Appl. Math. J. Chinese Univ.
2019, 34(4): 436-445

On growth of meromorphic solutions of some kind of non-homogeneous linear difference equations

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Abstract. In this paper, we investigate the growth of meromorphic solutions of some kind of non-homogeneous linear difference equations with special meromorphic coefficients. When there are more than one coefficient having the same maximal order and the same maximal type, the estimates on the lower bound of the order of meromorphic solutions of the involved equations are obtained. Meanwhile, the above estimates are sharpened by combining the relative results of the corresponding homogeneous linear difference equations.

§1 Introduction and main results

In this paper, we use the basic notations of Nevanlinna's value distribution theory (see e.g. [8, 10, 14]). In addition, we use the notation $\sigma(f)$ to denote the order of a meromorphic function $f(z)$ in the whole complex plane.

Recently, the research on the properties of meromorphic solutions of complex difference equations has become a subject of great interest from the viewpoint of Nevanlinna theory and its difference analogues. In particular, many scholars investigated the properties of meromorphic solutions of the homogeneous linear difference equation

$$A_k(z)f(z+\eta_k)+\cdots+A_1(z)f(z+\eta_1)+A_0(z)f(z)=0 \tag{1.1}$$

and its special case

$$A_k(z)f(z+k)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0, \tag{1.2}$$

where $k\in N_+$, $\eta_j(j=1,2,\cdots,k)$ are distinct non-zero complex constants, and obtained some results on the growth and value distribution of meromorphic solutions of (1.1) or (1.2).

Firstly, when the coefficients of (1.2) are polynomials or transcendental entire functions respectively, Chiang and Feng [6] obtained the following Theorems 1.A and 1.B.

Received: 2018-04-27. Revised: 2019-05-27.
MR Subject Classification: 30D35, 39B32.
Keywords: Complex linear difference equation, meromorphic solution, order.
Digital Object Identifier(DOI):https://doi.org/10.1007/s11766-019-3616-2.
Supported by the National Natural Science Foundation of China (No. 11761035) and the Natural Science Foundation of Jiangxi Province in China (No. 20171BAB201002).
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Theorem 1.A **([6])** Let $A_j(z)(j=0,1,\cdots,k)$ be polynomials such that there exists an integer $l(0\le l\le k)$ so that

$$\deg(A_l)>\max_{\substack{0\le j\le k\\ j\ne l}}\{\deg(A_j)\}$$

holds. Suppose $f(z)$ is a meromorphic solution to (1.2), then we have $\sigma(f)\ge 1$.

Theorem 1.B **([6])** Let $A_j(z)(j=0,1,\cdots,k)$ be entire functions such that there exists an integer $l(0\le l\le k)$ such that

$$\sigma(A_l)>\max_{\substack{0\le j\le k\\ j\ne l}}\{\sigma(A_j)\}.$$

If $f(z)$ is a meromorphic solution to (1.2), then we have $\sigma(f)\ge\sigma(A_l)+1$.

Obviously, the conditions in Theorems 1.A and 1.B show that there exists only one coefficient of (1.2) having the highest degree or the maximal order. Further, when there are more than one coefficient of (1.1) or (1.2) having the highest degree or the maximal order, Chen [5] and Laine-Yang [11] obtained the following Theorems 1.C and 1.D respectively.

Theorem 1.C **([5])** Let $A_j(z)(j=0,1,\cdots,k)$ be polynomials such that $A_0(z)A_k(z)\not\equiv 0$ and satisfy

$$\deg(A_0+A_1+\cdots+A_k)=\max_{0\le j\le k}\{\deg(A_j)\}\ge 1,$$

then every finite order meromorphic solution $f(z)(\not\equiv 0)$ of (1.2) satisfies $\sigma(f)\ge 1$.

Theorem 1.D **([11])** Let $A_j(z)(j=0,1,\cdots,k)$ be entire functions of finite order such that among those having the maximal order $\sigma=\max\limits_{0\le j\le k}\{\sigma(A_j)\}$, exactly one has its type strictly greater than the others. Then for any meromorphic solution of (1.1), we have $\sigma(f)\ge\sigma+1$.

Later, Liu-Mao [12] considered the case where there is more than one coefficient of (1.2) having the maximal order and the maximal type, and obtained the following Theorem 1.E.

Theorem 1.E **([12])** Let $A_j(z)=B_j(z)e^{P_j(z)}(j=0,1,\cdots,k)$, where $P_j(z)=\alpha_{jn}z^n+\cdots+\alpha_{j0}$ are polynomials with degree $n(\ge 1)$, $B_j(z)(\not\equiv 0)$ are entire functions of $\sigma(B_j)<n$. If $\alpha_{jn}(j=0,1,\cdots,k)$ are distinct complex numbers, then every meromorphic solution $f(z)(\not\equiv 0)$ of (1.2) satisfies $\sigma(f)\ge\max\limits_{0\le j\le k}\{\sigma(A_j)\}+1$.

