

Local time and Tanaka formula of G -martingales

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Abstract. The objective of this paper is to study the local time and Tanaka formula of symmetric G -martingales. We introduce the local time of G -martingales and show that it belongs to the G -expectation space $L_G^2(\Omega_T)$. By a localization argument, we obtain the bicontinuous modification of local time. Furthermore, we give the Tanaka formula for convex functions of G -martingales.

§1 Introduction

Motivated by probabilistic interpretations for fully nonlinear PDEs and financial problems with model uncertainty, Peng [13–15] systematically introduced the nonlinear G -expectation theory. Under the G -expectation framework, Peng constructed the corresponding G -Brownian motion, G -Itô's stochastic calculus and G -stochastic differential equations (G -SDEs). Readers can refer to [2, 4, 12] for further developments.

One of the most important notions under G -framework is the G -martingales, which are defined as the processes satisfying martingale property through conditional G -expectation. The representation theorem for G -martingales are obtained in [18–20]. The Lévy's characterization of G -martingales are investigated in [10, 21]. The developments of G -martingales have a deep connection with the settlement of G -backward stochastic differential equations (G -BSDEs), see [4].

This paper studies the local time and Tanaka formula of symmetric G -martingales. It generalizes the results in [3, 9] where the G -Brownian motion case was considered. Compared with the classical case, the integrand space for stochastic integral of G -martingales is not big enough because of nonlinearity. So we first introduce a proper integrand space $\bar{M}_G^2(0, T)$ which is bigger than the previous $M_G^2(0, T)$ when the quadratic variation of G -martingales is degenerate. Then, by proving some characterization results for $\bar{M}_G^2(0, T)$ and using the Krylov's estimate method as in [6], we construct the local time $L_t(a)$ for G -martingales and show that $L_t(a)$

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belongs to the G -expectation space $L_G^2(\Omega_t)$. Moreover, with the help of a localization argument, we prove that $L_t(a)$ has a modification which is continuous in a and t . Finally, we give the Tanaka formula for convex functions of G -martingales and state some basic properties of local time.

The paper is organized as follows. In Section 2, we recall some basic notions and results of G -expectation and G -martingales. In Section 3, we state the main results on local time and Tanaka formula of G -martingales.

§2 Preliminaries

In this section, we review some basic notions and results of G -expectation and G -martingales. More relevant details can be found in [13–15].

2.1 G -expectation space

Let Ω be a given nonempty set and \mathcal{H} be a linear space of real-valued functions on Ω such that if $X_1, \dots, X_d \in \mathcal{H}$, then $\varphi(X_1, X_2, \dots, X_d) \in \mathcal{H}$ for each $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, where $C_{b.Lip}(\mathbb{R}^d)$ is the space of bounded, Lipschitz functions on \mathbb{R}^d . \mathcal{H} is considered as the space of random variables.

Definition 2.1. A sublinear expectation $\hat{\mathbb{E}}$ on \mathcal{H} is a functional $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for each $X, Y \in \mathcal{H}$,

- (i) *Monotonicity:* $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ if $X \geq Y$;
- (ii) *Constant preserving:* $\hat{\mathbb{E}}[c] = c$ for $c \in \mathbb{R}$;
- (iii) *Sub-additivity:* $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;
- (iv) *Positive homogeneity:* $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Set $\Omega_T := C_0([0, T]; \mathbb{R}^d)$ the space of all \mathbb{R}^d -valued continuous paths $(\omega_t)_{t \geq 0}$ starting from origin, equipped with the supremum norm. Denote by $\mathcal{B}(\Omega_T)$ the Borel σ -algebra of Ω_T and $B_t(\omega) := \omega_t$ the canonical mapping. For each $t \in [0, T]$, we set $L_{ip}(\Omega_t) := \{\varphi(B_{t_1}, \dots, B_{t_k}) : k \in \mathbb{N}, t_1, \dots, t_k \in [0, t], \varphi \in C_{b.Lip}(\mathbb{R}^{k \times d})\}$.

Let $G : \mathbb{S}(d) \rightarrow \mathbb{R}$ be a given monotonic and sublinear function, where $\mathbb{S}(d)$ is the set of $d \times d$ symmetric matrices. Peng constructed the sublinear G -expectation space $(\Omega_T, L_{ip}(\Omega_T), \hat{\mathbb{E}})$, and under $\hat{\mathbb{E}}$, the canonical process $B_t = (B_t^1, \dots, B_t^d)$ is called a d -dimensional G -Brownian motion. The conditional G -expectation for $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$ at $t = t_j$, $1 \leq j \leq n$ is defined by

$$\hat{\mathbb{E}}_{t_j}[X] := \phi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_j} - B_{t_{j-1}}),$$

where $\phi(x_1, \dots, x_j) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_j, B_{t_{j+1}} - B_{t_j}, \dots, B_{t_n} - B_{t_{n-1}})]$.

