Appl. Math. J. Chinese Univ.
2019, 34(4): 379-392

Characterizations of product Hardy space associated to Schrödinger operators

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Abstract. Let L_1 and L_2 be the Schrödinger operators on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. By using different maximal functions and Littlewood-Paley g function on distinct variables, in this paper, some characterizations for functions in the product Hardy space $H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)$ associated to operators L_1 and L_2 are obtained.

§1 Introduction

It is well known that modern harmonic analysis played a very important role in partial differential equations. The theories of function spaces constitute the most of harmonic analysis. Thus, the characterizations of function spaces are very critical in harmonic analysis. For example, the classical Hardy spaces on $\mathbb{R}^n$ can be equivalently characterized via, such as, maximal functions, Lusin-area function, Littlewood-Paley g function, and atoms $[7,8,11,21,32]$, etc. The product Hardy spaces are the Hardy spaces on product domains, which were first introduced by Malliavin and Malliavin [26], and Gundy and Stein [15]. Then, the properties of these product spaces have been studied by Chang and Fefferman $[3,5]$. Other results for product spaces can be seen in $[4,6,7,12,13,23,30]$. We know that, the product Hardy space $H^1(\mathbb{R}^n\times\mathbb{R}^m)$ is also characterized in terms of the area function, maximal functions, atoms, and Riesz transforms $[3,5,15]$, etc.

In 2011, Li, Song and Tan [22] studied some new characterizations of the Hardy space H^1 on Euclidean product spaces $\mathbb{R}^n\times\mathbb{R}^m$ using different norms on distinct variables. They considered non-tangential maximal function and the Littlewood-Paley square function, as well as vertical maximal function, and obtained some characterizations of the product Hardy space $H^1(\mathbb{R}^n\times\mathbb{R}^m)$ as follows.

Received: 2016-01-31. Revised: 2019-05-01.
MR Subject Classification: 42B30, 42B35, 42B25.
Keywords: Product Hardy space, Schrödinger operator, characterization, Littlewood-Paley function, maximal function.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-019-3442-6.
Supported by the National Natural Science Foundation of China (11471176) and Natural Science Foundation of Shandong Province (BS2014SF002).
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Proposition 1.1 ([22]).

$$\begin{aligned}&H^1(\mathbb{R}^n\times\mathbb{R}^m)\simeq H^1_{\mathcal{N},\mathcal{S}}(\mathbb{R}^n\times\mathbb{R}^m)\simeq H^1_{\mathcal{S},\mathcal{N}}(\mathbb{R}^n\times\mathbb{R}^m)\\ &\simeq H^1_{\mathcal{N},g}(\mathbb{R}^n\times\mathbb{R}^m)\simeq H^1_{g,\mathcal{N}}(\mathbb{R}^n\times\mathbb{R}^m)\simeq H^1_{+,\mathcal{S}}(\mathbb{R}^n\times\mathbb{R}^m)\\ &\simeq H^1_{+,g}(\mathbb{R}^n\times\mathbb{R}^m)\simeq H^1_{\mathcal{S},+}(\mathbb{R}^n\times\mathbb{R}^m)\simeq H^1_{g,+}(\mathbb{R}^n\times\mathbb{R}^m).\end{aligned}$$

On the other hand, due to some important situations in which the theory of classical Hardy spaces is not applicable, the Hardy spaces associated to operators are introduced. Especially over the past ten years, many authors studied function spaces associated to operators, showed some characterizations of them [1,2,9,10,14,16–19,24,31], etc. In 2011 and 2012, Song and Tan discussed Hardy spaces associated to Schrödinger operators on product spaces. They obtained some characterizations of the Hardy space associated to Schrödinger operators on product domains, such as, atomic decomposition, the characterizations by Lusin area integral and the maximal functions [28,29].

A natural question is to establish some characterizations similar to Proposition 1.1 for the product Hardy space associated to Schrödinger operators $H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)$ by using different maximal functions and Littlewood-Paley functions on distinct variables.

The Schrödinger operators L_1 and L_2 are defined by

$$L_1=-\triangle_1+V_1,\quad\text{and}\quad L_2=-\triangle_2+V_2, \tag{1}$$

where $\triangle_1$ and $\triangle_2$ are the Laplacians, and V_1, V_2 are non-negative functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

As we know, the operator L_1 is a self-adjoint positive definite operator on $L^2(\mathbb{R}^n)$. Then from the Feynman-Kac formula, the kernel $p_{t_1}(x_1,y_1)$ of the semigroup $e^{-t_1L_1}$ satisfies the estimate

$$0\le p_{t_1}(x_1,y_1)\le\frac{1}{(4\pi t_1)^{n/2}}e^{-\frac{|x_1-y_1|^2}{4t_1}}. \tag{2}$$

For the self-adjoint positive definite operator L_2 on $L^2(\mathbb{R}^m)$, the kernel $p_{t_2}(x_2,y_2)$ of $e^{-t_2L_2}$ satisfies the similar estimate to (2).

Given a function f on $\mathbb{R}^n\times\mathbb{R}^m$, the area integral function Sf associated to operators L_1 and L_2 is defined by

$$Sf(x_1,x_2)=\Big(\iint_{\substack{|y_1-x_1|<t_1,\\|y_2-x_2|<t_2}}\Big|t_1^2L_1\mathrm{e}^{-t_1^2L_1}\otimes t_2^2L_2\mathrm{e}^{-t_2^2L_2}f(y_1,y_2)\Big|^2\frac{dy_1dt_1}{t_1^{n+1}}\frac{dy_2dt_2}{t_2^{m+1}}\Big)^{1/2}.$$

Definition 1.1. *Suppose that L_1 and L_1 are the Schrödinger operators as in (1). The Hardy space $H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)$ associated to L_1 and L_2 is defined as the completion of*

$$\big\{f\in L^2(\mathbb{R}^n\times\mathbb{R}^m):\|Sf\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}<\infty\big\}$$

with respect to the norm $\|f\|_{H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)}=\|Sf\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}$.

We known that if operators L_1 and L_2 are the Laplacians $\triangle_1$ and $\triangle_2$ on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, it follows from the area integral characterization by using convolution that the Hardy space $H^1(\mathbb{R}^n\times\mathbb{R}^m)$ coincides with the space $H^1_{\triangle_1,\triangle_2}(\mathbb{R}^n\times\mathbb{R}^m)$, and their norms are equivalent [3,4,12], etc.