In addition, Liu-Mao [12] also discussed the growth of meromorphic solutions of the special case of the non-homogeneous linear difference equation

$$A_k(z)f(z+\eta_k)+\cdots+A_1(z)f(z+\eta_1)+A_0(z)f(z)=A_{k+1}(z), \tag{1.3}$$

where $k\in N_+$, $\eta_j(j=1,2,\cdots,k)$ are distinct non-zero complex constants.

Similar to Theorem 1.E, Huang-Chen-Li [9] considered (1.3), and obtained the following Theorem 1.F.

Theorem 1.F **([9])** Suppose that $A_j(z)(j=0,1,\cdots,k+1)$ are meromorphic functions satisfying $\sigma=\max\limits_{0\le j\le k+1}\{\sigma(A_j)\}>0$. Denote $I^*=\{j\in\{0,1,\cdots,k+1\}:\sigma(A_j)=\sigma\}$ and suppose that $A_j(z)=B_j(z)e^{a_jz^\sigma}(j\in I^*)$, where $a_j\in\mathbb{C}\setminus\{0\}(j\in I^*)$ and $B_j(z)(j\in I^*)$ are meromorphic functions with finite order $\sigma(B_j) < \sigma(j \in I^*)$. If the constants $a_j(j \in I^*)$ are distinct, then each non-trivial meromorphic solution $f(z)$ of (1.3) satisfies $\sigma(f) \geq \sigma$.

Noting the non-zero complex constants $a_j(j \in I^*)$ are required to be distinct in Theorem 1.F,(which is similar in [12]) we consider to improve the conditions, that is, admit some of $a_j(j \in I^*)$ are the same. Under this condition, Belaïdi-Habib [1] considered the case of complex linear differential equations. Inspired by the above results, we proceed to consider (1.3), and obtain the following Theorem 1.1 and Corollaries 1.1, 1.2 under more general conditions.

For your convenience, we denote

$$I = \{0, 1, \cdots, k\} = I_1 \cup I_2 \cup I_3 \cup I_4,$$

where $c_j(j \in I)$ are real constants, and

$$I_1 = \{j \in I : c_j > 1\}, \quad I_2 = \{j \in I : 0 < c_j < 1\},$$
$$I_3 = \{j \in I : c_j < 0\}, \quad I_4 = \{j \in I : c_j = 1\}.$$

Theorem 1.1 Suppose that $A_j(z) = B_j(z)e^{b_j z}(j \in I), A_{k+1}(z) = B_{k+1}(z)e^{az}$, where $B_j(z)(j \in I \cup \{k+1\})$ are meromorphic functions satisfying $\max\limits_{j \in I \cup \{k+1\}}\{\sigma(B_j)\} < 1, a$ and $b_j(j \in I)$ are non-zero complex constants satisfying $b_j = c_j a(j \in I)$. If there exists an integer $s(\in I_1 \neq \emptyset)$ such that $c_s > c_j(j \in I_1 \setminus \{s\})$, or exists an integer $l(\in I_3 \neq \emptyset)$ such that $c_l < c_j(j \in I_3 \setminus \{l\})$, and $B_j(z) \not\equiv 0(j = s, l, k+1)$, then every meromorphic solution $f(z)$ of (1.3) satisfies $\sigma(f) \geq 1$.

Corollary 1.1 Suppose that the conditions in Theorem 1.1 hold, and $B_j(z)(j \in I \cup \{k+1\})$ are entire functions. Then every meromorphic solution $f(z)$ of (1.3) satisfies $\sigma(f) \geq 2$, except at most one meromorphic solution $f_0(z)$ satisfying $1 \leq \sigma(f_0) < 2$.

Corollary 1.2 Suppose that $A_j(z) = B_j(z)e^{P_j(z)}(j \in I), A_{k+1}(z) = B_{k+1}(z)e^{P(z)}$, where $P_j(z) = b_{jn}z^n + \cdots + b_{j1}z + b_{j0}(j \in I), P(z) = a_n z^n + \cdots + a_1 z + a_0$ are polynomials with degree $n(\geq 1)$ and $b_{jn} = c_j a_n(j \in I)$, $B_j(z)(j \in I \cup \{k+1\})$ are meromorphic functions satisfying $\max\limits_{j \in I \cup \{k+1\}}\{\sigma(B_j)\} < n$. If there exists an integer $p(\in I_1 \neq \emptyset)$ such that $c_p > c_j(j \in I_1 \setminus \{p\})$, or exists an integer $q(\in I_3 \neq \emptyset)$ such that $c_q < c_j(j \in I_3 \setminus \{q\})$, and $B_j(z) \not\equiv 0(j = p, q, k+1)$, then every meromorphic solution $f(z)$ of (1.3) satisfies $\sigma(f) \geq n$.