For each $p \geq 1$, we denote by $L_G^p(\Omega_t)$ the completion of $L_{ip}(\Omega_t)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. The G -expectation $\hat{\mathbb{E}}[\cdot]$ and conditional G -expectation $\hat{\mathbb{E}}_t[\cdot]$ can be extended continuously to $L_G^1(\Omega_T)$.

The following is the representation theorem for G -expectation.

Theorem 2.2 (see [1, 5]). *There exists a family \mathcal{P} of weakly compact probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that*

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad \text{for each } X \in L_G^1(\Omega_T).$$

\mathcal{P} is called a set that represents $\hat{\mathbb{E}}$.

Given \mathcal{P} that represents $\hat{\mathbb{E}}$, we define the capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad \text{for each } A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is said to be polar if $c(A) = 0$. A property is said to hold “quasi-surely” (q.s.) if it holds outside a polar set. In the following, we do not distinguish between two random variables X and Y if $X = Y$ q.s.

Set

$$\mathcal{L}(\Omega_T) := \{X \in \mathcal{B}(\Omega_T) : E_P[X] \text{ exists for each } P \in \mathcal{P}\}.$$

We extend the G -expectation to $\mathcal{L}(\Omega_T)$, still denote it by $\hat{\mathbb{E}}$, by setting

$$\hat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X], \quad \text{for } X \in \mathcal{L}(\Omega_T).$$

Then clearly, $L_G^p(\Omega_T) \subset \mathcal{L}(\Omega_T)$.

Definition 2.3. *A real function X on Ω_T is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set O with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.*

Definition 2.4. *We say that $X : \Omega_T \mapsto \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega_T \mapsto \mathbb{R}$ such that $X = Y$, q.s.*

We have the following characterization result of the space $L_G^p(\Omega_t)$, which can be seen as a counterpart of Lusin’s theorem in the nonlinear expectation theory.

Theorem 2.5. *For each $p \geq 1$, we have*

$$L_G^p(\Omega_t) = \{X \in \mathcal{B}(\Omega_t) : \lim_{N \rightarrow \infty} \hat{\mathbb{E}}[|X|^p I_{|X| \geq N}] = 0 \text{ and } X \text{ has a quasi-continuous version}\}.$$

Definition 2.6. *A process is a family of random variables $X = (X_t)_{t \in [0, T]}$ such that for all $t \in [0, T]$, $X_t \in L_G^1(\Omega_T)$. We say that process Y is a modification of process X if for each $t \in [0, T]$, $X_t = Y_t$, q.s.*

More generally, we can consider the process $(X_t)_{t \in [0, T]}$ taking values in Banach space E , i.e., for each $t \in [0, T]$, $X_t : \Omega_T \rightarrow E$ is measurable. The following is the Banach-valued Kolmogorov criterion for continuous modification with respect to capacity.

Lemma 2.7. *Let $(X_t)_{t \in [0, T]}$ be a process taking values in Banach space E . Assume that there exist positive constants α, β and c such that*

$$\hat{\mathbb{E}}[|X_t - X_s|^\alpha] \leq c|t - s|^{1+\beta}.$$

Then X admits a continuous modification \tilde{X} such that

$$\hat{\mathbb{E}} \left[\left(\sup_{s \neq t} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\gamma} \right)^\alpha \right] < \infty,$$

for every $\gamma \in (0, \beta/\alpha)$.

Now we give the definition of G -martingales.

Definition 2.8. A process $\{M_t\}$ is called a G -martingale if $M_t \in L_G^1(\Omega_t)$ and $\hat{\mathbb{E}}_s[M_t] = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both G -martingales, we call $\{M_t\}$ a symmetric G -martingale.

Remark 2.9. If M is a symmetric G -martingale, then under each P , it is a classical martingale.

In the following, we give the stochastic calculus with respect to a kind of martingales as well as its quadratic variation process. In this paper, we always assume that M is a symmetric martingale satisfying:

(H) $M_t \in L_G^2(\Omega_t)$ for each $t \geq 0$ and there exists a nonnegative constant $\Lambda > 0$ such that

$$\hat{\mathbb{E}}_t[|M_{t+s} - M_t|^2] \leq \Lambda s, \quad \text{for each } t, s \geq 0. \tag{1}$$

Remark 2.10. For $M_t \in L_G^2(\Omega_t)$, by the G -martingale representation theorem (see, e.g., [18, 20]), M_t can be represented as the integral of G -Brownian motion. From this, we know that M_t is continuous.

