Appl. Math. J. Chinese Univ.
2019, 34(4): 460-467

New conditions for pattern solutions of a Brusselator model

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Abstract. This paper is devoted to establishing a critical value of the concentration of one intermediary reactant which determines whether pattern solutions of a class of Brusselator models exist or not. We introduce a new method to compute the degree index of the related linear operator so that the obtained sufficient conditions are easier to verify than those in the known references. The proofs mainly rely on Leray-Schauder degree theory, implicit function theorem and analytical techniques.

§1 Introduction

The standard Brusselator model describing autocatalytic oscillating chemical reactions is in the form of

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = a - (b+1)u + u^2 v, \ x > 0, \ t > 0, \\ \dfrac{\partial v}{\partial t} - d_2\Delta v = bu - u^2 v, \ x > 0, \ t > 0, \end{cases} \tag{1.1}$$

which was introduced by Prigogine and Lefever [13] in 1968, where $(x,t) \in \Omega \times [0,+\infty)$ and Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$; $u(x,t)$ and $v(x,t)$ represent the concentration of two intermediary reactants having the diffusion rates $d_1, d_2 > 0$; $a, b > 0$ are fixed concentrations. In recent decades (1.1) has been extensively studied both numerically and analytically. Twizell *et al* [14] obtained the numerical solution of the initial-value problems by developing a second-order method. Peńa and Garcia [11] investigated the stability of stripes and hexagons towards spatial perturbations by establishing a generalized amplitude equation. Kang [6] discussed the dynamics of the local map of a discrete version of the Brusselator model, focusing on asymptotic behaviors of bounded trajectories inside the Julia set. Ma and Hu [9] had a global bifurcation and stability analysis for the steady states. More research results on model (1.1) can be found in [12][2][4]. For the study on positive steady states of nonlinear reaction-diffusion systems, there is a great deal of research results, see for example [1][3][5][10] and the references therein.

Received: 2018-02-07. Revised: 2018-12-14.
MR Subject Classification: 35K20, 35K55, 35K57, 92C40.
Keywords: Brusselator model, pattern solution, degree index theory.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-019-3597-1.
The work of M. Ma was supported by the National Natural Science Foundation of China (No. 11671359) and the Science Foundation of Zhejiang Sci-Tech University under Grant No. 15062173-Y. The work of Jichneg Tao was supported by the provincial Natural Science Foundation of Zhejiang (LY16A010009).
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Based on the results in [4], the conditions for the existence of pattern solutions of (1.1) can be derived, which is the same as that in [12], however, in practice it is difficult to verify them. In this paper, we study the existence and the non-existence of pattern solutions of model (1.1) subject to the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 \text{ on } \partial\Omega \text{ for } t>0 \tag{1.2}$$

and the initial data

$$u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ x\in\Omega, \tag{1.3}$$

where ν denotes the outward unit normal vector on $\partial\Omega$, $u_0(x), v_0(x)$ are non-negative continuous functions in $\overline{\Omega}$. The obtained conditions are very easy to be verified and different from that established in the known references [12][2][4].

The organization of this paper is as follows. In Section 2, we give preliminaries. In Section 3, we obtain the conditions for the occurrence of pattern formation by employing the homotopy invariance of degree index.

§2 preliminaries

The corresponding steady state problem of (1.1)-(1.3) is

$$\begin{cases} -d_1\Delta u=a-(b+1)u+u^2v,\ x\in\Omega, \\ -d_2\Delta v=bu-u^2v,\ x\in\Omega, \\ \dfrac{\partial u}{\partial \nu}=\dfrac{\partial v}{\partial \nu}=0,\ x\in\partial\Omega. \end{cases} \tag{2.1}$$

It is obvious that (2.1) has a unique constant solution $(a, b/a)$. The boundedness of solutions to (2.1) can be directly derived from Theorem 3.3 in [4]. But, for the convenience of readers, here we give its detailed proof by using Lemma 2.1 which is extracted from [7][8].