Recently, motivated by the Proposition 1.1, constituting by two of non-tangential maximal function, vertical maximal function and Lusin area integral, we obtained some characterizations for functions in the product Hardy space $H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m)$ associated to L_1 and L_2 by using different maximal functions and Lusin area integral on distinct variables. We have the following proposition.

Proposition 1.2 ([25]). *Suppose that L_1 and L_2 are the Schrödinger operators as in (1). Then*

$$\begin{aligned} H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m) &\simeq H^1_{L_1,L_2,NS_p}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,S_pN}(\mathbb{R}^n \times \mathbb{R}^m) \\ &\simeq H^1_{L_1,L_2,S_p,+}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,+,S_p}(\mathbb{R}^n \times \mathbb{R}^m). \end{aligned}$$

In this paper, we will consider the combination among non-tangential maximal function, vertical maximal function and Littlewood-Paley g function to establish the similar characterizations for functions in the product Hardy space $H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m)$ associated to L_1 and L_2 also by different characterized on distinct variables. These results are complementary for Proposition 1.2 respect to Proposition 1.1.

Suppose that L_1 and L_2 are the Schrödinger operators as in (1). By using different Littlewood-Paley functions on distinct variables, we define $f_{\mathcal{N},g}$ and $f_{+,g}$ functions as

$$f_{\mathcal{N},g}(x_1,x_2) = \sup_{|y_1-x_1|<t_1} \Big(\int_0^\infty |e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(y_1,x_2)|^2 \frac{dt_2}{t_2} \Big)^{1/2},$$

$$f_{+,g}(x_1,x_2) = \sup_{t_1>0} \Big(\int_0^\infty |e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(x_1,x_2)|^2 \frac{dt_2}{t_2} \Big)^{1/2}.$$

Similarly, we define product spaces $H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m)$ and $H^1_{L_1,L_2,+,g}(\mathbb{R}^n \times \mathbb{R}^m)$ by

$$H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in L^2(\mathbb{R}^n \times \mathbb{R}^m) : f_{\mathcal{N},g} \in L^1(\mathbb{R}^n \times \mathbb{R}^m)\},$$

$$H^1_{L_1,L_2,+,g}(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in L^2(\mathbb{R}^n \times \mathbb{R}^m) : f_{+,g} \in L^1(\mathbb{R}^n \times \mathbb{R}^m)\},$$

with the norms

$$\|f\|_{H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n\times\mathbb{R}^m)} = \|f_{\mathcal{N},g}\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)} \text{ and } \|f\|_{H^1_{L_1,L_2,+,g}(\mathbb{R}^n\times\mathbb{R}^m)} = \|f_{+,g}\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)},$$

respectively.

Then, the main result of this article, the characterizations for functions in product Hardy space $H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m)$ associated to the Schrödinger operators L_1 and L_2, is given by

$$H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,+,g}(\mathbb{R}^n \times \mathbb{R}^m).$$

This result will be proved in Section 3. In the next section, we will recall some definitions, and introduce some important lemmas.

§2 Definitions and lemmas

In this section, in order to obtain our main result, we recall the definitions of the atomic product Hardy space, tent space, and introduce some key lemmas.

Suppose that $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ is an open set with finite measure. Denote by $m(\Omega)$ the maximal dyadic subrectangles of Ω. Let $m_1(\Omega)$ denote those dyadic subrectangles $R \subseteq \Omega$, $R = I \times J$ that are maximal in the first variable. In other words, if $O = I_1 \times J \supseteq R$ is a dyadic subrectangle of Ω , then $I = I_1$. Define $m_2(\Omega)$ for the second variable similarly.

In [28], Song, Tan and Yan introduced the definition of product $(1,2)$-atom associated to the Schrödinger operators, and established the atomic decomposition for the product Hardy space associated to Schrödinger operators.

Definition 2.1 ([28]). *Let L_1 and L_2 be the Schrödinger operators as in (1). A function $a(x_1,x_2) \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$ is called a product $(1,2)$-atom if it satisfies*
(1) supp $a \subset \Omega$, *where Ω is an open set of $\mathbb{R}^n \times \mathbb{R}^m$ with finite measure;*
(2) a can be further decomposed into $a = \sum_{R\in m(\Omega)} a_R$, where for each $R \in m(\Omega)$, there exists a function b_R belonging to the domain of $L_1 \otimes L_2$ in $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ such that

(i) $a_R = \big(L_1 \otimes L_2\big) b_R$;

(ii) $\mathrm{supp}\big(L_1^j \otimes L_2^k\big) b_R \subset 10R, \quad j,\, k = 0, 1;$

(iii) $||a||_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \le |\Omega|^{-1/2}$ *and*

$$\sum_{R\in m(\Omega)} \sum_{j,k=0}^{1} \ell(I)^{4j-4}\ell(J)^{4k-4} \Big\| \big(L_1^j \otimes L_2^k\big) b_R \Big\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \le |\Omega|^{-1},$$

where L_i^0 denotes the identity operator, $i = 1, 2$.

The atomic product Hardy space $H^1_{L_1,L_2,at}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined as follows.

Definition 2.2 ([28]). *Let L_1 and L_2 be the Schrödinger operators as in (1). The atomic product Hardy space $H^1_{L_1,L_2,at}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined as follows. We say that $f = \sum_j \lambda_j a_j$ is a product atomic $(1,2)$-representation of f if $\{\lambda_j\}_j \in \ell^1$, each a_j is a product $(1,2)$-atom, and the sum converges in $L^2(\mathbb{R}^n \times \mathbb{R}^m)$. Then*

$$\mathbb{H}^1_{L_1,L_2,at}(\mathbb{R}^n \times \mathbb{R}^m) = \{f : f \text{ has a product atomic (1,2)-representation}\},$$

with the norm given by

$$\|f\|_{\mathbb{H}^1_{L_1,L_2,at}} = \inf\Big\{ \sum_{j=0}^{\infty} |\lambda_j| : f = \sum_j \lambda_j a_j \text{ is a product atomic (1,2)-representation}\Big\}.$$

The space $H^1_{L_1,L_2,at}(\mathbb{R}^n \times \mathbb{R}^m)$ is then defined as the completion of $\mathbb{H}^1_{L_1,L_2,at}(\mathbb{R}^n \times \mathbb{R}^m)$ with respect to this norm.

Then, we recall Journé's covering lemma and some useful results.