On the other hand, we consider the difference operators $\Delta^j f(j \in N_+)$ instead of the shift operators $f(z + \eta_j)(j \in N_+)$ in (1.3), and consider the growth of meromorphic solutions of the corresponding non-homogeneous linear difference equation. Here, for a non-zero complex constant c, the difference operators $\Delta^j f(j \in N_+)$ are defined as follows (see e.g. [2]),

$$\Delta f(z) = \Delta^1 f(z) = f(z+c) - f(z),$$
$$\Delta^{j+1} f(z) = \Delta(\Delta^j f(z)) = \Delta^j f(z+c) - \Delta^j f(z), j \in N_+.$$

By combining the reasoning method in Theorem 1.1 and the application of Lemma 2.2 and Remark 2.1, we obtain the following Theorem 1.2 and Corollary 1.3.

Theorem 1.2 Suppose that $A_j(z)(j \in I \cup \{k+1\})$ are defined as in Theorem 1.1. If $0 \in I_1$

and $c_0 > c_j (j \in I_1 \setminus \{0\})$, or $0 \in I_3$ and $c_0 < c_j (j \in I_3 \setminus \{0\})$, and $B_0(z)B_{k+1}(z) \not\equiv 0$, then every meromorphic solution $f(z)$ of the difference equation

$$A_k(z)\Delta^k f(z) + \cdots + A_1(z)\Delta f(z) + A_0(z)f(z) = A_{k+1}(z) \tag{1.4}$$

satisfies $\sigma(f) \geq 1$.

Corollary 1.3 Suppose that $A_j(z)(j \in I \cup \{k+1\})$ are defined as in Corollary 1.2. If $0 \in I_1$ and $c_0 > c_j (j \in I_1 \setminus \{0\})$, or $0 \in I_3$ and $c_0 < c_j (j \in I_3 \setminus \{0\})$, and $B_0(z)B_{k+1}(z) \not\equiv 0$, then every non-trivial meromorphic solution $f(z)$ of (1.4) satisfies $\sigma(f) \geq n$.

Wu-Zheng [13] also discussed the corresponding homogeneous linear difference equation to (1.4), and obtained the following Theorem 1.G.

***Theorem 1.G* ([13])** Let $d_j \in \mathbb{C}(j = 0, 1, \cdots, k)$ such that $d_0 \neq d_j$ and $|d_0| \geq |d_j| (j = 1, 2, \cdots, k)$, $h_j(z)(\not\equiv 0)(j = 0, 1, \cdots, k)$ be meromorphic functions with order less than n, then every meromorphic solution $f(z)(\not\equiv 0)$ of the difference equation

$$h_k(z)e^{d_k z^n}\Delta^k f(z) + \cdots + h_1(z)e^{d_1 z^n}\Delta f(z) + h_0(z)e^{d_0 z^n} f(z) = 0$$

satisfies $\sigma(f) \geq n+1$.

By combining Theorem 1.G, we consider the growth of meromorphic solutions of (1.4) further, and obtain the following Corollary 1.4.

Corollary 1.4 Suppose that the conditions in Theorem 1.2 hold, and $c_i \neq c_j (i \neq j)$, then every meromorphic solution $f(z)$ of (1.4) satisfies $\sigma(f) \geq 2$, except at most one meromorphic solution $f_0(z)$ satisfying $1 \leq \sigma(f_0) < 2$.

§2 Lemmas for proofs of main results

***Lemma 2.1* ([3])** Suppose that $P(z) = (\alpha+\beta i)z^n + \cdots$ is a polynomial with degree $n(\geq 1)$, $\omega(z)(\not\equiv 0)$ is a meromorphic function with $\sigma(\omega) < n$. Let $g(z) = \omega(z)e^{P(z)}, z = re^{i\theta}, \delta(P,\theta) = \alpha\cos n\theta - \beta \sin n\theta$, then for any given $\varepsilon > 0$, there exists a set $H \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi)\setminus(H_0 \cup H)$, there is $r_0 = r_0(\theta, \varepsilon)(> 0)$, such that for $r > r_0$, we have

(i) if $\delta(P,\theta) > 0$, then $\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1+\varepsilon)\delta(P,\theta)r^n\}$;

(ii) if $\delta(P,\theta) < 0$, then $\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}$;

where $H_0 = \{\theta \in [0, 2\pi) : \delta(P,\theta) = 0\}$ is a finite set.