Lemma 2.1. *Let $H\in C^1(\overline{\Omega}\times\mathbb{R})$. The following statements are true.*

(a) If $\xi\in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies

$$\Delta\xi+H(x,\xi)\ge 0 \quad in\ \Omega, \quad \frac{\partial\xi}{\partial\nu}\le 0 \quad on \quad \partial\Omega,$$

and $\xi(x_0)=\max_{\overline{\Omega}}\xi$, then $H(x_0,\xi(x_0))\ge 0$.

(b) If $\xi\in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies

$$\Delta\xi+H(x,\xi)\le 0 \quad in \quad \Omega, \quad \frac{\partial\xi}{\partial\nu}\ge 0 \quad on \quad \partial\Omega,$$

and $\xi(x_0)=\min_{\overline{\Omega}}\xi$, then $H(x_0,\xi(x_0))\le 0$.

Lemma 2.2. *Assume that $(u(x), v(x))$ is a non-negative solution of the system (2.1), then we have*

$$\frac{a}{b+1}\le u\le a+\frac{d_2b(b+1)}{d_1a}, \qquad \frac{b}{a+\frac{d_2b(b+1)}{d_1a}}\le v\le\frac{b(b+1)}{a}.$$

Proof. We first suppose that $x_0\in\Omega$ satisfies $u(x_0)=\min_{x\in\Omega}u(x)$. Then, Lemma 2.1 (b) leads to

$$a-(b+1)u(x_0)+u^2(x_0)v(x_0)\le 0,$$

which implies $u(x_0)\ge\frac{a}{b+1}$, and thus

$$u(x)\ge\frac{a}{b+1}, \qquad x\in\Omega. \tag{2.2}$$

Next, by setting $x_1\in\Omega$ such that $v(x_1)=\max_{x\in\Omega}v(x)$, and notice of Lemma 2.1(a), we have

$bu(x_1) - u^2(x_1)v(x_1) \geq 0$. From (2.2) it follows that

$$v(x) \leq \frac{b(b+1)}{a}, \quad x \in \Omega. \tag{2.3}$$

By adding the first two equations in (2.1) and letting $w = d_1u + d_2v$, we have the following system

$$\begin{cases} \Delta w + a - u = 0, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Set $x_2 \in \Omega$ satisfying $w(x_2) = \max_{x\in\Omega} w(x)$. Then Lemma 2.1$(a)$ yields $u(x_2) \leq a$. Furthermore, by (2.3), we have

$$u(x) \leq \frac{1}{d_1}w(x) \leq \frac{1}{d_1}w(x_2) \leq a + \frac{d_2b(b+1)}{d_1a}, \text{ for all } x \in \Omega. \tag{2.5}$$

Again, by applying Lemma 2.1(b), at the minimum points of v, we have $bu - u^2v \leq 0$. This inequality together with (2.5) implies

$$v(x) \geq \frac{b}{a + \frac{d_2b(b+1)}{d_1a}}, \text{ for all } x \in \Omega. \tag{2.6}$$

By (2.2)-(2.6), the results of the lemma hold true. □

According to the above lemma, we can easily verify the following result.

Proposition 2.3. *Let positive constants a, β_1, d_1, and d_2 be fixed. Then for all*

$$0 < b \leq \beta_1,$$

there exist two positive constants B_1 and B_2 depending on a, β_1, d_1, and d_2 such that all solutions (u, v) of (2.1) satisfy

$$B_1 < u, v < B_2, \text{ in } \overline{\Omega}.$$

Furthermore, by the standard elliptic regularity, from Proposition 2.3 it follows that

Proposition 2.4. *Let positive constants a, β_1 d_1, and d_2 be fixed. Then there exists a constant $B = B(a, \beta_1, d_1, d_2) > 0$ such that any solution (u, v) of (2.1) satisfies*

$$\| u \|_{C^k(\overline{\Omega})} + \| v \|_{C^k(\overline{\Omega})} \leq B \qquad \text{for all } b \in (0, \beta_1],$$

where k is any positive integer.