Lemma 2.1 ([20, 27]). *Let $\Omega^* = \{x \in \mathbb{R}^n \times \mathbb{R}^m : \mathcal{M}(\mathcal{X}_\Omega)(x) > 1/2\}$, where $\mathcal{M}$ denotes the strong maximal operator. For any $R = I \times J$, suppose $\gamma_1(R) = \sup\limits_{I_1\subset I, I\times J\subset\Omega^*} \frac{|I_1|}{|I|}$, $\gamma_2(R)$ is similarly. Then for any $\delta > 0$,*

$$\sum_{R\in m_2(\Omega)} |R|{\gamma_1}^{-\delta}(R) \le c_\delta|\Omega| \qquad \text{and} \qquad \sum_{R\in m_1(\Omega)} |R|{\gamma_2}^{-\delta}(R) \le c_\delta|\Omega|. \tag{3}$$

Lemma 2.2 ([25]). *Suppose that T is a bounded sublinear operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$, and for every product $(1,2)$-atom $a(x)$ on product domains, $\|Ta\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)} \le C$, with constant C independent of a. Then for any decomposition of f in Definition 2.2,*

$$\|Tf\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)} \le C\|f\|_{H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)}.$$

In the following, we shall assume that $\varphi \in C_0^1(\mathbb{R}^n)$ is a nonnegative, radial and nonincreasing function, $\varphi = 1$ on $B(0,1/2)$, supp $\varphi \subset B(0,1)$ and $\int \varphi(x)dx = 1$. Let ψ be a function with the same support as φ and mean value 0.

Lemma 2.3 ([25]). *Let $f \in L^2(\mathbb{R}^n\times\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^n)$, $u(x,t) = \mathrm{e}^{-t_1\sqrt{L_1}}\otimes\mathrm{e}^{-t_2\sqrt{L_2}}f(x_1,x_2)$. Then*

$$\begin{aligned}&\iint_{\mathbb{R}^{n+1}_+} \big|t_1\nabla_{X_1}u(x,t)\big|^2\big|\varphi_{t_1} * g(x_1)\big|^2\,\frac{dx_1\,dt_1}{t_1}\\ &\le \int_{\mathbb{R}^n} |t_2\sqrt{L_2}\mathrm{e}^{-t_2\sqrt{L_2}}f(x)|^2|g(x_1)|^2\,dx_1 + \iint_{\mathbb{R}^{n+1}_+}\big|u(x,t)\big|^2\big|\psi_{t_1} * g(x_1)\big|^2\,\frac{dx_1\,dt_1}{t_1}.\end{aligned} \tag{4}$$

If $x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$, let $\Gamma(x)$ denote the product cone $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$ where $\Gamma(x_1) = \{(y_1,t_1) \in \mathbb{R}^{n+1}_+ : |y_1 - x_1| < t_1\}$ and $\Gamma(x_2) = \{(y_2,t_2) \in \mathbb{R}^{m+1}_+ : |y_2 - x_2| < t_2\}$. Before giving the following lemma, we define the non-tangential fnction, Littlewood-Paley square function and Littlewood-Paley $\mathcal{G}$ function associated with L_1 and L_2 as follows.

$$\begin{aligned}\mathcal{N}(f)(x) &= \sup_{|x_1-y_1|<t_1,|x_2-y_2|<t_2} |e^{-t_1\sqrt{L_1}}\otimes e^{-t_2\sqrt{L_2}}f(y_1,y_2)|,\\ \mathcal{S}(f)(x) &= \Big(\iint_{\Gamma(x)} |t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}}\otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}f(y_1,y_2)|^2\frac{dy_1dt_1}{t_1^{n+1}}\frac{dy_2dt_2}{t_2^{m+1}}\Big)^{1/2},\\ \mathcal{G}(f)(x) &= \Big(\int_0^\infty\int_0^\infty |t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}}\otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}f(x)|^2\frac{dt_1dt_2}{t_1t_2}\Big)^{1/2}.\end{aligned}$$

Then, similarly, we can also define product spaces $H^1_{L_1,L_2,\mathcal{N}}(\mathbb{R}^n\times\mathbb{R}^m)$, $H^1_{L_1,L_2,\mathcal{S}}(\mathbb{R}^n\times\mathbb{R}^m)$ and $H^1_{L_1,L_2,\mathcal{G}}(\mathbb{R}^n\times\mathbb{R}^m)$ associated to Schrödinger operators L_1 and L_2 as

$$\begin{aligned}H^1_{L_1,L_2,\mathcal{N}}(\mathbb{R}^n\times\mathbb{R}^m) &= \{f \in L^2(\mathbb{R}^n\times\mathbb{R}^m) : \mathcal{N}(f) \in L^1(\mathbb{R}^n\times\mathbb{R}^m)\},\\ H^1_{L_1,L_2,\mathcal{S}}(\mathbb{R}^n\times\mathbb{R}^m) &= \{f \in L^2(\mathbb{R}^n\times\mathbb{R}^m) : \mathcal{S}(f) \in L^1(\mathbb{R}^n\times\mathbb{R}^m)\},\\ H^1_{L_1,L_2,\mathcal{G}}(\mathbb{R}^n\times\mathbb{R}^m) &= \{f \in L^2(\mathbb{R}^n\times\mathbb{R}^m) : \mathcal{G}(f) \in L^1(\mathbb{R}^n\times\mathbb{R}^m)\}\end{aligned}$$

with norms

$$\begin{aligned}\|f\|_{H^1_{L_1,L_2,\mathcal{N}}(\mathbb{R}^n\times\mathbb{R}^m)} &= \|\mathcal{N}(f)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)},\\ \|f\|_{H^1_{L_1,L_2,\mathcal{S}}(\mathbb{R}^n\times\mathbb{R}^m)} &= \|\mathcal{S}(f)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)},\\ \|f\|_{H^1_{L_1,L_2,\mathcal{G}}(\mathbb{R}^n\times\mathbb{R}^m)} &= \|\mathcal{G}(f)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)},\end{aligned}$$

respectively.

Therefore, according to [25, Lemma 3.3 and Theorem 3.4], and the g function is equivalent to the area function on L^p, we can obtain the following lemma.

Lemma 2.4.
$$\begin{aligned}H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m) &\simeq H^1_{L_1,L_2,at}(\mathbb{R}^n\times\mathbb{R}^m) \simeq H^1_{L_1,L_2,\mathcal{N}}(\mathbb{R}^n\times\mathbb{R}^m)\\ &\simeq H^1_{L_1,L_2,\mathcal{S}}(\mathbb{R}^n\times\mathbb{R}^m) \simeq H^1_{L_1,L_2,\mathcal{G}}(\mathbb{R}^n\times\mathbb{R}^m).\end{aligned}$$

In order to obtain our main results, we recall the definition of tent space as well as its atomic decomposition.

