{u})}(v)||_\alpha^\alpha - ||X^{H(u)}(u) - X^{H(\widehat{u})}(u)||_\alpha^\alpha \\ &\quad - ||X^{H(v)}(v) - X^{H(\widehat{u})}(v)||_\alpha^\alpha \\ &\le ||X^{H(u)}(u) - X^{H(v)}(v)||_\alpha^\alpha \\ &\le 3\big(||X^{H(\widehat{u})}(u) - X^{H(\widehat{u})}(v)||_\alpha^\alpha + ||X^{H(u)}(u) - X^{H(\widehat{u})}(u)||_\alpha^\alpha \\ &\quad + ||X^{H(v)}(v) - X^{H(\widehat{u})}(v)||_\alpha^\alpha\big). \end{aligned} \tag{3.7}$$

From Lemma 17 in Ayache, Roueff and Xiao [3], we have

$$C_6 \sum_{l=1}^{N} |u_l - v_l|^{\alpha H_l(\widehat{u})} \le ||X^{H(\widehat{u})}(u) - X^{H(\widehat{u})}(v)||_\alpha^\alpha \le C_7 \sum_{l=1}^{N} |u_l - v_l|^{\alpha H_l(\widehat{u})}, \tag{3.8}$$

where the positive constants C_6, C_7 depending on a, b, ϵ and N.

According to Lemma 3.1, Hölder inequality and the conditions $C1$ and $C2$, there exists a positive constant δ small enough, such that for any $u, v \in [\epsilon, T]^N$ with $|u - v| < \delta$, then one has $|\widehat{u} - v| < \delta$ and

$$\begin{aligned} ||X^{H(\widehat{u})}(u) - X^{H(u)}(u)||_\alpha^\alpha &\le C|H(u) - H(\widehat{u})|^\alpha \\ &= C_8\Big(\sum_{j=1}^{N} |H_j(\widehat{u}) - H_j(u)|^2\Big)^{\alpha/2} \le C_9\Big(\sum_{j=1}^{N} |u_j - v_j|^{2M_j}\Big)^{\alpha/2} \\ &\le C_{10} \sum_{j=1}^{N} |u_j - v_j|^{\alpha M_j} \le C_{11} \sum_{j=1}^{N} |u_j - v_j|^{\alpha H_j(\widehat{u})}. \end{aligned} \tag{3.9}$$

Using the same way as (3.9), we get

$$||X^{H(\widehat{u})}(v) - X^{H(v)}(v)||_\alpha^\alpha \le C_{12} \sum_{j=1}^{N} |u_j - v_j|^{\alpha H_j(\widehat{u})}. \tag{3.10}$$

From condition $C1$, note that $\sup_{u_j} H_j(u_j) \le M_j$, combining (3.7)–(3.10), we obtain that (3.6) holds. This completes the proof. □

In the following, we decompose $X^{H(u)}(u)$ and get a multifractional sheets $Z_j(u)$ which satisfies one-sided sectorial local nondeterminism.

Let

$$g_j(u_j,v_j)=(u_j-v_j)_+^{H_j(u)-1/\alpha}-(-v_j)_+^{H_j(u)-1/\alpha}.$$

According to (2.3), for any $u\in\mathbb{R}_+^N$,

$$\begin{aligned}X^{H(u)}(u)&=\int_{(-\infty,u]/[0,u]}\prod_{j=1}^N g_j(u_j,v_j)M_\alpha(dv)+\int_{[0,u]}\prod_{j=1}^N g_j(u_j,v_j)M_\alpha(dv)\\&=\int_{(-\infty,u]/[0,u]}\prod_{j=1}^N g_j(u_j,v_j)M_\alpha(dv)+\int_{[0,u]}\prod_{j=1}^N(u_j-v_j)_+^{H_j(u)-1/\alpha}M_\alpha(dv).\end{aligned}\tag{3.11}$$

Denote

$$Y^{H(u)}(u)=\int_{[0,u]}\prod_{j=1}^N(u_j-v_j)_+^{H_j(u)-1/\alpha}M_\alpha(dv).$$