***Lemma 2.2* ([6])** Let $f(z)$ be a meromorphic function of finite order σ, and let $\eta_1, \eta_2 (\eta_1 \neq \eta_2)$ be two arbitrary complex numbers. Let $\varepsilon(> 0)$ be given, then there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E$, we have

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Remark 2.1 By Lemma 2.2, we have that

$$\begin{aligned}\left|\frac{\Delta^j f(z)}{f(z)}\right| &= \left|\frac{\sum\limits_{i=0}^{j} C_j^i(-1)^{j-i}f(z+ic)}{f(z)}\right| \leq \sum_{i=0}^{j} C_j^i \left|\frac{f(z+ic)}{f(z)}\right| \\ &\leq \sum_{i=0}^{j} C_j^i \exp\{r^{\sigma-1+\varepsilon}\} = 2^j \exp\{r^{\sigma-1+\varepsilon}\}, \quad j \in N_+.\end{aligned}$$

Lemma 2.3 **([7])** Let $f(z)$ be a meromorphic function, c be a non-zero complex constant, then we have that for $r \to \infty$,

$$(1+o(1))T(r-|c|,f) \leq T(r,f(z+c)) \leq (1+o(1))T(r+|c|,f).$$

Therefore, it follows that $\sigma(f(z+c)) = \sigma(f), \quad \mu(f(z+c)) = \mu(f)$.

Lemma 2.4 **([4])** Let $f(z)$ be a meromorphic function with $\sigma(f) = \sigma < \infty$, then for any given $\varepsilon(>0)$, there exists a subset $E \subset [0,+\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin [0,1] \cup E$ and sufficiently large r , we have

$$\exp\{-r^{\sigma+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\sigma+\varepsilon}\}.$$

§3 Proofs of main results

Proof of Theorem 1.1 On the contrary, we suppose that $\sigma(f) < 1$.

Denote $\alpha = \max\limits_{j\in I\cup\{k+1\}}\{\sigma(B_j)\}$, then $\alpha < 1$. Denote $H_0 = \{\theta \in [0,2\pi) : \delta(az,\theta) = 0\}$, then H_0 is a finite set. For any $\theta \in [0,2\pi)\backslash H_0$, we denote

$$\delta_s = \delta((c_s-1)az,\theta), \quad \delta_l = \delta((c_l-1)az,\theta),$$

$$\delta_1 = \max_{j\in I_1\backslash\{s\}}\{\delta((c_j-1)az,\theta)\}, \quad \delta_3 = \max_{j\in I_3\backslash\{l\}}\{\delta((c_j-1)az,\theta)\},$$

then $\delta_s \neq 0, \delta_l \neq 0, \delta_1 \neq 0, \delta_3 \neq 0$. In the following, we divide the proof into two cases.

Case (1) If there exists an integer $l(\in I_3 \neq \emptyset)$ such that $c_l < c_j (j \in I_3 \setminus \{l\})$.

It follows by Lemma 2.1 that for any given $\varepsilon(0 < \varepsilon < \min\{\frac{\delta_l-\delta_3}{\delta_l+2\delta_3}, 1-\alpha, 1-\sigma(f)\})$, there exists a set $H_1 \subset [0,2\pi)$ with linear measure zero such that for any $\theta \in [0,2\pi)\backslash(H_0 \cup H_1)$, there is $r_0 = r_0(\theta,\varepsilon)(>0)$ such that for $|z| = r > r_0$, the conclusions in Lemma 2.1 hold for $B_j(z)e^{(c_j-1)az}, j \in I \setminus I_4$.

Now, we can choose a ray $\arg z = \theta_1 \in [0,2\pi) \setminus (H_0 \cup H_1)$ such that $\delta(-az,\theta_1) > 0$. Clearly,

$$\delta((c_j-1)az,\theta_1) = (1-c_j)\delta(-az,\theta_1), \quad j \in I.$$

If $j \in I_1$, then $\delta((c_j-1)az,\theta_1) < 0$; if $j \in I_2$, then $0 < \delta((c_j-1)az,\theta_1) < \delta(-az,\theta_1)$; if $j \in I_3 \setminus \{l\}$, then $0 < \delta(-az,\theta_1) < \delta((c_j-1)az,\theta_1) \leq \delta_3 < \delta_l$. Therefore, it follows by Lemma 2.1 that for the above ε, there is $r_1 = r_1(\theta_1,\varepsilon)(>1)$ such that for all z satisfying $|z| = r > r_1$ and $\arg z = \theta_1$, we have

$$|B_l(z)e^{(c_l-1)az}| \geq \exp\{(1-\varepsilon)\delta_l r\}; \tag{3.1}$$

$$|B_j(z)e^{(c_j-1)az}| \le \exp\{(1-\varepsilon)\delta((c_j-1)az,\theta_1)r\} < 1, \quad j \in I_1; \tag{3.2}$$

$$\begin{aligned}|B_j(z)e^{(c_j-1)az}| &\le \exp\{(1+\varepsilon)\delta((c_j-1)az,\theta_1)r\} \\ &\le \exp\{(1+\varepsilon)\delta(-az,\theta_1)r\}, \quad j \in I_2;\end{aligned} \tag{3.3}$$

$$|B_j(z)e^{(c_j-1)az}| \le \exp\{(1+\varepsilon)\delta((c_j-1)az,\theta_1)r\} \le \exp\{(1+\varepsilon)\delta_3 r\}, \quad j \in I_3 \setminus \{l\}. \tag{3.4}$$

It follows by Lemma 2.2 that for the above ε, there exists a subset $E_1 \subset (1,+\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\exp\{-r^{\sigma(f)-1+\varepsilon}\} \le \Big|\frac{f(z+\eta_i)}{f(z+\eta_j)}\Big| \le \exp\{r^{\sigma(f)-1+\varepsilon}\}, \tag{3.5}$$

where $i, j \in I$, $i \ne j$, $\eta_i = 0$ when $i = 0$, and $\eta_j = 0$ when $j = 0$.