Definition 2.3 ([7]). *For any function $f(y,t)$ on $\mathbb{R}^{n+1}_+$, define*

$$\mathcal{A}(f)(x) = (\iint_{\Gamma(x)} |f(y,t)|^2 \frac{dy\,dt}{t^{n+1}})^{1/2}. \tag{5}$$

The tent space T^1_2 is defined as the space of functions f such that $\mathcal{A}(f) \in L^1(\mathbb{R}^n)$ with norm $\|f\|_{T^1_2} = \|\mathcal{A}(f)\|_{L^1(\mathbb{R}^n)}$.

Lemma 2.5 ([7]). *Suppose that a T^1_2-atom $a(x,t)$ is a function supported on $\widehat{Q}$ with*

$$\int_{\widehat{Q}} |a(x,t)|^2 \frac{dx\,dt}{t} \leq |Q|^{-1}, \tag{6}$$

where $\widehat{Q}$ is the tent of Q. The atomic decomposition of f in the tent space is:

$$f = \sum_j \lambda_j a_j, \quad \text{for any } f \in T^1_2(\mathbb{R}^{n+1}_+), \tag{7}$$

where every a_j is a T^1_2-atom, and furthermore, $\sum_j |\lambda_j| \leq C\|f\|_{T^1_2}$.

§3 Characterization of product Hardy space

With the above discussion, we show our main results in the following.

Theorem 3.1. $H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,+,g}(\mathbb{R}^n \times \mathbb{R}^m)$.

Proof. Since the proof of $H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,+,g}(\mathbb{R}^n \times \mathbb{R}^m)$ is similar to $H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m)$, we only show that

$$H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m) \simeq H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m). \tag{8}$$

Obviously, (8) is the direct result of the following two inclusions.

$$H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m) \subset H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m), \tag{9}$$

$$H^1_{L_1,L_2}(\mathbb{R}^n \times \mathbb{R}^m) \subset H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m). \tag{10}$$

To prove (9), by Lemma 2.4, it suffices to show that

$$\begin{aligned}&\iint \Big(\int_0^\infty \int_0^\infty |t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(x_1,x_2)|^2 \frac{dt_2}{t_2}\frac{dt_1}{t_1}\Big)^{1/2} dx_1dx_2 \\ &\leq C \iint \sup_{|y_1-x_1|<t_1} \Big(\int_0^\infty |e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(y_1,x_2)|^2 \frac{dt_2}{t_2}\Big)^{1/2} dx_1dx_2,\end{aligned} \tag{11}$$

for any $f \in H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n \times \mathbb{R}^m) \cap L^2(\mathbb{R}^n \times \mathbb{R}^m)$.

Let $\mathcal{B}$ denote the functions $\{F_{t_2,x_2}(y_1) : y_1 \in \mathbb{R}^n, t_2 \in (0,\infty), x_2 \in \mathbb{R}^m\}$ with norm

$$\|F_{t_2,x_2}(y_1)\|_{\mathcal{B}} = \Big(\int_0^\infty |F_{t_2,x_2}(y_1)|^2 \frac{dt_2}{t_2}\Big)^{1/2}.$$

Obviously, if we can prove the following two inequalities, then (11) holds.

$$\begin{aligned}&\iint \Big(\iint_{\Gamma(x_1)} \|t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(y_1,x_2)\|^2_{\mathcal{B}} \frac{dy_1dt_1}{{t_1}^{n+1}}\Big)^{1/2} dx_1dx_2 \\ &\leq C \iint \sup_{|y_1-x_1|<t_1} \Big(\int_0^\infty |e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(y_1,x_2)|^2 \frac{dt_2}{t_2}\Big)^{1/2} dx_1dx_2,\end{aligned} \tag{12}$$

$$\iint \Big(\int_0^\infty \int_0^\infty |t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(x_1,x_2)|^2 \frac{dt_2}{t_2}\frac{dt_1}{t_1}\Big)^{1/2} dx_1dx_2$$
$$\leq C \iint \Big(\iint_{\Gamma(x_1)} \|t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(x_1,x_2)\|_{\mathcal{B}}^2 \frac{dy_1dt_1}{t_1^{n+1}}\Big)^{1/2} dx_1dx_2. \tag{13}$$

Firstly, we prove (12). Notice that

$$|t_1\nabla_{Y_1}e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}|^2 \geq C|t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}|^2.$$

Let $F_{t_2,x_2}(y_1) = t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}f(y_1,x_2)$. Define the square function and non-tengential maximal function of F as

$$S(F)(x_1,x_2) = \Big(\iint_{\Gamma(x_1)} \|t_1\nabla e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(y_1,x_2)\|_{\mathcal{B}}^2 \frac{dy_1dt_1}{{t_1}^{n+1}}\Big)^{1/2}$$

and

$$F^*(x_1,x_2) = \sup_{|x_1-y_1|<t_1} \|e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}} f(y_1,x_2)\|_{\mathcal{B}},$$

respectively. Then, to prove (12), we only need to show that

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} S(F)(x_1,x_2)dx_1dx_2 \leq C\int_{\mathbb{R}^n\times\mathbb{R}^m} F^*(x_1,x_2)dx_1dx_2. \tag{14}$$

It is well know that, in order to obtain (14), it suffices to prove that for $\alpha>0$ and $x_2\in\mathbb{R}^m$, the following estimate (15) holds.

$$\int_{\{\mathcal{M}^{(1)}(\mathcal{X}_{F^*>\alpha})(x_1)<2^{-(n+1)}\}} \big(S(F)(x_1,x_2)\big)^2dx_1$$
$$\leq C\alpha^2|\{x_1 : F^*(x_1,x_2)>\alpha\}| + C\int_{\{x_1:F^*(x_1,x_2)\leq\alpha\}} \big(F^*(x_1,x_2)\big)^2dx_1. \tag{15}$$

Notice that for any $\alpha>0$, $x_2\in\mathbb{R}^m$,

$$\int_{\mathbb{R}^n} S(F)(x_1,x_2)dx_1 = \int_0^\infty |\{x_1 : S(F)(x_1,x_2)>\alpha\}|d\alpha,$$

and the following fact which was given in [22],

$$|\{x_1 : S(F)(x_1,x_2)>\alpha\}| \leq \frac{C}{\alpha^2}\int_{\{x_1:F^*(x_1,x_2)\leq\alpha\}} \big(F^*(x_1,x_2)\big)^2dx_1 + C|\{x_1 : F^*(x_1,x_2)>\alpha\}|.$$