Clearly, for any integers $n\ge 2$, $u^1,u^2,\cdots,u^n\in[\epsilon,T]^N$, $a_1,a_2,\cdots,a_n\in\mathbb{R}$, it is easy to obtain that

$$||\sum_{l=1}^n a_lX^{H(u^l)}(u^l)||_\alpha\ge||\sum_{l=1}^n a_lY^{H(u^l)}(u^l)||_\alpha.\tag{3.12}$$

Now, the rectangle $[0,u]$ can be decomposed into the disjoint union of sub-rectangles, for any $u\in[\epsilon,T]^N$,

$$[0,u]=[0,\epsilon]^N\cup\bigcup_{j=1}^N Q_j(u)\cup P(\epsilon,u),$$

where $Q_j(u)=\{s\in[0,T]^N:0\le s_i\le\epsilon \text{ if } i\ne j,\ \epsilon<s_j\le u_j\}$ and $P(\epsilon,u)$ can be written as a union of 2^N-N-1 sub-rectangles of $[0,u]$. Thus, we get

$$\begin{aligned}Y^{H(u)}(u)&=\int_{[0,\epsilon]^N}g(u,v)M_\alpha(dv)+\sum_{j=1}^N\int_{Q_j(u)}g(u,v)M_\alpha(dv)\\&\quad+\int_{P(\epsilon,u)}g(u,v)M_\alpha(dv)\\&=:Z_1(\epsilon,u)+\sum_{j=1}^N Z_j(u)+Z_3(\epsilon,u),\end{aligned}\tag{3.13}$$

where $g(u,v)$ was defined in (2.2). One can find that the stochastic process $Z_1(\epsilon,u)$, $Z_j(u)$ $(1\le j\le N)$ and $Z_3(\epsilon,u)$ are independent since they are defined over disjoint sets. The Lemma 3.3 proves that the stochastic process $Z_j(u)$ satisfies one-sided sectorial local nondeterminism. It will be very important to the proofs of the main results.

Lemma 3.3. *Assume $I\in\mathcal{A}$ is closed interval and $j\in\{1,2,\cdots,N\}$. For any integers $n\ge 2$ and $u^1,\cdots,u^n\in I$ satisfies $u_j^1\le u_j^2\le\cdots\le u_j^n$. Then, we have*

$$||Z_j(u^n)|Z_j(u^1),\cdots,Z_j(u^{n-1})||_\alpha^\alpha\ge C|u_j^n-u_j^{n-1}|^{\alpha H_j(u^n)},\tag{3.14}$$

where $u_j^0=0$, C denotes a constant only depending on ϵ and I.

Proof. In order to show (3.14) holds, we only need to show that there exists a constant C such that

$$||Z_j(u^n) - \sum_{l=1}^{n-1} a_l Z_j(u^l)||_\alpha^\alpha \geq C|u_j^n - u_j^{n-1}|^{\alpha H_j(u^n)},$$

where $a_l \in \mathbb{R}$ $(l = 1, 2, \cdots, n-1)$ (see, for example, Ayache, Wu and Xiao [4]).

Using the fact that $Z_j(u)$ has the independence on disjoint sets, we have

$$\begin{aligned}
||Z_j(u^n) - \sum_{l=1}^{n-1} a_l Z_j(u^l)||_\alpha^\alpha &= \int_{Q_j(u^n)/Q_j(u^{n-1})} |g(u^n, v)|^\alpha dv \\
&\quad + \int_{Q_j(u^{n-1})/Q_j(u^{n-2})} |g(u^n, v) - a_{n-1} g(u^{n-1}, v)|^\alpha dv + \cdots \\
&\quad + \int_{Q_j(u^1)} |g(u^n, v) - \sum_{l=1}^{n-1} a_l g(u^l, v)|^\alpha dv \\
&\geq \int_{Q_j(u^n)/Q_j(u^{n-1})} |g(u^n, v)|^\alpha dv \\
&\geq \int_0^\epsilon \cdots \int_{u_j^{n-1}}^{u_j^n} \cdots \int_0^\epsilon \prod_{k=1}^N (u_k^n - v_k)^{\alpha H_k(u^n) - 1} dv \\
&\geq C_{13} \int_{u_j^{n-1}}^{u_j^n} (u_j^n - v_j)^{\alpha H_j(u^n) - 1} dv_j \\
&\geq C_{14} |u_j^n - u_j^{n-1}|^{\alpha H_j(u^n)}.
\end{aligned} \tag{3.15}$$