For each $T > 0$ and $p \geq 1$, we define

$$M_G^{p,0}(0, T) := \left\{ \eta = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t) : N \in \mathbb{N}, 0 \leq t_0 \leq t_1 \leq \dots \leq t_N \leq T, \right. \\ \left. \xi_j \in L_G^p(\Omega_{t_j}), j = 0, 1, \dots, N \right\}.$$

For each $\eta \in M_G^{p,0}(0, T)$, set the norm $\|\eta\|_{M_G^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{\frac{1}{p}}$ and denote by $M_G^p(0, T)$ the completion of $M_G^{p,0}(0, T)$ under $\|\cdot\|_{M_G^p}$.

For $\eta \in M_G^{2,0}(0, T)$, define the stochastic integral with respect to M by

$$I(\eta) = \int_0^T \eta_t dM_t := \sum_{j=0}^{N-1} \xi_{t_j} (M_{t_{j+1}} - M_{t_j}) : M_G^{2,0}(0, T) \rightarrow L_G^2(\Omega_T).$$

The proof of following lemma is the same as that of Lemma 3.5 in Chap. III of [15], so we omit it.

Lemma 2.11. For each $\eta \in M_G^{2,0}(0, T)$, we have

$$\hat{\mathbb{E}} \left[\left| \int_0^T \eta_t dM_t \right|^2 \right] \leq \Lambda \hat{\mathbb{E}} \left[\int_0^T |\eta_t|^2 dt \right]. \tag{2}$$

By the above lemma, we can extend the integral continuously to $M_G^2(0, T)$. Indeed, for $\eta \in M_G^2(0, T)$, there exists a sequence $\{\eta^n\} \subset M_G^{2,0}(0, T)$ such that $\eta^n \rightarrow \eta$ in $M_G^2(0, T)$. By Lemma 2.11, $\int_0^T \eta_t^n dM_t$ is a Cauchy sequence in $L_G^2(\Omega_T)$, we define

$$\int_0^T \eta_t dM_t := L_G^2 - \lim_{n \rightarrow \infty} \int_0^T \eta_t^n dM_t.$$

It's easy to see $\int_0^t \eta_s dM_s$ is a symmetric G -martingale and (2) still holds.

Next we consider the quadratic variation of M . Let $\pi_t^N = \{t_0^N, \dots, t_N^N\}$ be a partition of $[0, t]$ and denote

$$\mu(\pi_t^N) := \max\{|t_{j+1}^N - t_j^N| : j = 0, 1, \dots, N - 1\}.$$

Consider

$$\begin{aligned} \sum_{j=0}^{N-1} (M_{t_{j+1}^N} - M_{t_j^N})^2 &= \sum_{j=0}^{N-1} (M_{t_{j+1}^N}^2 - M_{t_j^N}^2) - 2 \sum_{j=0}^{N-1} M_{t_j^N} (M_{t_{j+1}^N} - M_{t_j^N}) \\ &= M_t^2 - 2 \sum_{j=0}^{N-1} M_{t_j^N} (M_{t_{j+1}^N} - M_{t_j^N}). \end{aligned}$$

Letting $\mu(\pi_t^N) \rightarrow 0$, the right side converges to $M_t^2 - 2 \int_0^t M_s dM_s$ in $L_G^2(\Omega_T)$. So

$$\sum_{j=0}^{N-1} (M_{t_{j+1}^N} - M_{t_j^N})^2 \longrightarrow M_t^2 - 2 \int_0^t M_s dM_s \quad \text{in } L_G^2(\Omega_T).$$

We call this limit the quadratic variation of M and denote it by $\langle M \rangle_t$.

Remark 2.12. Since M satisfies (H), by a standard approximation argument, we can deduce that $(M_t)_{0 \leq t \leq T} \in M_G^2(0, T)$. Thus the integral $\int_0^t M_s dM_s$ is meaningful.

By the definition of $\langle M \rangle_t$, it is easy to obtain, for each $t, s \geq 0$,

$$\hat{\mathbb{E}}_t[\langle M \rangle_{t+s} - \langle M \rangle_t] = \hat{\mathbb{E}}_t[\langle M \rangle_{t+s} - \langle M \rangle_t] = \hat{\mathbb{E}}_t[M_{t+s}^2 - M_t^2] = \hat{\mathbb{E}}_t[|M_{t+s} - M_t|^2] \leq \Lambda s. \quad (3)$$

Remark 2.13. Note that $\langle M \rangle_t$ is q.s. defined, and under each $P \in \mathcal{P}$, it is also the classical quadratic variation $\langle M \rangle_t^P$ of martingale M .