Since $\sigma(f) < 1$, then by Lemma 2.3, we have

$$\sigma(f(z+\eta_j)) = \sigma(\frac{1}{f(z+\eta_j)}) = \sigma(f) < 1, \quad j \in I \setminus \{0\}.$$

It also follows by Lemma 2.4 that for the above ε, there exists a subset $E_2 \subset [0,+\infty)$ with finite linear measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$ and $r \to \infty$, we have

$$|B_j(z)| \le \exp\{r^{\alpha+\varepsilon}\}, \quad j \in I \cup \{k+1\}, \tag{3.6}$$

and

$$\exp\{-r^{\sigma(f)+\varepsilon}\} \le \Big|\frac{1}{f(z+\eta_j)}\Big| \le \exp\{r^{\sigma(f)+\varepsilon}\}, \tag{3.7}$$

where $j \in I$, and $\eta_j = 0$ when $j = 0$.

We divide (1.3) by $f(z+\eta_l)$ to get

$$\begin{aligned}|B_l(z)e^{(c_l-1)az}| \le\; & \sum_{j\in I_1}|B_j(z)e^{(c_j-1)az}|\Big|\frac{f(z+\eta_j)}{f(z+\eta_l)}\Big| + \sum_{j\in I_2}|B_j(z)e^{(c_j-1)az}|\Big|\frac{f(z+\eta_j)}{f(z+\eta_l)}\Big| \\ & + \sum_{j\in I_3\setminus\{l\}}|B_j(z)e^{(c_j-1)az}|\Big|\frac{f(z+\eta_j)}{f(z+\eta_l)}\Big| + \sum_{j\in I_4}|B_j(z)|\Big|\frac{f(z+\eta_j)}{f(z+\eta_l)}\Big| \\ & + \Big|\frac{B_{k+1}(z)}{f(z+\eta_l)}\Big|.\end{aligned} \tag{3.8}$$

Then by substituting (3.1)-(3.7) into (3.8), we deduce that for all z satisfying $\arg z = \theta_1$ and $|z| = r \notin [0, r_1] \cup (E_1 \cup E_2)$, we get

$$\begin{aligned}\exp\{(1-\varepsilon)\delta_l r\} &\le |B_l(z)e^{(c_l-1)az}| \\ &\le O(1)\exp\{r^{\sigma(f)-1+\varepsilon}\} + O(1)\exp\{(1+\varepsilon)\delta(-az,\theta_1)r\}\exp\{r^{\sigma(f)-1+\varepsilon}\} \\ &\quad + O(1)\exp\{(1+\varepsilon)\delta_3 r\}\exp\{r^{\sigma(f)-1+\varepsilon}\} + O(1)\exp\{r^{\alpha+\varepsilon}\}\exp\{r^{\sigma(f)-1+\varepsilon}\} \\ &\quad + \exp\{r^{\alpha+\varepsilon}\}\exp\{r^{\sigma(f)+\varepsilon}\} \\ &\le \exp\{(1+2\varepsilon)\delta_3 r\},\end{aligned} \tag{3.9}$$

a contradiction.

Case (2) If there exists an integer $s(\in I_1 \ne \emptyset)$ such that $c_s > c_j (j \in I_1 \setminus \{s\})$.

It follows by Lemma 2.1 that for any given $\varepsilon(0 < \varepsilon < \min\{\frac{\delta_s-\delta_1}{\delta_s+2\delta_1}, 1-\alpha, 1-\sigma(f)\})$, there exists a set $H_2 \subset [0, 2\pi)$ with linear measure zero such that for any $\theta \in [0, 2\pi)\backslash(H_0 \cup H_2)$, there is $r_0 = r_0(\theta,\varepsilon)(> 0)$ such that for $|z| = r > r_0$, the conclusions in Lemma 2.1 hold for

$B_j(z)e^{(c_j-1)az}, j \in I \setminus I_4$.