This completes the proof. □

By Lemma 3.2, Lemma 3.3 and Boufoussi, Dozzi and Guerbaz [6], it is easy to obtain the following inequality

$$||\sum_{l=1}^n a_l [Z_j(u^l) - Z_j(u^{l-1})]||_\alpha^\alpha \geq C \sum_{l=1}^n |a_l|^\alpha ||Z_j(u^l) - Z_j(u^{l-1})||_\alpha^\alpha$$

for Gaussian process, but not easy for stable process. Hence, we give different inequality below.

Lemma 3.4. *For any $n \geq 2$, there exists a constant $C > 0$ depending on n only, such that for every $j \in 1, 2, \cdots, N$*

$$||\sum_{l=1}^n a_l Z_j(u^l)||_\alpha \geq C \left(|a_1| ||Z_j(u^1)||_\alpha + \sum_{l=2}^n |a_l| \Big|\Big| Z_j(u^l) | Z_j(u^1), \cdots, Z_j(u^{l-1}) \Big|\Big|_\alpha \right), \tag{3.16}$$

where $a_l \in \mathbb{R}$, $u^l \in I \in \mathcal{A}$, $(l = 1, 2, \cdots, n)$ satisfies $u^1 < \cdots < u^n$.

Proof. Without loss of generality, we only need to prove $n = 2$. Let $\mathcal{A}_1$ be the subspace generated by $Z_j(u^1)$. Then the metric projection of $Z_j(u^2)$ on $\mathcal{A}_1$ can be written as $a_{21} Z_j(u^1)$ for some $a_{21} \in \mathbb{R}$. Then $Z_j(u^2) - a_{21} Z_j(u^1)$ and $\mathcal{A}_1$ are orthogonal. So, we can find that

$$\begin{aligned}
||a_1 Z_j(u^1) + a_2 Z_j(u^2)||_\alpha &= |a_2| \Big|\Big| (\frac{a_1}{a_2} + a_{21}) Z_j(u^1) + Z_j(u^2) - a_{21} Z_j(u^1) \Big|\Big|_\alpha \\
&\geq C \left(|a_1 + a_{21} a_2| \Big|\Big| Z_j(u^1) \Big|\Big|_\alpha + |a_2| \Big|\Big| Z_j(u^2) | Z_j(u^1) \Big|\Big|_\alpha \right).
\end{aligned}$$

The left proof for cases $n \geq 3$ are similar. □

Combining (3.12), (3.13) with the independence of $Z_j(u)$, one can obtain the following lemma.

Lemma 3.5. *For $n \geq 2$, $a_l \in \mathbb{R}$, $u^l \in I \in \mathcal{A}$ $(l = 1, 2, \cdots, n)$. Then, we have*

$$||\sum_{l=1}^{n} a_l X^{H(u^l)}(t^l)||_\alpha^\alpha \geq ||\sum_{l=1}^{n} a_l Y^{H(u^l)}(u^l)||_\alpha^\alpha \geq \sum_{j=1}^{N} ||\sum_{l=1}^{n} a_l Z_j(u^l)||_\alpha^\alpha, \tag{3.17}$$

where $a_l \in \mathbb{R}$, $u^l \in I$, $(l = 1, 2, \cdots, n)$ satisfies $u_j^1 < \cdots < u_j^n$ with $|u_j^1 - u_j^n| < \delta$.

Next, we study the existence and joint continuity of the local times of LMFSS $X = \{X^{H(u)}(u), u \in \mathbb{R}_+^N\}$.