Then by (15), integrating on both sides of the inequality above for α and x_2, we can obtain (14). Hence, it is remaining to prove (15). Note that

$$\int_{\{\mathcal{M}^{(1)}(\mathcal{X}_{F^*>\alpha})(x_1)<2^{-(n+1)}\}} \big(S(F)(x_1,x_2)\big)^2dx_1$$
$$\leq \int_0^\infty \int_{\{\mathcal{M}^{(1)}(\mathcal{X}_{F^*>\alpha})(x_1)<2^{-(n+1)}\}} \iint_{\Gamma(x_1)} |t_1\nabla e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)|^2 \frac{dy_1dt_1}{t_1^{n+1}}\frac{dx_1dt_2}{t_2}.$$

However,

$$\int_{\{\mathcal{M}^{(1)}(\mathcal{X}_{F^*>\alpha})(x_1)<2^{-(n+1)}\}} \iint_{\Gamma(x_1)} |t_1\nabla e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)|^2 \frac{dy_1dt_1}{t_1^{n+1}}dx_1$$
$$= \int_{R^*} |\nabla e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)|^2 |B(y_1,t_1)\cap\{\mathcal{M}^{(1)}(\mathcal{X}_{F^*>\alpha})(x_1)<2^{-(n+1)}\}| \frac{dy_1dt_1}{t_1^{n-1}}$$
$$\leq C\int_{R^*} |t_1\nabla e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)|^2 \frac{dy_1dt_1}{t_1},$$

where $R^* = \{(y_1,t_1) : |B(y_1,t_1)\cap\{z : F^*(z,x_2)>\alpha\}| \le 2^{-(n+1)}|B(y_1,t_1)|\}$.
Therefore,

$$\int_{\{\mathcal{M}^{(1)}(\mathcal{X}_{F^*>\alpha})(x_1)<2^{-(n+1)}\}} \big(S(F)(x_1,x_2)\big)^2 dx_1 \le \int_{R^*} \|t_1\nabla e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)\|_{\mathcal{B}}^2 \frac{dy_1dt_1}{t_1}.$$

It is easy to check that if $|B(y_1,t_1)\cap\{z : F^*(z,x_2)>\alpha\}| \le 2^{-(n+1)}|B(y_1,t_1)|$, then $g*\varphi_{t_1}(y_1)>C$ for some constant $C>0$, where $\varphi\in C_0^1(\mathbb{R}^n)$ is as in Lemma 2.3 and $g(x)=\chi_{\{F^*(x_1,x_2)\le\alpha\}}(x)$. Together with Lemma 2.3, we have

$$\begin{aligned}
&\int_{\{\mathcal{M}^{(1)}(\mathcal{X}_{F^*>\alpha})(x_1)<2^{-(n+1)}\}} \big(S(F)(x_1,x_2)\big)^2 dx_1\\
\le\ & C\int_{R^*} \|t_1\nabla e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)\|_{\mathcal{B}}^2|\varphi_{t_1}*g(y_1)|\frac{dy_1dt_1}{t_1}\\
\le\ & C\int_0^\infty\int_{\mathbb{R}^{n+1}_+} |t_1\nabla e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)|^2|\varphi_{t_1}*g(y_1)|\frac{dy_1dt_1}{t_1}\frac{dt_2}{t_2}\\
\le\ & C\int_0^\infty\int_{\mathbb{R}^n} |F_{t_2,x_2}(y_1)|^2|g(y_1)|^2dy_1\frac{dt_2}{t_2}\\
&+C\int_0^\infty\int_{\mathbb{R}^{n+1}_+} |e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)|^2|\psi_{t_1}*g(y_1)|^2\frac{dy_1dt_1}{t_1}\frac{dt_2}{t_2}\\
=\ & C\int_{\mathbb{R}^n}\|F_{t_2,x_2}(x_1)\|_{\mathcal{B}}^2|g(x_1)|^2dx_1\\
&+C\int_{\mathbb{R}^{n+1}_+} \|e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)\|_{\mathcal{B}}^2|\psi_{t_1}*g(y_1)|^2\frac{dy_1dt_1}{t_1}\\
=\ & M_1+M_2.
\end{aligned}$$

For the term M_1, by the definitions of $g(x)$ and $F^*(x_1,x_2)$, we obtain

$$M_1 \le C\int_{\{x_1:F^*(x_1,x_2)\le\alpha\}} \big(F^*(x_1,x_2)\big)^2dx_1.$$

For M_2, we only need to consider $\psi_{t_1}*g(y_1)\ne 0$. In this case, $B(y_1,t_1)\cap\{z : F^*(z,x_2)\le\alpha\}\ne\emptyset$. Thus, there exists a point $z_1^0\in\mathbb{R}^n$ such that $|z_1^0-y_1|<t_1$ and $F^*(z_1^0,x_2)\le\alpha$. Therefore,

$$\|e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)\|_{\mathcal{B}} \le \sup_{|z_1^0-z_1|<s_1}\|e^{-t_1\sqrt{L_1}}F_{t_2,x_2}(y_1)\|_{\mathcal{B}} = F^*(z_1^0,x_2)\le\alpha.$$

Hence, by the cancellation condition of ψ, we can get that

$$\begin{aligned}
M_2 &\le C\alpha^2\int_{\mathbb{R}^{n+1}_+}|\psi_{t_1}*\chi_{F^*(\cdot,x_2)\le\alpha}(y_1)|^2\frac{dy_1dt_1}{t_1}\\
&\le C\alpha^2|\{y_1 : F^*(y_1,x_2)>\alpha\}|.
\end{aligned}$$

Combining with the estimates of M_1 and M_2, we can see that (15) holds, which implies (12) is valid.

Secondly, we prove (13). In order to do this, we first show the fact that

$$\Big\|\Big(\int_0^\infty|F(x,t)|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^1(\mathbb{R}^n)} \le C\Big\|\Big(\iint_{\Gamma(x)}|F(y,t)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\Big\|_{L^1(\mathbb{R}^n)}. \tag{16}$$

According to the Definition 2.3, the right hand of the inequality (16) is $F(x,t)$'s T_2^1 norm. By the atomic decomposition of the tent space (7), in order to prove (16), it suffices to show that for any T_2^1-atom $a(x,t)$ supported on $\hat{Q}$, there exists a constant C independent of a, satisfying

$$\Big\|\Big(\int_0^\infty |a(x,t)|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^1(\mathbb{R}^n)} \le C.$$

Actually, by Hölder's inequality and the definition of T_2^1-atom, we have

$$\begin{aligned}\Big\|\Big(\int_0^\infty |a(x,t)|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^1(\mathbb{R}^n)} &= \int_Q\Big(\int_0^{l(Q)} |a(x,t)|^2\frac{dt}{t}\Big)^{1/2}dx\\ &\le \Big(\int_{\widehat{Q}}|a(x,t)|^2\frac{dtdx}{t}\Big)^{1/2}|Q|^{1/2} \le C|Q|^{-1/2}|Q|^{1/2}\le C.\end{aligned}$$

Thus, (16) holds.