The following lemma is extracted from [12][4] where they did not present a detailed proof.

Lemma 2.5. *Let the positive constants a, d_1, and d_2 be fixed. Then there exists a constant $\beta = \beta(a, d_1, d_2)$ such that system (2.1) has no nonnegative pattern solutions for all $b \in (0, \beta)$.*

Proof. On the premise that the positive constants a, d_1, and d_2 are fixed, we let $\{\alpha_n\} \subset (0, \infty)$ satisfy$\alpha_n \to 0$ as $n \to \infty$. We claim that if (u_n, v_n) is a nonnegative solution of (2.1) corresponding to $b = \alpha_n$, then it follows that

$$(u_n, v_n) \to (a, 0) \text{ in } C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \text{ as } n \to \infty. \tag{2.7}$$

Indeed, by Proposition 2.4, the sequence (u_n, v_n) is bounded in $C^3(\overline{\Omega}) \times C^3(\overline{\Omega})$. Thus, passing to a subsequence if necessary, (u_n, v_n) tends to some point $(\overline{u}, \overline{v})$ as $n \to \infty$ in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. For (2.1), letting $n \to \infty$, then $(\overline{u}, \overline{v})$ satisfies

$$\begin{cases} -d_1\Delta\overline{u} = a - \overline{u} + \overline{u}^2\overline{v}, \ x \in \Omega, \\ d_2\Delta\overline{v} = \overline{u}^2\overline{v}, \ x \in \Omega, \\ \dfrac{\partial\overline{u}}{\partial\nu} = \dfrac{\partial\overline{v}}{\partial\nu} = 0, \ x \in \partial\Omega. \end{cases} \tag{2.8}$$

Integrating the first two equations in (2.8) on Ω, we have

$$\int_\Omega \overline{u}dx = a|\Omega| + \int_\Omega \overline{u}^2\overline{v}dx \ \text{ and } \ \int_\Omega \overline{u}^2\overline{v}dx = 0, \tag{2.9}$$

respectively, where $|\Omega|$ represents the measure of the integral range Ω. Then we have $\int_\Omega \overline{u}dx = a|\Omega|$, and $\overline{v} = 0$. Thus, the first equation in (2.8) reduces to

$$-d_1\Delta\overline{u} = a - \overline{u}, \ x \in \Omega.$$

Multiplying the above equation by $a - \overline{u}$ and integrating on Ω give

$$d_1 \int_\Omega |\nabla(a-\overline{u})|^2 dx = -\int_\Omega (a-\overline{u})^2 dx.$$

It is easy to see that $\overline{u} = a$. Therefore, the claim follows.

We now define a map $P : \mathbb{R} \times H_n^2(\Omega) \times H_n^2(\Omega) \to L^2(\Omega)$ as

$$P(b,u,v) = \begin{pmatrix} \Delta u + (a-(b+1)u+u^2v)/d_1 \\ \Delta v + (bu-u^2v)/d_2 \end{pmatrix}, \tag{2.10}$$

where $H_n^2(\Omega) = \{\omega \in W^{2,2}(\Omega) : \frac{\partial\omega}{\partial\nu} = 0\}$. Obviously, $P(b,u,v) = \mathbf{0}$ just corresponds to system (2.1). Therefore, by using the same argument as in the proof of the above claim, we know $P(0,u,v) = \mathbf{0}$ has the unique solution $(a,0)$. The Fréchet derivative of $P(0,u,v) = \mathbf{0}$ at $(a,0)$ is

$$D_{(u,v)}P(0,a,0) = \begin{pmatrix} \Delta - 1/d_1 & a^2/d_1 \\ 0 & \Delta - a^2/d_2 \end{pmatrix},$$

which is invertible. Thus, according to the Implicit Function Theorem, there exist positive constants α_0, γ such that $(b, a, b/a)$ is the unique solution of $P(b,u,v) = \mathbf{0}$ in $[0,\alpha_0] \times B_\gamma(a,0)$, where $B_\gamma(a,0)$ represents the open ball in $H_n^2(\Omega) \times H_n^2(\Omega)$ with the radius γ and centered at $(a,0)$.