Theorem 3.1. *Let $X = \{X^{H(u)}(u), u \in \mathbb{R}_+^N\}$ be the LMFSS, $I = [\epsilon, 1]^N$ and $\overline{H_l} = \max_{u \in I} H_l(u)$ for $l = 1, \cdots, N$. If $d < \sum_{l=1}^{N} \frac{1}{\overline{H_l}}$, then X admits an L^2-integrable local time $L(\cdot, I)$ almost surely.*

Proof. The proof is similar to the proof of Theorem 4.1 in Xiao [24]. So we give a sketch of the proof. The occupation measure μ_I is

$$\mu_I(\cdot) = \lambda_N\{u \in I : X(u) \in \cdot\},$$

and the Fourier transform of the occupation measure μ_I is

$$\widehat{\mu}_I(\xi) = \int_T e^{i\langle \xi, X^{H(u)}(u)\rangle} du.$$

Applying the Fubini's theorem twice, we have

$$\mathbb{E}\int_{\mathbb{R}^d} |\widehat{\mu}_I(\xi)|^2 d\xi = \int_I\int_I\int_{\mathbb{R}^d} \mathbb{E}\exp(i\langle \xi, X^{H(u)}(u) - X^{H(v)}(v)\rangle) d\xi du dv.$$

Denote

$$\mathcal{J}(T) = \int_I\int_I\int_{\mathbb{R}^d} \mathbb{E}\exp(i\langle \xi, X^{H(u)}(u) - X^{H(v)}(v)\rangle) d\xi du dv.$$

In order to complete the proof of the Theorem, by Theorem 21.9 in Geman and Horowitz [11], it is enough to prove $\mathcal{J}(T) < \infty$. Using Lemma 3.3 and Lemma 3.5, we have

$$\begin{aligned}\mathcal{J}(T) &= \int_I\int_I \frac{dvdu}{||X^{H(u)}(u) - X^{H(v)}(v)||_\alpha^d} \\ &\leq C_{15}\int_I\int_I \frac{dvdu}{(\sum_{l=1}^{N} ||Z_l(u) - Z_l(v)||_\alpha)^d} \\ &\leq C_{16}\int_I\int_I \frac{dvdu}{(\sum_{l=1}^{N} |u_l - v_l|^{H_l(v)})^d} < \infty,\end{aligned} \tag{3.18}$$

since $d < \sum_{l=1}^{N} \frac{1}{\overline{H_l}}$. □

The following result gives the joint continuity of the local times of $\{X^{H(u)}(u), u \in \mathbb{R}_+^N\}$.

Theorem 3.2. *Let $X = \{X^{H(u)}(u), u \in \mathbb{R}_+^N\}$ be the LMFSS with values in $\mathbb{R}^d$. Let $I = [\epsilon, 1]^N$ and $\overline{H_l} = \max_{u \in I} H_l(u)$ for $l = 1, \cdots, N$. If $d < \sum_{l=1}^{N} \frac{1}{\overline{H_l}}$, then X has a jointly continuous local time on I.*

In order to prove Theorem 3.2, we need some preliminaries. First, we give some moment estimates for the local times of X.

By (25.5) and (25.7) in Geman and Horowitz [11]: for any $x, y \in \mathbb{R}^d$, $I \in \mathcal{A}$ and any integers $n \geq 1$,

$$\begin{aligned}\mathbb{E}[L(x,I)^n] = (2\pi)^{-nd} \int_{I^n} \int_{\mathbb{R}^{nd}} & \exp\big(- i \sum_{k=1}^{n} \langle v^k, x\rangle\big) \\ & \times \mathbb{E} \exp\big(i \sum_{k=1}^{n} \langle v^k, X(u^k)\rangle\big) d\overline{v} d\overline{u},\end{aligned} \tag{3.19}$$

and for any even integers $n \geq 2$,

$$\begin{aligned}\mathbb{E}[\big(L(x,I) - L(y,I)\big)^n] = (2\pi)^{-nd} \int_{I^n} \int_{\mathbb{R}^{nd}} & \prod_{k=1}^{n} [e^{-i\langle v^k, x\rangle} - e^{-i\langle v^k, y\rangle}] \\ & \times \mathbb{E} \exp\big(i \sum_{k=1}^{n} \langle v^j, X(u^k)\rangle\big) d\overline{v} d\overline{u},\end{aligned} \tag{3.20}$$

where $\overline{v} = (v^1, \cdots, v^n)$, $\overline{u} = (u^1, \cdots, u^n)$ and each $v^k = (v^k_1, \cdots, v^k_d) \in \mathbb{R}^d$, $u^k \in I$.