Theorem 2.14 (B-D-G inequality). *There exists some constant $C > 0$ such that*

$$\hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} \left|\int_0^t \eta_s dM_s\right|^2\right] \leq C \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^2 d\langle M \rangle_t\right].$$

§3 Main results

We first introduce a bigger integrand space for the stochastic calculus of G -martingales M . This space plays an important role in the construction of local time.

For $p \geq 1$ and $\eta \in M_G^{p,0}(0, T)$, we define a new norm $\|\eta\|_{\bar{M}_G^p} = (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p d\langle M \rangle_t])^{\frac{1}{p}}$ and denote the completion of $M_G^{p,0}(0, T)$ under the norm $\|\cdot\|_{\bar{M}_G^p}$ by $\bar{M}_G^p(0, T)$.

We have the following result concerning the relationship between two spaces $M_G^p(0, T)$ and $\bar{M}_G^p(0, T)$.

Lemma 3.1. *We have*

$$\|\eta\|_{\bar{M}_G^p} \leq \Lambda^{\frac{1}{p}} \|\eta\|_{M_G^p} \text{ for each } \eta \in M_G^{p,0}(0, T), \quad \text{and} \quad M_G^p(0, T) \subset \bar{M}_G^p(0, T). \quad (4)$$

If moreover there exists a constant $0 < \lambda \leq \Lambda$ such that $\hat{\mathbb{E}}_t[|M_{t+s} - M_t|^2] \geq \lambda s$, then the quadratic variation of M is non-degenerate, i.e., $\hat{\mathbb{E}}_t[\langle M \rangle_{t+s} - \langle M \rangle_t] \geq \lambda s$, which implies

$$\|\eta\|_{\bar{M}_G^p} \geq \lambda^{\frac{1}{p}} \|\eta\|_{M_G^p} \text{ for each } \eta \in M_G^{p,0}(0, T), \quad \text{and} \quad M_G^p(0, T) = \bar{M}_G^p(0, T).$$

Proof. We only prove (4) since the proof for the second part is similar. We only need to show $\|\eta\|_{\bar{M}_G^p} \leq \Lambda^{\frac{1}{p}} \|\eta\|_{M_G^p}$ for $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t) \in M_G^{p,0}(0, T)$. By the sub-linearity of $\hat{\mathbb{E}}$, we have

$$\begin{aligned} \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p d\langle M \rangle_t\right] &= \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} |\xi_j|^p (\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_j})\right] \\ &\leq \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} |\xi_j|^p (\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_j} - \Lambda(t_{j+1} - t_j))\right] + \Lambda \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} |\xi_j|^p (t_{j+1} - t_j)\right] \end{aligned} \tag{5}$$

Note that

$$\begin{aligned} \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} |\xi_j|^p (\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_j} - \Lambda(t_{j+1} - t_j))\right] &\leq \sum_{j=0}^{N-1} \hat{\mathbb{E}}[|\xi_j|^p (\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_j} - \Lambda(t_{j+1} - t_j))] \\ &= \sum_{j=0}^{N-1} \hat{\mathbb{E}}[|\xi_j|^p \hat{\mathbb{E}}_{t_j}[\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_j} - \Lambda(t_{j+1} - t_j)]] \\ &\leq 0. \end{aligned}$$

Combining this with (5), we get the desired result. □

The following counterexample shows that $\bar{M}_G^p(0, T)$ is really bigger than $M_G^p(0, T)$.

Example 3.2. Let $p = 1$ and $M_t \equiv 0$. Then $\|\eta\|_{\bar{M}_G^p} = 0$, and thus, every progressive measurable process belongs to $\bar{M}_G^1(0, T)$. But such a process may not be in $M_G^1(0, T)$, see [6].

For $\eta \in \bar{M}_G^{2,0}(0, T)$, by a similar analysis as in Proposition 4.5 in Chap. III of [15], we have

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dM_t\right)^2\right] = \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^2 d\langle M \rangle_t\right].$$

Then the definition of integral $\int_0^T \eta_t dM_t$ can be extended continuously to $\bar{M}_G^2(0, T)$. Moreover, on $M_G^2(0, T)$, this definition coincides with the one in Section 1.

In the following, C always denotes a generic constant which is free to vary from line to line. By a standard argument, we can obtain a regular version of the stochastic integral.

Proposition 3.3. For $\eta \in \bar{M}_G^2(0, T)$, there exists a modification of $\int_0^t \eta_s dM_s$ such that $t \rightarrow \int_0^t \eta_s dM_s$ is continuous.