We now take a sequence of positive real numbers $\{\alpha_n\}$ satisfying $\alpha_n \to 0$ as $n \to \infty$. Let (u_n, v_n) be a solution of (2.1) with $b = \alpha_n$ and a, d_1, d_2 being fixed. Then, it follows that $P(\alpha_n, u_n, v_n) = \mathbf{0}$. The above claim implies that $(u_n, v_n) \to (a,0)$ in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ as $n \to \infty$. This means that the solution (α_n, u_n, v_n) will lie in $(0,\alpha_0) \times B_\gamma(a,0)$ when n is large enough. Therefore, when $b = \alpha_n$ is small enough, system (2.1) has only the constant solution $(a,0)$. The proof is complete. □

Lemma 2.5 shows the non-existence of pattern solutions for (2.1) when the parameter b is sufficiently small.

§3 Existence of pattern solutions

This section is devoted to establishing the conditions for the existence of pattern solutions of (1.1)-(1.3). To proceed, we first present some properties of negative Laplace operator $-\Delta$. Denote by

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots < \lambda_k \le \cdots, \tag{3.1}$$

the eigenvalues (counting multiplicity) of $-\Delta$ with zero Neumann boundary condition. Furthermore, let $\{\Phi_k\}$ be the corresponding $L^\infty-$ normalized eigenfunctions. Then it follows that

$$-\Delta\Phi_k = \lambda_k\Phi_k \ \text{ in } \ \Omega, \quad \frac{\partial\Phi_k}{\partial\nu} = 0 \ \text{ on } \ \partial\Omega, \quad \|\Phi_k\|_{L^\infty(\Omega)} = 1. \tag{3.2}$$

We now let the Banach space $\mathbf{X}$ be

$$\mathbf{X} = \left\{(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) : \ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 \ \text{ on } \partial\Omega\right\}$$

and

$$\mathbf{X}^+ = \left\{(u,v) \in \mathbf{X} :\ u,v > 0 \text{ in } \ C(\overline{\Omega})\right\}.$$

System (2.1) can be rewritten as

$$W = (I-\Delta)^{-1}[W + G(b,W)], \qquad W \in \mathbf{X}^+, \tag{3.3}$$

where $W = (u,v)$,

$$G(b,W) = \begin{pmatrix} \frac{1}{d_1}(a-(b+1)u+u^2v) \\ \frac{1}{d_2}(bu-u^2v) \end{pmatrix}$$

and $(I-\Delta)^{-1}$ represents the inverse of $I-\Delta$ with homogenous Neumann boundary conditions; moreover, we have

$$F(b,W) = 0, \qquad W \in \mathbf{X}^+ \tag{3.4}$$

with

$$F(b,W) = W - (I-\Delta)^{-1}[W+G(b,W)], W \in \mathbf{X}^+.$$

Thus, looking for solutions of (2.1) is equivalent to finding zero points of the operator F. We will apply the topological degree theory to prove the existence of non-constant zero points of F. It is easy to have that the linearized operator of $F(b,.)$ around the constant fixed point $W^* = (a, b/a)$ is

$$\nabla F(b,W^*) = I - (I-\Delta)^{-1}(I + \nabla G(W^*)), \tag{3.5}$$

where

$$\nabla G(W^*) = \begin{pmatrix} \frac{1}{d_1}(b-1) & \frac{1}{d_1}a^2 \\ -\frac{1}{d_2}b & -\frac{1}{d_2}a^2 \end{pmatrix}.$$