In the following, the property of one-sided sectorial local nondeterminism of Z_j proved in Lemma 3.3 plays an essential role in the proofs of Lemmas 3.6 and 3.7.

Lemma 3.6. *Let the conditions in Theorem 3.2 hold and γ is the unique integer in $1, \cdots, N$ such that*

$$\sum_{i=1}^{\gamma-1} \frac{1}{\overline{H}_i} \leq d \leq \sum_{i=1}^{\gamma} \frac{1}{\overline{H}_i}. \tag{3.21}$$

Then, for any $x \in \mathbb{R}^d$, $T = [a, a + \langle h\rangle] \subseteq I$ with $h > 0$ small enough and any integer $n \geq 1$, we have

$$\mathbb{E}[L(x,T)^n] \leq C h^{n\beta_\gamma}, \tag{3.22}$$

where $\beta_\gamma = N - \gamma - \overline{H}_\gamma d + \sum_{l=1}^{\gamma} \overline{H}_\gamma / \overline{H}_l$, $C > 0$ only depending on n, N, d, $\overline{H}$ and I.

Proof. By (3.19), for every interval $T = \prod_{j=1}^{N} [a_j, a_j + h_j] \subseteq I$,

$$\mathbb{E}[L(x,T)^n] = (2\pi)^{-nd} \int_{T^n} \prod_{k=1}^{d} \{ \int_{\mathbb{R}^n} \exp\big(- || \sum_{j=1}^{n} v^j_k X^{H(u^j)}(u^j) ||^\alpha_\alpha \big) dV_k \} d\overline{u}, \tag{3.23}$$

where $V_k = (v^1_k, \cdots, v^n_k) \in \mathbb{R}^n$, since $X_1, \cdots, X_d$ are independent and identically distributed.

Denote

$$\begin{aligned} J_k := & \int_{\mathbb{R}^n} \exp\big(- || \sum_{j=1}^{n} v^j_k X^{H(u^j)}(u^j) ||^\alpha_\alpha \big) dV_k \\ \leq & \int_{\mathbb{R}^n} \exp\big(- \sum_{l=1}^{N} || \sum_{j=1}^{n} v^j_k Z_l(u^j) ||^\alpha_\alpha \big) dV_k \\ \leq & \int_{\mathbb{R}^n} \exp\big(- \sum_{l=1}^{\gamma} || \sum_{j=1}^{n} v^j_k Z_l(u^j) ||^\alpha_\alpha \big) dV_k. \end{aligned} \tag{3.24}$$

It is clear that, for all $1 \leq l \leq N$, there is a permutation π_l of $\{1, ..., n\}$ satisfying

$$a_l \leq t_l^{\pi_l(1)} \leq t_l^{\pi_l(2)} \leq \cdots \leq t_l^{\pi_l(n)} \leq a_l + h_l.$$

By Lemma 3.3, Lemma 3.4, for every $1 \le l \le N$,

$$\begin{aligned} \|\sum_{j=1}^{n} v_k^j Z_l(u^j)\|_\alpha^\alpha &\ge c_n \sum_{j=1}^{n} |w_{k,l}^j|^\alpha (u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)})^{\alpha H_l(u^{\pi_l(j)})} \\ &\ge c_n \sum_{j=1}^{n} |w_{k,l}^j|^\alpha (u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)})^{\alpha \overline{H}_l}, \end{aligned} \tag{3.25}$$

here c_n is a positive constant depending on n and

$$(w_{k,l}^1, \cdots, w_{k,l}^n) = (v_k^{\pi_l(1)}, \cdots, v_k^{\pi_l(n)}) A_l,$$

$A_l = (a_{ij})$ $n \times n$ is an lower triangle matrix with $a_{ii} = 1$ for all $1 \le i \le n$.