Proof. Denote $I(\eta)_t = \int_0^t \eta_s dM_s$. We can take a sequence $\eta^n \in \bar{M}_G^{2,0}(0, T)$ such that $\eta^n \rightarrow \eta$ in $\bar{M}_G^2(0, T)$. It is easy to see that $t \rightarrow \int_0^t \eta_s^n dM_s$ is continuous. By the B-D-G inequality,

$$\hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} |I(\eta^n)_t - I(\eta^m)_t|^2\right] \leq C \hat{\mathbb{E}}\left[\int_0^T |\eta^n - \eta^m|^2 d\langle M \rangle_t\right]. \tag{6}$$

From the Markov inequality (see [1]), for each $a > 0$,

$$\hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} |I(\eta^n)_t - I(\eta^m)_t| \geq a\right] \leq \frac{1}{a^2} \hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} |I(\eta^n)_t - I(\eta^m)_t|^2\right]. \tag{7}$$

Combining (6) and (7), and using the Borel-Cantelli lemma (see [15]), one can extract a subsequence $I(\eta^{n_k})_t$ converging q.s. uniformly. We denote this limit by Y_t , and it is a continuous

modification of $I(\eta)_t$. □

Remark 3.4. Henceforth, we will only consider the continuous modification of $\int_0^t \eta_s dM_s$.

Let us state some characterization results for the space $\bar{M}_G^p(0, T)$, which are important for our future discussion.

Lemma 3.5. Assume $X \in \bar{M}_G^p(0, T)$. Then for each $\varphi \in C_{Lip}(\mathbb{R})$, we have $\varphi(X_t)_{0 \leq t \leq T} \in \bar{M}_G^p(0, T)$.

Proof. We can find a sequence $X_t^n = \sum_{j=0}^{N_n-1} \xi_j^n(\omega) I_{[t_j^n, t_{j+1}^n)}(t)$, where $\xi_j \in L_G^p(\Omega_{t_j^n})$, such that $X^n \rightarrow X$ under the norm $\|\cdot\|_{\bar{M}_G^p}$. Note that $\varphi(\xi_j^n(\omega)) \in L_G^p(\Omega_{t_j^n})$ by Theorem 2.5, then we have

$$\varphi(X_t^n) = \sum_{j=0}^{N_n-1} \varphi(\xi_j^n) I_{[t_j^n, t_{j+1}^n)}(t) \in \bar{M}_G^{p,0}(0, T).$$

Now the desired result follows from the observation that

$$\hat{\mathbb{E}}\left[\int_0^T |\varphi(X_t) - \varphi(X_t^n)|^p d\langle M \rangle_t\right] \leq L_\varphi^p \hat{\mathbb{E}}\left[\int_0^T |X_t - X_t^n|^p d\langle M \rangle_t\right] \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where L_φ is the Lipschitz constant of φ . □

Proposition 3.6. Assume $\eta \in \bar{M}_G^p(0, T)$. Then

$$\hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p I_{\{|\eta_t| > N\}} d\langle M \rangle_t\right] \rightarrow 0, \quad \text{as } N \rightarrow 0.$$

Proof. It suffices to prove the case that $\eta_t = \sum_{j=0}^{n-1} \xi_j I_{[t_j, t_{j+1})}(t) \in M_G^{p,0}(0, T)$, where $\xi_j \in L_G^p(\Omega_{t_j})$. We take bounded, continuous functions φ_N such that $I_{\{|x| > N\}} \leq \varphi_N \leq I_{\{|x| > N-1\}}$. Then by Theorem 2.5,

$$\begin{aligned} \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p I_{\{|\eta_t| > N\}} d\langle M \rangle_t\right] &= \hat{\mathbb{E}}\left[\sum_{i=1}^{n-1} |\xi_i|^p I_{\{|\xi_i| > N\}} (\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i})\right] \\ &\leq \sum_{i=1}^{n-1} \hat{\mathbb{E}}[|\xi_i|^p \varphi_N(\xi_i) (\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i})] \\ &= \sum_{i=1}^{n-1} \hat{\mathbb{E}}[|\xi_i|^p \varphi_N(\xi_i)] \hat{\mathbb{E}}_{t_i}[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i}] \\ &\leq \Lambda \sum_{i=1}^{n-1} \hat{\mathbb{E}}[|\xi_i|^p I_{\{|\xi_i| > N-1\}}] (t_{i+1} - t_i) \rightarrow 0, \quad \text{as } N \rightarrow \infty. \end{aligned}$$

□

Recall that we always assume that $(M_t)_{t \geq 0}$ is a symmetric G -martingale satisfying (H). The following Krylov's estimate can be used to show that a kind of processes belong to $\bar{M}_G^p(0, T)$.