Suppose $F = (\int_0^\infty |t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}}\otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}f(x_1,x_2)|^2\frac{dt_2}{t_2})^{1/2}$. Substituting F into (16), then it tells us

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big(\int_0^\infty\int_0^\infty |t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}}\otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}f(x_1,x_2)|^2\frac{dt_2}{t_2}\frac{dt_1}{t_1}\Big)^{1/2}dx_1\\ &\le C\int_{\mathbb{R}^n}\Big(\iint_{\Gamma(x_1)}\int_0^\infty |t_1\sqrt{L_1}e^{-t_1\sqrt{L_1}}\otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}f(x_1,x_2)|^2\frac{dt_2}{t_2}\frac{dy_1dt_1}{t_1^{n+1}}\Big)^{1/2}dx_1.\end{aligned}$$

Therefore, (13) holds. Combining with (12), we know that (11) is correct. Then (9) is proved.

To prove (10), by Lemma 2.4, we only need to prove

$$H^1_{L_1,L_2,at}(\mathbb{R}^n\times\mathbb{R}^m)\subset H^1_{L_1,L_2,\mathcal{N},g}(\mathbb{R}^n\times\mathbb{R}^m). \tag{17}$$

From estimate (2), we know that for every $k\in\mathbb{N}$, there exists a constant C_k such that the kernel $p_{t,k}$ of the operator $(t\sqrt{L})^k e^{-t\sqrt{L}}$ satisfies

$$|p_{t,k}(x,y)|\le C_k\frac{t}{(t+|x-y|)^{n+1}},\quad \forall\, t>0,\ x,y\in\mathbb{R}^n. \tag{18}$$

Thus, for any $f\in L^2(\mathbb{R}^n\times\mathbb{R}^m)$, using the kernel estimate (18) and the fact that non-tangential maximal function is dominated by Hardy-Littlewood maximal operator on $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\|f_{\mathcal{N},g}\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} &= \iint\sup_{|y_1-x_1|<t_1}\int_0^\infty |e^{-t_1\sqrt{L_1}}\otimes t_2\sqrt{L_2}e^{-t_2\sqrt{L_2}}f(y_1,x_2)|^2\frac{dt_2}{t_2}dx_1dx_2\\ &\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\int_0^\infty \mathcal{M}^{(1)}(t_2\sqrt{L_2}\mathrm{e}^{-t_2\sqrt{L_2}}f(\cdot,x_2))^2(x_1)\,\frac{dt_2}{t_2}dx_1dx_2\\ &\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|f(x_1,x_2)|^2dx_1dx_2 = C\|f\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)},\end{aligned} \tag{19}$$

where $\mathcal{M}^{(1)}$ is the Hardy-Littlewood maximal operator on the first variable which is bounded on $L^2(\mathbb{R}^n)$, and the third step also uses the L^2 boundedness of g function.

For any $f\in H^1_{L_1,L_2,at}(\mathbb{R}^n\times\mathbb{R}^m)$, suppose $f(x)=\sum_j\lambda_j a_j(x)$, where each a_j is a product $(1,2)$-atom and $\sum_j|\lambda_j|<\infty$. Then noting that Lemma 2.2, it is enough to show that for every product $(1,2)$-atom a, there exists a constant C independent of a, such that

$$\|a_{\mathcal{N},g}\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}\le C. \tag{20}$$

Suppose that

$$a = a(x_1,x_2) = \sum_{R\in m(\Omega)} a_R = \sum_{R\in m(\Omega)}(L_1\otimes L_2)b_R$$

is a product $(1,2)$-atom supported in some open set Ω of $\mathbb{R}^n\times\mathbb{R}^m$. For any $R=I\times J\in m(\Omega)$, let $l(I), l(J)$ be the side-length of I and J, respectively. Suppose I_1 is the biggest dyadic cube containing I such that $I_1\times J\subset\Omega^*=\{x\in\mathbb{R}^n\times\mathbb{R}^m:\mathcal{M}(\mathcal{X}_\Omega)(x)>1/2\}$, then $I_1\times J\in m_1(\Omega^*)$. Let J_1 be the biggest dyadic cube such that $J_1\supseteq J$ and $I_1\times J_1\subset\Omega^{**}=\{x\in\mathbb{R}^n\times\mathbb{R}^m:\mathcal{M}(\mathcal{X}_{\Omega^*})(x)>1/2\}$. Let $\widetilde{R}$ be the 10-fold dilate of $I_1\times J_1$ concentric with $I_1\times J_1$. Obviously, the boundedness of the strong maximal function shows that $|\cup\widetilde{R}|\le C|\Omega^{**}|\le C|\Omega^*|\le C|\Omega|$. Set

$$\|a_{\mathcal{N},g}\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}=\int_{\cup\widetilde{R}}a_{\mathcal{N},g}(x)dx+\int_{(\cup\widetilde{R})^c}a_{\mathcal{N},g}(x)dx. \tag{21}$$

Then, (19) and the size condition of $(1,2)$-atom a tell us

$$\int_{\cup\widetilde{R}}a_{\mathcal{N},g}(x)dx\le C|\cup\widetilde{R}|^{1/2}\|a_{\mathcal{N},g}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\le C|\Omega|^{1/2}\|a\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\le C. \tag{22}$$

To estimate the second part of the right hand in (21), we write

$$\int_{(\cup\widetilde{R})^c}a_{\mathcal{N},g}(x)dx\le\sum_{R\in m(\Omega)}\int_{x_1\notin10I_1}(a_R)_{\mathcal{N},g}(x)dx+\sum_{R\in m(\Omega)}\int_{x_2\notin10J_1}(a_R)_{\mathcal{N},g}(x)dx=\mathrm{D}+\mathrm{E}.$$

We only estimate the term D, since the estimate of E is similarly. Let

$$\mathrm{D}=\sum_{R\in m(\Omega)}\Big(\int_{x_1\notin10I_1}\int_{x_2\in10J}+\int_{x_1\notin10I_1}\int_{x_2\notin10J}\Big)(a_R)_{\mathcal{N},g}(x)dx=\mathrm{D}_1+\mathrm{D}_2.$$