Thus, the linearized eigenvalue problem

$$\nabla F(b,W^*)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \mu \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{3.6}$$

can be expanded as

$$\begin{cases} -(1-\mu)\Delta\varphi = \left[\mu + \dfrac{1}{d_1}(b-1)\right]\varphi + \dfrac{1}{d_1}a^2\psi, \text{ in } \Omega, \\ -(1-\mu)\Delta\psi = -\dfrac{1}{d_2}b\varphi + \left(\mu - \dfrac{1}{d_2}a^2\right)\psi, \text{ in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0, \text{ on } \partial\Omega. \end{cases} \tag{3.7}$$

By (3.1) and (3.2), nontrivial solutions of (3.7) are in the form of

$$\varphi = \sum_{k=0}^{\infty} h_k\Phi_k, \qquad \psi = \sum_{k=0}^{\infty} l_k\Phi_k. \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$\begin{pmatrix} \mu + \frac{b-1}{d_1} - (1-\mu)\lambda_k & \frac{a^2}{d_1} \\ -\frac{b}{d_2} & \mu - \frac{a^2}{d_2} - (1-\mu)\lambda_k \end{pmatrix}\begin{pmatrix} h_k \\ l_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.9}$$

for each $k = 0,1,2,\cdots$. It is well known that (3.9) admits nontrivial solutions if and only if

$$\left(\mu - (1-\mu)\lambda_k + \frac{b-1}{d_1}\right)\left(\mu - (1-\mu)\lambda_k - \frac{a^2}{d_2}\right) + \frac{ba^2}{d_1d_2} = 0. \tag{3.10}$$

We give a notation Γ as

$$\Gamma = \mu - (1-\mu)\lambda_k. \tag{3.11}$$

From (3.10) it follows that

$$\Gamma^{\pm}(b) = \frac{1}{2}\left[\left(\frac{a^2}{d_2} - \frac{b-1}{d_1}\right) \pm \sqrt{\left(\frac{a^2}{d_2} - \frac{b-1}{d_1}\right)^2 - \frac{4a^2}{d_1d_2}}\,\right]. \tag{3.12}$$

Through a simple computation, we have

Proposition 3.1. *Let positive constants a, d_1, and d_2 be fixed,*

(i) if $b > \left(1+\sqrt{\frac{d_1}{d_2}}a\right)^2$, then $\Gamma^+(b) < 0$ and $\Gamma^-(b) < 0$;

(ii) if $b < \left(1-\sqrt{\frac{d_1}{d_2}}a\right)^2$, then $\Gamma^+(b) > 0$ and $\Gamma^-(b) > 0$.

By Proposition 2.3, there exist positive numbers B_1 and B_2 depending on a, b, d_1 and d_2 such that for any $\bar{b} \le b$, any solution (u, v) of (2.1) with $\bar{b}$ replacing b satisfies

$$B_1 < u, v < B_2 \quad \text{in } \overline{\Omega}.$$

Set

$$\mathcal{M} = \{(u, v) \in X : B_1 < u, v < B_2 \quad \text{in } \overline{\Omega}\} \tag{3.13}$$

and $\Psi(b, .) : \mathcal{M} \to C(\overline{\Omega} \times C(\overline{\Omega})$ by

$$\Psi(b, W) = (I - \Delta)^{-1}[W + G(b, W)], \tag{3.14}$$

where G is defined in (3.3). Obviously, $\Psi(b, .)$ is a compact operator for any fixed b.