Theorem 3.7. (Krylov's estimate) [8, 11, 17] There exists some constant C depending on Λ and T such that, for each $p \geq 1$ and Borel function g ,

$$\hat{\mathbb{E}}\left[\int_0^T |g(M_t)| d\langle M \rangle_t\right] \leq C \left(\int_{\mathbb{R}} |g(x)|^p dx\right)^{1/p}.$$

Proof. We outline the proof for the convenience of readers. For any $P \in \mathcal{P}$, $(M_t)_{t \geq 0}$ is a martingale. By Hölder's inequality,

$$E_P[\int_0^T |g(M_t)|d\langle M \rangle_t] \leq C_1(E_P[\int_0^T |g(M_t)|^p d\langle M \rangle_t])^{1/p}, \tag{8}$$

where $C_1 = (\hat{\mathbb{E}}[\langle M \rangle_T])^{(p-1)/p}$. Let $L_t^P(a)$ be the corresponding local time at a of M under P . By the classical Tanaka formula (see, e.g., [16]),

$$L_T^P(a) = |M_T - a| - |M_0 - a| - \int_0^T \text{sgn}(M_t - a)dM_t, \text{ P-a.s.}$$

Taking expectation on both sides, we get

$$0 \leq E_P[L_T^P(a)] = E_P[|M_T - a|] - |M_0 - a| \leq E_P[|M_T - M_0|] \leq \hat{\mathbb{E}}[|M_T - M_0|].$$

Applying the occupation time formula under P , we obtain

$$E_P[\int_0^T |g(M_t)|^p d\langle M \rangle_t] = E_P[\int_{\mathbb{R}} |g(a)|^p L_T^P(a) da] = \int_{\mathbb{R}} |g(a)|^p E_P[L_T^P(a)] da \leq C_2 \int_{\mathbb{R}} |g(a)|^p da, \tag{9}$$

where $C_2 = \hat{\mathbb{E}}[|M_T - M_0|]$. Combining (8) and (9), we have

$$E_P[\int_0^T |g(M_t)|d\langle M \rangle_t] \leq C(\int_{\mathbb{R}} |g(a)|^p da)^{\frac{1}{p}}.$$

Note that C is independent of P , the desired result follows by taking supremum over $P \in \mathcal{P}$ in the above inequality. □

Lemma 3.8. *Assume φ', φ are Borel measurable and $\varphi' = \varphi$ a.e. Then for each $p \geq 1$, we have $\varphi'(M.) = \varphi(M.)$ in $\bar{M}_G^p(0, T)$, i.e., $\|\varphi'(M.) - \varphi(M.)\|_{\bar{M}_G^p} = 0$.*

Proof. By Theorem 3.7, we get $\hat{\mathbb{E}}[\int_0^T |\varphi' - \varphi|^p(M_t)d\langle M \rangle_t] \leq C\|\varphi' - \varphi\|_{L^p(\mathbb{R})}^p = 0$. □

The following is a kind of dominated convergence result for G -martingales.

Proposition 3.9. *Assume $(\varphi^n)_{n \geq 1}$ is a sequence of Borel measurable functions such that φ^n is linear growth uniformly, i.e., $|\varphi^n(x)| \leq C(1 + |x|)$, $n \geq 1$ for some constants C . If $\varphi^n \rightarrow \varphi$ a.e., then*

$$\lim_{n \rightarrow \infty} \hat{\mathbb{E}}[\int_0^T |\varphi^n(M_t) - \varphi(M_t)|^2 d\langle M \rangle_t] = 0.$$

Proof. By Lemma 3.8, without loss of generality, we may assume $|\varphi(x)| \leq C(1 + |x|)$. For any $N > 0$, we have

$$\begin{aligned} \hat{\mathbb{E}}[\int_0^T |\varphi^n(M_t) - \varphi(M_t)|^2 d\langle M \rangle_t] &\leq \hat{\mathbb{E}}[\int_0^T |\varphi^n(M_t) - \varphi(M_t)|^2 I_{\{|M_t| \leq N\}} d\langle M \rangle_t] \\ &\quad + \hat{\mathbb{E}}[\int_0^T |\varphi^n(M_t) - \varphi(M_t)|^2 I_{\{|M_t| > N\}} d\langle M \rangle_t]. \end{aligned} \tag{10}$$