For D_1, by using Hölder's inequality and the L^2 boundedness of g function on $\mathbb{R}^m$, we have

$$\begin{aligned}\mathrm{D}_1\quad&\le C\sum_{R\in m(\Omega)}|J|^{1/2}\int_{x_1\notin10I_1}\|(a_R)_{\mathcal{N},g}(x_1,\cdot)\|_{L^2(\mathbb{R}^m)}dx_1\\&\le C\sum_{R\in m(\Omega)}|J|^{1/2}\int_{x_1\notin10I_1}\Big(\int_{\mathbb{R}^m}\sup_{|x_1-y_1|<t_1,t_1<l(I)}|e^{-t_1\sqrt{L_1}}a_R(y_1,x_2)|^2dx_2\Big)^{1/2}dx_1\\&\quad+C\sum_{R\in m(\Omega)}|J|^{1/2}\int_{x_1\notin10I_1}\Big(\int_{\mathbb{R}^m}\sup_{|x_1-y_1|<t_1,t_1\ge l(I)}|e^{-t_1\sqrt{L_1}}a_R(y_1,x_2)|^2dx_2\Big)^{1/2}dx_1\\&=\mathrm{D}_{1,1}+\mathrm{D}_{1,2}.\end{aligned}$$

Let x_I be the center of cube I. Noting that $x_1\notin10I_1$, $z_1\in I$, if $|x_1-y_1|<t_1<l(I)$, then $|y_1-z_1|\sim|x_1-x_I|$. It follows from estimate (18) that

$$\begin{aligned}&|e^{-t_1\sqrt{L_1}}a_R(\cdot,x_2)(y_1)|\le C\int_{\mathbb{R}^n}\frac{t_1}{(t_1+|y_1-z_1|)^{n+1}}|a_R(z_1,x_2)|dz_1\\&\le C\frac{l(I)}{|x_1-x_I|^{n+1}}\|a_R(\cdot,x_2)\|_{L^1(\mathbb{R}^n)}\le C|I|^{1/2}\frac{l(I)}{|x_1-x_I|^{n+1}}\|a_R(\cdot,x_2)\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Thus, by Lemma 2.1 and the size condition of product $(1,2)$-atom, we can obtain that

$$\begin{aligned}\mathrm{D}_{1,1}\quad&\le\quad C\sum_{R\in m(\Omega)}|J|^{1/2}|I|^{1/2}\int_{x_1\notin10I_1}\frac{l(I)}{|x_1-x_I|^{n+1}}dx_1\|a_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\\&\le\quad C\sum_{R\in m(\Omega)}|J|^{1/2}|I|^{1/2}\|a_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\int_{10l(I_1)}^{\infty}\frac{l(I)}{r^{n+1}}r^{n-1}dr\\&\le\quad C\sum_{R\in m(\Omega)}|R|^{1/2}\|a_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\frac{l(I)}{l(I_1)}\\&\le\quad C\Big(\sum_{R\in m(\Omega)}|R|\gamma_1^{-2}(R)\Big)^{1/2}\Big(\sum_{R\in m(\Omega)}\|a_R\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\Big)^{1/2}\le C.\end{aligned}$$

For the term $D_{1,2}$, since $x_1 \notin 10I_1$, $|x_1 - y_1| < t_1$, $l(I) \le t_1$ and $z_1 \in I$, it is easy to check that $t_1 + |y_1 - z_1| \ge |x_1 - x_I|/2$. Therefore, we apply the definition of the product $(1,2)$-atom to obtain

$$\begin{aligned}
|e^{-t_1\sqrt{L_1}} a_R(y_1,x_2)| &\le \Big(\frac{l(I)}{t_1}\Big)^2 |t_1^2 L_1 e^{-t_1\sqrt{L_1}} l(I)^{-2}(L_1^0 \otimes L_2^1) b_R(y_1,x_2)| \\
&\le C\Big(\frac{l(I)}{t_1}\Big)^2 \int_{\mathbb{R}^n} \frac{t_1}{(t_1 + |y_1 - z_1|)^{n+1}} \Big| l(I)^{-2}(L_1^0 \otimes L_2^1) b_R(z_1,x_2)\Big| dz_1 \\
&\le C \frac{l(I)}{|x_1 - x_I|^{n+1}} \|l(I)^{-2}(L_1^0 \otimes L_2^1) b_R(\cdot,x_2)\|_{L^1(\mathbb{R}^n)}. \qquad (23)
\end{aligned}$$

Thus, combining with Lemma 2.1, we have

$$\begin{aligned}
D_{1,2} &\le C \sum_{R\in m(\Omega)} |J|^{1/2}|I|^{1/2} \int_{x_1 \notin 10I_1} \frac{l(I)}{|x_1 - x_I|^{n+1}} dx_1 \|l(I)^{-2}(L_1^0 \otimes L_2^1) b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \\
&\le C \sum_{R\in m(\Omega)} |J|^{1/2}|I|^{1/2} \|l(I)^{-2}(L_1^0 \otimes L_2^1) b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \int_{10l(I_1)}^{\infty} \frac{l(I)}{r^2} dr \\
&\le C \sum_{R\in m(\Omega)} |R|^{1/2} \|l(I)^{-2}(L_1^0 \otimes L_2^1) b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \frac{l(I)}{l(I_1)} \\
&\le C\Big(\sum_{R\in m(\Omega)} |R|\gamma_1^{-2}(R)\Big)^{1/2} \Big(\sum_{R\in m(\Omega)} l(I)^{-4} \|(L_1^0 \otimes L_2^1) b_R\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\Big)^{1/2} \\
&\le C|\Omega|^{1/2}|\Omega|^{-1/2} \le C.
\end{aligned}$$

Hence, the estimates of $D_{1,1}$ and $D_{1,2}$ show that $D_1 \le C$.