Now, we can choose a ray $\arg z = \theta_2 \in [0, 2\pi) \setminus (H_0 \cup H_2)$ such that $\delta(-az, \theta_2) < 0$. Clearly,

$$\delta((c_j - 1)az, \theta_2) = (1 - c_j)\delta(-az, \theta_2), j \in I.$$

If $j \in I_1 \setminus \{s\}$, then $0 < \delta((c_j - 1)az, \theta_2) \le \delta_1 < \delta_s$; if $j \in I_2 \cup I_3$, $\delta((c_j - 1)az, \theta_2) < 0$. Therefore, it follows by Lemma 2.1 that for the above ε, there is $r_2 = r_2(\theta_2, \varepsilon)(> 1)$ such that for all z satisfying $|z| = r > r_2$ and $\arg z = \theta_2$, we have

$$|B_s(z)e^{(c_s-1)az}| \ge \exp\{(1-\varepsilon)\delta_s r\}; \tag{3.10}$$

$$\begin{aligned} |B_j(z)e^{(c_j-1)az}| &\le \exp\{(1+\varepsilon)\delta((c_j-1)az, \theta_2)r\} \\ &\le \exp\{(1+\varepsilon)\delta_1 r\}, \quad j \in I_1 \setminus \{s\}; \end{aligned} \tag{3.11}$$

$$|B_j(z)e^{(c_j-1)az}| \le \exp\{(1-\varepsilon)\delta((c_j-1)az, \theta)r\} < 1, \quad j \in I_2 \cup I_3. \tag{3.12}$$

We divide (1.3) by $f(z+\eta_s)$ to get

$$\begin{aligned} &|B_s(z)e^{(c_s-1)az}| \\ \le &\sum_{j\in I_1\setminus\{s\}} |B_j(z)e^{(c_j-1)az}|\Big|\frac{f(z+\eta_j)}{f(z+\eta_s)}\Big| + \sum_{j\in I_2\cup I_3} |B_j(z)e^{(c_j-1)az}|\Big|\frac{f(z+\eta_j)}{f(z+\eta_s)}\Big| \\ &+ \sum_{j\in I_4} |B_j(z)|\Big|\frac{f(z+\eta_j)}{f(z+\eta_s)}\Big| + \Big|\frac{B_{k+1}(z)}{f(z+\eta_s)}\Big|. \end{aligned} \tag{3.13}$$

Then by substituting (3.5)-(3.7) and (3.10)-(3.12) into (3.13), we deduce that for all z satisfying $\arg z = \theta_2$ and $|z| = r \notin [0, r_2] \cup (E_1 \cup E_2)$, we have

$$\begin{aligned} \exp\{(1-\varepsilon)\delta_s r\} &\le |B_l(z)e^{(c_s-1)az}| \\ &\le O(1)\exp\{(1+\varepsilon)\delta_1 r\}\exp\{r^{\sigma(f)-1+\varepsilon}\} + O(1)\exp\{r^{\sigma(f)-1+\varepsilon}\} \\ &\quad + O(1)\exp\{r^{\alpha+\varepsilon}\}\exp\{r^{\sigma(f)-1+\varepsilon}\} + \exp\{r^{\alpha+\varepsilon}\}\exp\{r^{\sigma(f)+\varepsilon}\} \\ &\le \exp\{(1+2\varepsilon)\delta_1 r\}, \end{aligned} \tag{3.14}$$

a contradiction.

Therefore, we have $\sigma(f) \ge 1$.

The proof of Theorem 1.1 is complete.

Proof of Corollary 1.1 We consider the homogeneous linear difference equation

$$B_k(z)e^{c_k az}f(z+\eta_k) + \cdots + B_1(z)e^{c_1 az}f(z+\eta_1) + B_0(z)e^{c_0 az}f(z) = 0, \tag{3.15}$$

where k, $\eta_j(j = 1, 2, \cdots, k)$, $B_j(z)(j = 0, 1, \cdots, k)$, $c_j(j = 0, 1, \cdots, k)$ and a are defined as in Theorem 1.1.

Case (1) If there exists an integer $l(\in I_3 \ne \emptyset)$ such that $c_l < c_j(j \in I_3 \setminus \{l\})$, then we rewrite (3.15) as follows:

$$\begin{aligned} &B_l(z)e^{(c_l-c_m)az}f(z+\eta_l) + \sum_{j\in I_1\cup I_2} B_j(z)e^{(c_j-c_m)az}f(z+\eta_j) \\ &+ \sum_{j\in I_3\setminus\{l\}} B_j(z)e^{(c_j-c_m)az}f(z+\eta_j) + \sum_{j\in I_4} B_j(z)e^{(1-c_m)az}f(z+\eta_j) = 0, \end{aligned} \tag{3.16}$$

where $m \in I_1$ and $c_m \ge c_j(j \in I_1 \setminus \{m\})$. (If the above m are more than one, we can choose any one of them arbitrarily.)