Theorem 3.2. *Let $\bar{b} \le b$ with $b \in (0, \beta)$ as in Lemma 2.5 and assume $\beta = \left(1-\sqrt{\frac{d_1}{d_2}}a\right)^2$. Then we have*

$$\deg(I - \Psi(\bar{b}, .), \mathcal{M}, \mathbf{0}) = 1.$$

Proof. Obviously, zero points of the operator $I - \Psi(\bar{b}, .) = F(\bar{b}, .)$ are just the solutions of (2.1) by replacement of b with $\bar{b}$. According to the definition of $\mathcal{M}$ and Proposition 2.3, we know that $I - \Psi(\bar{b}, .)$ has no zero points on $\partial\mathcal{M}$. Therefore, the Leray-Schauder topological degree $\deg(I - \Psi(\bar{b}, .), \mathcal{M}, \mathbf{0})$ is well defined. By Lemma 2.5, it is obvious that $I - \Psi(\bar{b}, .)$ has the unique zero point W^* in $\mathcal{M}$. Hence,

$$\deg(I - \Psi(\bar{b}, .), \mathcal{M}, \mathbf{0}) = \text{index}(I - \Psi(\bar{b}, .), W^*). \tag{3.15}$$

By the index formula, we have

$$\text{index}(I - \Psi(\bar{b}, .), W^*) = (-1)^\sigma, \tag{3.16}$$

where σ is the number of negative eigenvalues (counting multiplicity) of $I - \nabla\Psi(\bar{b}, .) = \nabla F(\bar{b}, .)$. Noting (3.11) and Proposition 3.1 *(ii)*, we have

$$\mu_k(\bar{b}) = \frac{\lambda_k + \Gamma^\pm(\bar{b})}{\lambda_k + 1} > 0, \qquad k = 0, 1, 2, \cdots,$$

which implies that the result is true. The proof is complete. □

Based on the Proposition 3.1, the result below can be directly verified.

Proposition 3.3. *Let $b_c = \left(1+\sqrt{\frac{d_1}{d_2}}a\right)^2$. Assume $b > b_c$, then the following statements hold true.*

(i) $\Gamma^+(b)$ is a monotone increasing function in b and satisfies

$$\lim_{b \to b_c} \Gamma^+(b) = -2a\sqrt{\frac{d_1}{d_2}}, \qquad \lim_{b \to \infty} \Gamma^+(b) = 0.$$

(ii) $\Gamma^-(b)$ is a monotone decreasing function in b and satisfies

$$\lim_{b \to b_c} \Gamma^-(b) = -2a\sqrt{\frac{d_1}{d_2}}, \qquad \lim_{b \to \infty} \Gamma^-(b) = -\infty.$$

Set

$$\sigma_1(b) = \text{the number of} \{k \in \mathbb{N} \cup \{0\} : \Gamma^+(b) < -\lambda_k\} \tag{3.17}$$

and

$$\sigma_1(b) = \text{the number of} \{k \in \mathbb{N} \cup \{0\} : \Gamma^-(b) < -\lambda_k\}. \tag{3.18}$$

Thus, the number of negative eigenvalues (counting multiplicity) of (3.6) is

$$\sigma(b) = \sigma_1(b) + \sigma_2(b);$$

moreover, it is obvious that $\Gamma^+(b) < 0 = -\lambda_0$ and $\Gamma^-(b) < 0 = -\lambda_0$, thus $\sigma(b) \geq 2$ for any $b > b_c$. By Proposition 3.3, if we let

$$b^k = \sup\{b > b_c : \Gamma^+(b) < -\lambda_k\} \quad \text{and} \quad b_j = \inf\{b > b_c : \Gamma^-(b) < -\lambda_j\},$$

then the two positive sequences $\{b^k\}_{k=1}^{j_0}$ and $\{b_j\}_{j=j_0}^{\infty}$ satisfy, respectively,

$$b^1 \geq b^2 \geq \cdots b^k \geq \cdots \to b_c \quad (k \to j_0)$$

and

$$b_c \leq b_{j_0+1} \leq \cdots b_j \leq \cdots \to \infty \quad (j \to \infty),$$

where $j_0 = \max\{j : \Gamma^+(b_c) = \Gamma^-(b_c) = -2a\sqrt{\frac{d_1}{d_2}} < -\lambda_j\}$. The above analysis yields

$$\sigma_1(b) = \begin{cases} j_0+1, & b \in (b_c, b^{j_0}), \\ k+1, & b \in (b^{k+1}, b^k), \\ 1, & b \in (b^1, \infty). \end{cases} \quad \text{and} \quad \sigma_2(b) = \begin{cases} j+1, & b \in (b_j, b_{j+1}), \\ j_0+1, & b \in (b_c, b_{j_0+1}), \end{cases} \tag{3.19}$$

where $b^k \neq b^{k+1}$ and $b_j \neq b_{j+1}$. Now we state the main results.