From (3.21) and Lemma 4.5 in Xiao [24], we can find that there exist $\rho_1, \cdots, \rho_\gamma \ge 1$, such that

$$\sum_{l=1}^{\gamma} \frac{1}{\rho_l} = 1, \quad \frac{H_l d}{\rho_l} < 1, \quad \forall \ l = 1, 2, \cdots, \gamma$$

and

$$(1 - \frac{1}{n}) \sum_{l=1}^{\gamma} \frac{H_l d}{\rho_l} \le H_\gamma d + \gamma - \sum_{l=1}^{\gamma} \frac{H_\gamma}{H_l}.$$

By (3.24) and (3.25), using the Hölder inequality and change of variables

$$w_{k,l}^j (u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)})^{\overline{H}_l} = w_{k,l}^j,$$

we can see

$$\begin{aligned} J_k &\le \int_{\mathbb{R}^n} \prod_{l=1}^{\gamma} \exp\big(- c_{n,1} \sum_{j=1}^{n} |w_{k,l}^j|^\alpha (u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)})^{\alpha \overline{H}_l} \big) dV_k \\ &\le \prod_{l=1}^{\gamma} \left[\int_{\mathbb{R}^n} \exp\big(- c_{n,2}\, \rho_l \sum_{j=1}^{n} |w_{k,l}^j|^\alpha (u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)})^{\alpha \overline{H}_l} \big) dV_k \right]^{\frac{1}{\rho_l}} \\ &\le c_{n,3} \prod_{l=1}^{\gamma} \prod_{j=1}^{n} \frac{1}{(u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)})^{\overline{H}_l/\rho_l}}, \end{aligned} \tag{3.26}$$

where $c_{n,1},\ c_{n,2},\ c_{n,3}$ are positive constants depending on n and $\gamma \in \{1, \cdots, N\}$.

Combining (3.23), (3.24) with (3.26), we have

$$\begin{aligned} \mathbb{E}[L(x,T)^n] &\le c_{n,4} \sum_{\pi_1, \cdots, \pi_N} \int_{\Gamma(\pi_1, \cdots, \pi_N)} \prod_{l=1}^{\gamma} \prod_{j=1}^{n} \frac{1}{(u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)})^{\overline{H}_l/\rho_l}} d\overline{u} \\ &\le c_{n,5} \prod_{l=1}^{\gamma} h_l^{n(1-(1-\frac{1}{n})\overline{H}_l d/\rho_l)} \cdot \prod_{l=\gamma+1}^{N} h_l^n. \end{aligned} \tag{3.27}$$

Now we consider the special case when $T = [a, a + \langle h \rangle]$ is a cube, that is

$$h_1 = \cdots = h_N = h.$$

By (3.27), we obtain that

$$\mathbb{E}[L(x,T)^n] \le C_{17} h^{n(N-(1-\frac{1}{n})\sum_{l=1}^{\gamma} \overline{H}_l d/\rho_l)} \le C_{18} h^{n\beta_\gamma}, \tag{3.28}$$

where $\sum_{l=1}^{\gamma} \frac{1}{\rho_l} = 1,\ \gamma \in \{1, \cdots, N\},\ \beta_\gamma = N - \gamma - \overline{H}_\gamma d + \sum_{l=1}^{\gamma} \overline{H}_\gamma / \overline{H}_l$. This completes the proof. □

Lemma 3.7. *Let the conditions in Theorem 3.2 hold, γ is the unique integer in $\{1,\cdots,N\}$ satisfying (3.21), then for any subintervals $T=[a,a+\langle h\rangle]\subseteq I$ with $h>0$, any $w,v\in\mathbb{R}^d$ with $|w-v|\le 1$,*

$$\mathbb{E}[\big(L(w,T)-L(v,T)\big)^n]\le C|w-v|^{n\kappa}h^{n(\beta_\gamma-\overline{H}_\gamma\kappa)}, \tag{3.29}$$

where $\alpha_\gamma=\sum_{l=1}^{\gamma}\frac{1}{H_l}-d$, $\kappa\in(0,1\wedge\frac{\alpha_\gamma}{2\gamma})$, $C>0$ only depending on $n,\ \kappa,\ N,\ d,\ \overline{H}$ and I.