Case (2) If there exists an integer $s(\in I_1 \neq \emptyset)$ such that $c_s > c_j(j \in I_1 \setminus \{s\})$, then we rewrite (3.15) as follows:

$$\begin{aligned} & B_s(z)e^{(c_s-c_n)az}f(z+\eta_s) + \sum_{j\in I_1\setminus\{s\}} B_j(z)e^{(c_j-c_n)az}f(z+\eta_j) \\ & + \sum_{j\in I_2\cup I_3} B_j(z)e^{(c_j-c_n)az}f(z+\eta_j) + \sum_{j\in I_4} B_j(z)e^{(1-c_n)az}f(z+\eta_j) = 0, \end{aligned} \tag{3.17}$$

where $n \in I_3$ and $c_n \leq c_j(j \in I_1 \setminus \{n\})$. (If the above n are more than one, we can choose any one of them arbitrarily.)

It follows by the conditions in Theorem 1.1 that for both Case (1) and Case (2), there is only one coefficient, $B_l(z)e^{(c_l-c_m)az}$ in (3.16) or $B_s(z)e^{(c_s-c_n)az}$ in (3.17), having the maximal type, that is, the conditions in Theorem 1.D hold. Therefore, it follows by Theorem 1.D that every meromorphic solution $f(z)(\not\equiv 0)$ of (3.15) satisfies $\sigma(f) \geq 2$.

Let $f_1(z)$ and $f_2(z)$ be two distinct meromorphic solutions of (1.3), and satisfy $\sigma(f_i) < 2, i = 1, 2$, then $f_1(z)-f_2(z)(\not\equiv 0)$ is a meromorphic solution of the corresponding homogeneous equation (3.15) and satisfies $\sigma(f_1-f_2) < 2$, which is a contradiction with the fact $\sigma(f_1-f_2) \geq 2$. Thus, every meromorphic solution $f(z)$ satisfies $\sigma(f) \geq 2$, except at most one meromorphic solution $f_0(z)$ satisfying $\sigma(f_0) < 2$. It follows by Theorem 1.1 that $\sigma(f_0) \geq 1$. Therefore, every meromorphic solution $f(z)$ satisfies $\sigma(f) \geq 2$, except at most one meromorphic solution $f_0(z)$ satisfying $1 \leq \sigma(f_0) < 2$.

The proof of Corollary 1.1 is complete.

Proof of Corollary 1.2 We use the similar reasoning method as the one in Theorem 1.1 to prove as follows.

On the contrary, we suppose that $\sigma(f) < n$.

Denote $\beta = \max_{j\in I\cup\{k+1\}}\{\sigma(B_j)\}$, then $\beta < n$. For any $\theta \in [0, 2\pi)\setminus H_0^*$, we denote

$$\delta_p = \delta((c_p-1)P(z), \theta), \quad \delta_q = \delta((c_q-1)P(z), \theta),$$

$$\delta_1^* = \max_{j\in I_1\setminus\{p\}}\{\delta((c_j-1)P(z), \theta)\}, \quad \delta_3^* = \max_{j\in I_3\setminus\{q\}}\{\delta((c_j-1)P(z), \theta)\},$$

where $H_0^* = \{\theta \in [0, 2\pi) : \delta(P(z), \theta) = 0\}$ is a finite set.

Case (1) If there exists an integer $q(\in I_3 \neq \emptyset)$ such that $c_q < c_j(j \in I_3 \setminus \{q\})$, then we can choose a ray $\arg z = \theta_3 \in [0, 2\pi) \setminus (H_0^* \cup H_3)$ such that $\delta(-P(z), \theta_3) > 0$, and Lemma 2.1 holds, where $H_3 \subset [0, 2\pi)$ has linear measure zero. For all z satisfying $|z| = r$ and $\arg z = \theta_3$, we rewrite (3.1)-(3.4) as follows:

$$|B_q(z)e^{(c_q-1)b_nz^n(1+o(1))}| \geq \exp\{(1-\varepsilon)\delta_q r^n\}; \tag{3.18}$$

$$|B_j(z)e^{(c_j-1)b_nz^n(1+o(1))}| \leq \exp\{(1-\varepsilon)\delta((c_j-1)P(z), \theta_3)r^n\} < 1, \quad j \in I_1; \tag{3.19}$$

$$\begin{aligned} |B_j(z)e^{(c_j-1)b_nz^n(1+o(1))}| &\leq \exp\{(1+\varepsilon)\delta((c_j-1)P(z), \theta_3)r^n\} \\ &\leq \exp\{(1+\varepsilon)\delta(-P(z), \theta_3)r^n\}, \quad j \in I_2; \end{aligned} \tag{3.20}$$

$$\begin{aligned} |B_j(z)e^{(c_j-1)b_nz^n(1+o(1))}| &\leq \exp\{(1+\varepsilon)\delta((c_j-1)P(z), \theta_3)r^n\} \\ &\leq \exp\{(1+\varepsilon)\delta_3^* r^n\}, \quad j \in I_3 \setminus \{q\}. \end{aligned} \tag{3.21}$$

Thus, similar as the proof of Theorem 1.1, we have

$$\exp\{(1-\varepsilon)\delta_q r^n\} \le \exp\{(1+2\varepsilon)\delta_3^* r^n\},$$

a contradiction.