According to Theorem 3.7, we can find a constant C' such that

$$\hat{\mathbb{E}}[\int_0^T |\varphi^n(M_t) - \varphi(M_t)|^p I_{\{|M_t| \leq N\}} d\langle M \rangle_t] \leq C' \int_{\{|x| \leq N\}} |\varphi^n(x) - \varphi(x)|^p dx,$$

which converges to 0, as $n \rightarrow \infty$ by the Lebesgue's dominated convergence theorem. On the other hand, note that $M_t = \hat{\mathbb{E}}_t[M_T]$, by an approximation argument, we have $(M_t)_{t \leq T} \in$

$M_G^2(0, T) \subset \bar{M}_G^2(0, T)$. Then, by the linear growth condition on φ^n and φ , and Proposition 3.6, we get

$$\hat{\mathbb{E}}\left[\int_0^T |\varphi^n(M_t) - \varphi(M_t)|^2 I_{\{|M_t| > N\}} d\langle M \rangle_t\right] \leq C \hat{\mathbb{E}}\left[\int_0^T (1 + |M_t|)^2 I_{\{|M_t| > N\}} d\langle M \rangle_t\right] \rightarrow 0,$$

as $N \rightarrow 0$.

First letting $n \rightarrow \infty$ and then letting $N \rightarrow \infty$ in (10), we get the desired result. □

By the Krylov’s estimate and Proposition 3.9, we can show that $\bar{M}_G^2(0, T)$ contains a lot of processes that we may interest in. Such kind of processes are important for the construction of local time.

Proposition 3.10. *For each Borel measurable function φ of linear growth, we have $(\varphi(M_t))_{t \leq T} \in \bar{M}_G^2(0, T)$.*

Proof. We take a sequence of Lipschitz continuous functions $(\varphi^n)_{n \geq 1}$, such that φ^n converges to φ a.e. and $|\varphi^n(x)| \leq C(1 + |x|)$. Then by Proposition 3.9, we have

$$\lim_{n \rightarrow \infty} \hat{\mathbb{E}}\left[\int_0^T |\varphi^n - \varphi|^2(M_t) d\langle M \rangle_t\right] = 0.$$

Since $(\varphi^n(M_t))_{t \leq T} \in \bar{M}_G^2(0, T)$ for each n by Lemma 3.5, we derive that $(\varphi(M_t))_{t \leq T} \in \bar{M}_G^2(0, T)$, and this completes the proof. □

Now we can define the local time of G -martingale M . For each $P \in \mathcal{P}$, by the classical Tanaka formula under P ,

$$|M_t - a| = |M_0 - a| + \int_0^t \text{sgn}(M_s - a) dM_s + L_t^P(a), \quad P\text{-a.s.}, \tag{11}$$

where $L_t^P(a)$ is the local time of martingale M_t at a under P . According to Proposition 3.10, we have $(\text{sgn}(M_s - a))_{s \leq t} \in \bar{M}_G^2(0, t)$. This implies that $\int_0^t \text{sgn}(M_s - a) dM_s \in L_G^2(\Omega_t)$. We define the local time for G -martingale M by

$$L_t(a) := |M_t - a| - |M_0 - a| - \int_0^t \text{sgn}(M_s - a) dM_s \in L_G^2(\Omega_t).$$

Then (11) gives that

$$L_t(a) = L_t^P(a), \quad P\text{-a.s.} \tag{12}$$

The local time always possesses a bicontinuous modification.

Theorem 3.11. *There exists a modification of the process $\{L_t(a) : t \in [0, T], a \in \mathbb{R}\}$ such that $L_t(a)$ is bicontinuous, i.e., the map $(a, t) \rightarrow L_t(a)$ is continuous.*

Proof. It suffices to prove that $\hat{M}_t^a := \int_0^t \text{sgn}(M_s - a) dM_s$ has such kind of modification. Let $N > 0$ be given. For each integer $n \geq 1$, we define stopping time

$$\tau_N = \inf\{s \geq 0 : |M_s - M_0|^n + \langle M \rangle_s^{\frac{n}{2}} \geq N\}.$$

Denote $\bar{M}_t = M_{\tau_N \wedge t}$. Under each $P \in \mathcal{P}$, from the classical optional sampling theorem, \bar{M}_t is a martingale. We denote the corresponding local time of \bar{M} by $\bar{L}_t^P(a)$. Then by classical B-D-G inequality,

$$E_P[|\bar{L}_T^P(a)|^n] \leq C_n(|\bar{M}_T - \bar{M}_0|^n + |\langle \bar{M} \rangle_T^{\frac{n}{2}}|) \leq C_n N,$$

where C_n is a constant depending on n and may vary from line to line. For $x < y$, from occupation formula under P and Hölder's inequality, we have

$$\begin{aligned} & E_P[\sup_{0 \leq t \leq T} |\int_0^t \operatorname{sgn}(\bar{M}_u - x) d\bar{M}_u - \int_0^t \operatorname{sgn}(\bar{M}_u - y) d\bar{M}_u|^{2n}] \\ & \leq C_n E_P[|\int_0^T I_{[x,y)}(\bar{M}_t) d\langle \bar{M} \rangle_t^P|^n] \\ & \leq C_n E_P[|\int_x^y \bar{L}_T^P(a) da|^n] \\ & \leq C_n (y-x)^n E_P[\frac{1}{y-x} \int_x^y |\bar{L}_T^P(a)|^n da] \\ & \leq C_n N (y-x)^n. \end{aligned}$$