Let us estimate the term D_2. Suppose x_J is the center of J, write

$$\begin{aligned}
(a_R)^2_{\mathcal{N},g}(x) \le\ & \sup_{|x_1-y_1|<t_1, t_1 \le l(I)} \Big(\int_0^{l(J)} + \int_{l(J)}^{\infty}\Big) |e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2} e^{-t_2\sqrt{L_2}} a_R(y_1,x_2)|^2 \frac{dt_2}{t_2} \\
& + \sup_{|x_1-y_1|<t_1, t_1 > l(I)} \Big(\int_0^{l(J)} + \int_{l(J)}^{\infty}\Big) |e^{-t_1\sqrt{L_1}} \otimes t_2\sqrt{L_2} e^{-t_2\sqrt{L_2}} a_R(y_1,x_2)|^2 \frac{dt_2}{t_2} \\
=\ & D_{2,1} + D_{2,2} + D_{2,3} + D_{2,4}.
\end{aligned}$$

Let

$$\begin{aligned}
A_1 &= \int_0^{l(J)} |t_2\sqrt{L_2} e^{-t_2\sqrt{L_2}} a_R(y_1,x_2)|^2 \frac{dt_2}{t_2}, \\
A_2 &= \int_{l(J)}^{\infty} |t_2\sqrt{L_2} e^{-t_2\sqrt{L_2}} a_R(y_1,x_2)|^2 \frac{dt_2}{t_2}.
\end{aligned}$$

Since $x_2 \notin 10J$, when $t_2 \le l(J)$, it follows from (18) that

$$\begin{aligned}
A_1 &\le C \int_0^{l(J)} \Big(\int_J \frac{t_2}{(t_2 + |z - x_2|)^{m+1}} |a_R(y_1,z)| dz\Big)^2 \frac{dt_2}{t_2} \\
&\le C \frac{1}{|x_2 - x_J|^{2(m+1)}} \|a_R(y_1,\cdot)\|^2_{L^1(\mathbb{R}^m)} \int_0^{l(J)} t_2 dt_2 \\
&= C \frac{l(J)^2}{|x_2 - x_J|^{2(m+1)}} \|a_R(y_1,\cdot)\|^2_{L^1(\mathbb{R}^m)}.
\end{aligned}$$

When $t_2 > l(J)$, by the definition of the product $(1,2)$-atom as well as (18), we obtain

$$
\begin{aligned}
A_2 \ &\leq \ \int_{l(J)}^{\frac{|x_2-x_J|}{4}} |\Big(\frac{l(J)}{t_2}\Big)^2 (t_2\sqrt{L_2})^3 e^{-t_2\sqrt{L_2}} l(J)^{-2}(L_1^1\otimes L_2^0)b_R(y_1,x_2)|^2\frac{dt_2}{t_2} \\
&\quad + \int_{\frac{|x_2-x_J|}{4}}^{\infty} |\Big(\frac{l(J)}{t_2}\Big)^2 (t_2\sqrt{L_2})^3 e^{-t_2\sqrt{L_2}} l(J)^{-2}(L_1^1\otimes L_2^0)b_R(y_1,x_2)|^2\frac{dt_2}{t_2} \\
&\leq \ C\int_{l(J)}^{\frac{|x_2-x_J|}{4}} \Big(\int_J \frac{l(J)^2 t_2}{(t_2+|z-x_2|)^{m+1}}|l(J)^{-2}(L_1^1\otimes L_2^0)b_R(y_1,z)|dz\Big)^2\frac{dt_2}{t_2^5} \\
&\quad + C\int_{\frac{|x_2-x_J|}{4}}^{\infty} \Big(\int_J \frac{l(J)^2 t_2}{(t_2+|z-x_2|)^{m+1}}|l(J)^{-2}(L_1^1\otimes L_2^0)b_R(y_1,z)|dz\Big)^2\frac{dt_2}{t_2^5} \\
&\leq \ C\frac{l(J)^4}{|x_2-x_J|^{2(m+1)}}\|l(J)^{-2}(L_1^1\otimes L_2^0)b_R(y_1,\cdot)\|^2_{L^1(\mathbb{R}^m)}\Big(\int_{l(J)}^{\infty}\frac{1}{t_2^3}dt_2\Big) \\
&\leq \ C\frac{l(J)^2}{|x_2-x_J|^{2(m+1)}}\|l(J)^{-2}(L_1^1\otimes L_2^0)b_R(y_1,\cdot)\|^2_{L^1(\mathbb{R}^m)}.
\end{aligned}
$$

Then, by the estimates of A_1 and A_2, we can get the estimates of $\mathrm{D}_{2,i}$, $i=1,2,3,4$. For example, in the following, we estimate $\mathrm{D}_{2,4}$. According to the estimate of A_2 and (23),

$$
\begin{aligned}
\mathrm{D}_{2,4} \ &\leq \ \sup_{|y_1-x_1|<t_1,t_1>l(I)} \frac{l(J)^2}{|x_2-x_J|^{2(m+1)}}\|l(J)^{-2}(e^{-t_1\sqrt{L_1}}L_1^1\otimes L_2^0)b_R(y_1,\cdot)\|^2_{L^1(\mathbb{R}^m)} \\
&\leq \ C\frac{l(J)^2}{|x_2-x_J|^{2(m+1)}}\frac{l(I)^2}{|x_1-x_I|^{2(n+1)}}\|l(J)^{-2}l(I)^{-2}(L_1^0\otimes L_2^0)b_R(y_1,\cdot)\|^2_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}.
\end{aligned}
$$

Taking together the estimates of $\mathrm{D}_{2,i}$, $i=1,2,3,4$, we have

$$
(a_R)_{\mathcal{N},g}(x) \leq C\frac{l(I)}{|x_1-x_I|^{n+1}}\frac{l(J)}{|x_2-x_J|^{m+1}}|R|^{1/2}\sum_{k,j=0}^{1} l(I)^{2k-2}l(J)^{2j-2}\|(L_1^k\otimes L_2^j)b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}.
$$

Hence, by Lemma 2.1 and the definition of product $(1,2)$-atom, as well as Hölder's inequality, we obtain

$$
\begin{aligned}
\mathrm{D}_2 \ &\leq \ \sum_{R\in m(\Omega)}\frac{l(I)}{l(I_1)}|R|^{1/2}\sum_{k,j=0}^{1} l(I)^{2k-2}l(J)^{2j-2}\|(L_1^k\otimes L_2^j)b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \\
&\leq \ C\Big(\sum_{R\in m(\Omega)}|R|\gamma_1^{-2}(R)\Big)^{1/2}\Big(\sum_{R\in m(\Omega)}\sum_{k,j=0}^{1} l(I)^{4k-4}l(J)^{4j-4}\|(L_1^k\otimes L_2^j)b_R\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\Big)^{1/2} \\
&\leq \ C|\Omega|^{-1/2}|\Omega|^{1/2}\leq C.
\end{aligned}
$$

Combining with the estimate of D_1, we estimate the term D. Then together with (22), we know that (20) is proved. Thus, (17) holds. Therefore, (10) is proved. Hence, the proof of Theorem 3.1 is finished. □

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