Case (2) If there exists an integer $p(\in I_1 \neq \emptyset)$ such that $c_p > c_j(j \in I_1 \setminus \{p\})$, then we can choose a ray $\arg z = \theta_4 \in [0, 2\pi) \setminus (H_0^* \cup H_4)$ such that $\delta(-P(z), \theta_4) < 0$, and Lemma 2.1 holds, where $H_4 \subset [0, 2\pi)$ has linear measure zero. For all z satisfying $|z| = r$ and $\arg z = \theta_4$, we rewrite (3.10)-(3.12) as follows:

$$|B_p(z)e^{(c_p-1)b_n z^n(1+o(1))}| \ge \exp\{(1-\varepsilon)\delta_p r^n\}; \tag{3.22}$$

$$\begin{aligned} |B_j(z)e^{(c_j-1)b_n z^n(1+o(1))}| &\le \exp\{(1+\varepsilon)\delta((c_j-1)P(z), \theta_4)r^n\} \\ &\le \exp\{(1+\varepsilon)\delta_1^* r^n\}, \quad j \in I_1 \setminus \{p\}; \end{aligned} \tag{3.23}$$

$$|B_j(z)e^{(c_j-1)b_n z^n(1+o(1))}| \le \exp\{(1-\varepsilon)\delta((c_j-1)P(z), \theta_4)r^n\} < 1, \quad j \in I_2 \cup I_3. \tag{3.24}$$

Thus, similar as the method of proof of Theorem 1.1, we have

$$\exp\{(1-\varepsilon)\delta_q r^n\} \le \exp\{(1+2\varepsilon)\delta_1^* r^n\},$$

a contradiction.

Therefore, we have $\sigma(f) \ge n$.

The proof of Corollary 1.2 is complete.

Proof of Theorem 1.2 On the contrary, we suppose that $\sigma(f) < 1$. We divide (1.4) by $f(z)$ to get

$$-A_0(z) = A_k(z)\frac{\Delta^k f(z)}{f(z)} + \cdots + A_1(z)\frac{\Delta f(z)}{f(z)} - \frac{A_{k+1}(z)}{f(z)}. \tag{3.25}$$

By Remark 2.1, we have

$$\left|\frac{\Delta^j f(z)}{f(z)}\right| \le O(\exp\{r^{\sigma(f)-1+\varepsilon}\}), \quad j \in I \setminus \{0\}. \tag{3.26}$$

Case (1) If $0 \in I_3$ and $c_0 < c_j(j \in I_3 \setminus \{0\})$, then we can choose a ray $\arg z = \theta_5 \in [0, 2\pi) \setminus H_5$ such that $\delta(-az, \theta_5) > 0$, and Lemma 2.1 holds, where $H_5 \subset [0, 2\pi)$ has linear measure zero. By combining (3.1)-(3.4) (where we take θ_5 instead of θ_1), (3.6)-(3.7), and (3.25)-(3.26), we have for all z satisfying $\arg z = \theta_5, |z| = r \notin [0, 1] \cup (E_1 \cup E_2)$ and $r \to \infty$,

$$\exp\{(1-\varepsilon)\delta_0 r\} \le \exp\{(1+2\varepsilon)\delta_3 r\}, \tag{3.27}$$

where $\delta_0 = \delta((c_0-1)az, \theta)$. By a similar reasoning method as the one in Theorem 1.1, we can get a contradiction from (3.27).

Case (2) If $0 \in I_1$ and $c_0 > c_j(j \in I_1 \setminus \{0\})$, then we can choose a ray $\arg z = \theta_6 \in [0, 2\pi) \setminus H_6$ such that $\delta(-az, \theta_6) < 0$, and Lemma 2.1 holds, where $H_6 \subset [0, 2\pi)$ has linear measure zero. By combining (3.6)-(3.7), (3.10)-(3.12), and (3.25)-(3.26), we have for all z satisfying $\arg z = \theta_6, |z| = r \notin [0, 1] \cup (E_1 \cup E_2)$ and $r \to \infty$,

$$\exp\{(1-\varepsilon)\delta_0 r\} \le \exp\{(1+2\varepsilon)\delta_1 r\}, \tag{3.28}$$

where $\delta_0 = \delta((c_0-1)az, \theta)$. By a similar reasoning method as the one in Theorem 1.1, we can get a contradiction from (3.28).

Therefore, we have $\sigma(f) \ge 1$.

The proof of Theorem 1.2 is complete.

Proofs of Corollaries 1.3 and 1.4 The proofs of Corollaries 1.3 and 1.4 are similar as the ones of Corollaries 1.2 and 1.1 respectively.

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