Note that, $\hat{M}_{t \wedge \tau_N}^a = \int_0^t \operatorname{sgn}(\bar{M}_u - a) d\bar{M}_u$, P -a.s. Thus,

$$\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\hat{M}_{t \wedge \tau_N}^x - \hat{M}_{t \wedge \tau_N}^y|^{2n}] \leq C_n N (y-x)^n.$$

Applying Lemma 2.7 to

$$a \rightarrow \hat{M}_{t \wedge \tau_N}^a \in E := C([0, T]; \mathbb{R}),$$

we obtain that $\hat{M}_{t \wedge \tau_N}^a$ has a bicontinuous version for each N , which implies that \hat{M}_t^a has a bicontinuous version. □

Now we give the Tanaka formula for convex functions of G -martingales.

Theorem 3.12. *Let f be a convex function such that left derivative f'_- satisfies the linear growth condition. Then*

$$f(M_t) - f(M_0) = \int_0^t f'_-(M_s) dM_s + \frac{1}{2} \int_{\mathbb{R}} L_t(a) df'_-(a), \quad q.s. \tag{13}$$

where df'_- is the Lebesgue-Stieltjes measure of f'_- . Moreover, the integral $\int_{\mathbb{R}} L_t(a) df'_-(a)$ belongs to $L_G^1(\Omega_t)$.

Proof. According to Proposition 3.10, we have $(f'_-(M_s))_{s \leq t} \in \bar{M}_G^2(0, t)$. Note that, under each $P \in \mathcal{P}$, $\int_0^t f'_-(M_s) dM_s$ is also the stochastic integral with respect to martingale M_t and $L_t(a)$ is the local time of M_t . By the classical Tanaka formula for martingales, we have

$$f(M_t) - f(M_0) = \int_0^t f'_-(M_s) dM_s + \frac{1}{2} \int_{\mathbb{R}} L_t(a) df'_-(a), \quad P\text{-a.s.}$$

Since the four terms in the above identity both q.s. defined, we deduce that the above formula holds q.s.

Since convex function f is continuous, we know that $f(M_t)$ is quasi-continuous. Moreover, the linear growth condition of f'_- implies that $|f(x)| \leq C(1 + |x|^2)$ by Problem 3.6.21 (6.46) in [7]. Thus,

$$\hat{\mathbb{E}}[|f(M_t)| I_{\{|f(M_t)| > N\}}] \leq C \hat{\mathbb{E}}[(1 + |M_t|^2) I_{\{|M_t|^2 > \frac{N}{C} - 1\}}] \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Then from Theorem 2.5, we deduce that $f(M_t) \in L_G^1(\Omega_t)$, which, together with (13), implies $\int_{\mathbb{R}} L_t(a) df'_-(a) \in L_G^1(\Omega_t)$. □

Remark 3.13. For G -Brownian motion, the moments of every order and the quadratic variation are both finite. So in the case that M is a G -Brownian motion, the similar proof can give a better version for Proposition 3.9, 3.10 and Theorem 3.12 where the linear growth condition on the corresponding functions is replaced by the polynomial growth condition.

Finally, we list some useful properties of local time, which follow directly from applying the classical ones under each $P \in \mathcal{P}$.

Proposition 3.14. *We have*

- (i) *The measure $dL_t(a)$ grows only when $M = a$: $\int_{\mathbb{R}_+} I_{\{M_t \neq a\}} dL_t(a) = 0$, $q.s.$;*
- (ii) *Occupation time formula: for each bounded or positive Borel measurable function g , we have $\int_0^T g(M_t) d\langle M \rangle_t = \int_{\mathbb{R}} g(a) L_T^P(a) da$, $q.s.$;*
- (iii) *For the bicontinuous version of $L_t(a)$, the following representation hold:*

$$L_t(a) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I_{[a, a+\varepsilon)}(M_s) d\langle M \rangle_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I_{(a-\varepsilon, a+\varepsilon)}(M_s) d\langle M \rangle_t, q.s.$$

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