Proof. Assume κ is a constant defined in Lemma 3.7, by elementary inequalities

$$|e^{ix}-1|\le 2^{1-\kappa}|x|^\kappa,\quad \forall\ x\in\mathbb{R}$$

and triangle-type inequalities, we have

$$\prod_{k=1}^{n}\left|e^{-i\langle x^k,w\rangle}-e^{-i\langle x^k,v\rangle}\right|\le 2^{n(1-\kappa)}|w-v|^{n\kappa}\sum\prod_{k=1}^{n}|x^k_{j_k}|^\kappa, \tag{3.30}$$

where $\sum$ is occupied by all the sequences $(j_1,\cdots,j_n)\in\{1,\cdots,d\}^n$ and $x^k,\ w,\ v\in\mathbb{R}^d,\ k=1,\cdots,n$.

Combining (3.20), (3.30) with Lemma 3.5, we get

$$\begin{aligned}
&\mathbb{E}[(L(w,T)-L(v,T))^n]\\
&\le |w-v|^{n\kappa}\sum\int_{T^n}d\overline{u}\int_{\mathbb{R}^{nd}}\prod_{m=1}^{n}|x^m_{j_m}|^\kappa\exp\left[-||\sum_{k=1}^{n}\langle x^k,X^{H(u^k)}(u^k)\rangle||^\alpha_\alpha\right]d\overline{x}\\
&=|w-v|^{n\kappa}\sum\int_{T^n}d\overline{u}\int_{\mathbb{R}^{nd}}\prod_{m=1}^{n}|x^m_{j_m}|^\kappa\prod_{j=1}^{d}\exp\left[-||\sum_{k=1}^{n}x^k_jX_j^{H(u^k)}(u^k)||^\alpha_\alpha\right]d\overline{x}\\
&\le |w-v|^{n\kappa}\sum\int_{T^n}d\overline{u}\int_{\mathbb{R}^{nd}}\prod_{m=1}^{n}|x^m_{j_m}|^\kappa\prod_{j=1}^{d}\exp\left[-\sum_{l=1}^{N}||\sum_{k=1}^{n}x^k_jZ_l(u^k)||^\alpha_\alpha\right]d\overline{x}\\
&\le |w-v|^{n\kappa}\sum\int_{T^n}d\overline{u}\int_{\mathbb{R}^{nd}}\prod_{m=1}^{n}|x^m_{j_m}|^\kappa\prod_{j=1}^{d}\exp\left[-\sum_{l=1}^{\gamma}||\sum_{k=1}^{n}x^k_jZ_l(u^k)||^\alpha_\alpha\right]d\overline{x}\\
&=|w-v|^{n\kappa}\sum\int_{T^n}d\overline{u}\prod_{j=1}^{d}\int_{\mathbb{R}^n}\prod_{k=1}^{n}|x^k_j|^{\kappa\eta^k_j}\exp\left[-\sum_{l=1}^{\gamma}||\sum_{k=1}^{n}x^k_jZ_l(u^k)||^\alpha_\alpha\right]dx_j,
\end{aligned} \tag{3.31}$$

where

$$\eta^k_j=\begin{cases}1, & \text{if}\quad j=j_k,\\ 0, & \text{if}\quad j\ne j_k,\end{cases} \tag{3.32}$$

$x_j=(x^1_j,\cdots,x^n_j)\in\mathbb{R}^n$ and $\sum_{j=1}^d\eta^k_j=1$. Let $\mathcal{M}$ be defined by

$$\mathcal{M}=\prod_{j=1}^{d}\int_{\mathbb{R}^n}\prod_{k=1}^{n}|x^k_j|^{\kappa\eta^k_j}\exp\left[-\sum_{l=1}^{\gamma}||\sum_{k=1}^{n}x^k_jZ_l(u^k)||^\alpha_\alpha\right]dx_j. \tag{3.33}$$