Theorem 3.4. *Let the positive constants a, d_1, and d_2 be fixed. System (2.1) has at least one pattern solution provided that*

$$b \in (b^{k+1}, b^k) \cap (b_j, b_{j+1}) \quad \textit{with} \quad b^k \neq b^{k+1}, \quad b_j \neq b_{j+1} \textit{ and } \quad k+j \quad \textit{is odd.} \tag{3.20}$$

Proof. Define $\overline{\Psi}(t,.) : [0,1] \times \mathcal{M} \to C(\overline{\Omega}) \times C(\overline{\Omega})$ by

$$\overline{\Psi}(t, W) = (I - \Delta)^{-1} \begin{pmatrix} u + \frac{1}{d_1}(a - [(1-t)\overline{b} + tb]u + u^2 v) \\ v + \frac{1}{d_2}[[(1-t)\overline{b} + tb]u - u^2 v] \end{pmatrix},$$

where $\overline{b}$ is taken as in Theorem 3.2. Then $\overline{\Psi}(t,.)$ is a compact operator, and by Proposition 2.3, any fixed point W of $\overline{\Psi}(t, W)$ satisfies $W \notin \partial\mathcal{M}$ for all $t \in [0,1]$. Thus, the homotopy invariance of degree index yields that

$$\deg(I - \overline{\Psi}(1,.), \mathcal{M}, \mathbf{0}) = \deg(I - \overline{\Psi}(0,.), \mathcal{M}, \mathbf{0}). \tag{3.21}$$

Obviously, $I - \overline{\Psi}(0,\cdot) = I - \Psi(\overline{b},\cdot) = F(\overline{b},\cdot)$. Therefore, by Theorem 3.2, we have

$$\deg(I - \overline{\Psi}(0,.), \mathcal{M}, \mathbf{0}) = \operatorname{index}(I - \Psi(\overline{b},.), W^*) = 1. \tag{3.22}$$

On the other hand, if system (2.1) has no other solutions except the constant one W^*, then under the given conditions, from (3.19) it follows that

$$\begin{aligned} &\deg(I - \overline{\Psi}(1,.), \mathcal{M}, \mathbf{0}) = \operatorname{index}(I - \overline{\Psi}(1,.), W^*) \\ = \;&\operatorname{index}(F(b,.), W^*) = (-1)^{\sigma_1(b)+\sigma_2(b)} = (-1)^{k+j} = -1, \end{aligned}$$

which contradicts with (3.21) and (3.22), and so our assumption is incorrect. Hence there exists at least one non-constant solution in system (2.1), that is, pattern formation occurs in system (2.1). The proof is complete. □

By Theorems 3.2 and 3.4, the following corollary is quite straightforward.

Corollary 3.5. *Let the positive constants a, d_1 and d_2 be fixed. Then we have*

(i) If $b \in (b^1, \infty) \cap (b_j, b_{j+1})$ with $j \geq j_0$ and j is odd, then there exists at least one pattern solution in (2.1).

(ii) If $b \in (b_c, b^{j_0}) \cap (b_c, b_{j_0+1})$, then further studies are needed to determine whether (2.1) has a pattern solution.

(iii) If $b < \left(1 - \sqrt{\frac{d_1}{d_2}}a\right)^2$*, then further studies are needed to determine whether (2.1) has a pattern solution.*

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