Similar to the proof of Lemma 3.6, by Lemma 3.3, Lemma 3.4, one can estimate the upper bound of $\exp\big[-\sum_{l=1}^{\gamma}||\sum_{k=1}^{n}x^k_jZ_l(u^k)||^\alpha_\alpha\big]$. Since $\int_{\mathbb{R}}x^ke^{-cx^\alpha}dx<\infty$, then use the same way as in the proof of Lemma 3.6 and change of variables $w^k_{j,l}(u_l^{\pi_l(k)}-u_l^{\pi_l(k-1)})^{\overline{H}_l}=w^k_{j,l}$, where

$$(w^1_{j,l},\cdots,w^n_{j,l})=(x_j^{\pi_l(1)},\cdots,x_j^{\pi_l(n)})A_l,$$

$A_l=(a_{ij})$ $n\times n$ is an lower triangle matrix with $a_{ii}=1$ for all $1\le i\le n$. Thus, there exists a constant C such that

$$\mathcal{M} \le C \prod_{l\neq l_0}^{\gamma}\prod_{k=1}^{n} \frac{1}{|u_l^k - u_l^{k-1}|^{(\overline{H}_l d)/\rho_l}} \cdot \prod_{k=1}^{n} \frac{1}{|u_l^k - u_l^{k-1}|^{\kappa \overline{H}_{l_0}}},$$

where ρ_l was defined in the proof of Lemma 3.6. Then, substitute the right hand side of (3.31) by above upper bound of $\mathcal{M}$, we have

$$\begin{aligned}&\mathbb{E}[(L(w,T) - L(v,T))^n] \\ &\le |w-v|^{n\kappa} \prod_{l\neq l_0}^{\gamma} h_l^{n(1-(1-\frac{1}{n})\overline{H}_l d/\rho_l)} h_{l_0}^{-\kappa \overline{H}_{l_0}} \prod_{l=\gamma+1}^{N} h_l^n \\ &\le |w-v|^{n\kappa} h^{n(\beta_\gamma - \overline{H}_\gamma \kappa)}.\end{aligned} \tag{3.34}$$

This completes the proof. □

Proof of Theorem 3.2 Note that, by Lemma 3.6 and Lemma 3.7, for every fixed interval $I \in \mathcal{A}$ such that $I \subseteq T$, $X^{H(u)}(u)$ has local time $L(x,T)$ a.s., that is continuous for all $x \in \mathbb{R}^d$.

To prove the joint continuity, observe that for all $x, y \in \mathbb{R}^d$ and $s, t \in T$ with $|t-s| < \delta$, where $\delta > 0$ is the same as in Lemma 3.2, then

$$\begin{aligned}&\mathbb{E}[(L(x,[a,s]) - L(y,[a,t]))^n] \\ &\quad \le C\left(\mathbb{E}[(L(x,[a,s]) - L(x,[a,t]))^n] + \mathbb{E}[(L(x,[a,t]) - L(y,[a,t]))^n]\right).\end{aligned}$$

Hence the joint continuity of the local time of X follows from a multiparameter version of Kolmogorov's continuity theorem (see Khoshnevisan [12]).

□

It follows from Lemma 3.6 and Lemma 3.7 and Ehm [10] and Xiao [23] that we have the following uniform Hölder conditions for the local time $L(x,\cdot)$.

Theorem 3.3. *Let $d < \sum_{i=1}^{N} \frac{1}{H_i}$. Then there exists constant $C > 0$ independent of $x \in \mathbb{R}^d$ such that for almost all $u \in I$,*

$$\limsup_{h\to 0} \frac{L(x, V(u,h))}{\psi_u(h)} \le C, \tag{3.35}$$

where $V(u,h)$ is the open or closed ball, $\psi_u(h) = h^{\beta_{\kappa(u)}}(\log\log(1/h))^{N-\beta_{\kappa(u)}}$ and $\beta_{\kappa(u)} = N - \kappa(u) - H_{\kappa(u)} d + \sum_{l=1}^{\kappa(u)} \frac{H_{\kappa(u)}(u)}{H_l(u)}$.

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