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A modified Tikhonov regularization method for a Cauchy problem of a time fractional diffusion equation

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Abstract. In this paper, we consider a Cauchy problem of the time fractional diffusion equation (TFDE) in $x \in [0, L]$. This problem is ubiquitous in science and engineering applications. The ill-posedness of the Cauchy problem is explained by its solution in frequency domain. Furthermore, the problem is formulated into a minimization problem with a modified Tikhonov regularization method. The gradient of the regularization functional based on an adjoint problem is deduced and the standard conjugate gradient method is presented for solving the minimization problem. The error estimates for the regularized solutions are obtained under H^p norm priori bound assumptions. Finally, numerical examples illustrate the effectiveness of the proposed method.

§1 Introduction

Time fractional diffusion equations have attracted wide attentions in the recent decade which can be used to describe anomalous diffusion phenomena (superdiffusion and subdiffusion phenomena) instead of classical diffusion procedure. If the initial concentration distribution and boundary conditions are given, a complete recovery of the unknown solution is attainable from solving a well-posed forward problem [7,8]. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We may only know the noisy data on a part of the boundary or at some interior points of the concerned domain. These situations will lead to inverse problems, i.e. fractional inverse diffusion problems (FIDP) in finite domain or half space. The research of FIDP in half space can be found in [6, 10, 13, 16, 19, 22–24]. For the case of finite domain, we usually aim to determine the Cauchy data on inaccessible boundary from the known data of accessible boundary.

In the early paper [12] about the Cauchy problems, Murio solved time fractional inverse heat problems based on space marching and finite difference method, but without analysis.

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Zheng et al. in [23, 25] solved the Cauchy problems of the time fractional diffusion equations on a strip domain with a Fourier truncation method and a convolution regularization method. They showed the ill-posedness of the Cauchy problems and derived error estimates for the approximations under a prior bound of the solution. Xu et al. in [21] proved a conditional stability of a Cauchy problem with a special time fractional derivative order $\alpha = 1/2$. In [1], a kernel-based meshless method has been proposed for a Cauchy problem of the time-fractional diffusion equation on a one-dimensional bounded domain. The recovery of a nonlinear boundary condition from the lateral Cauchy data was studied in [14] by using an integral equation approach and a convergent fixed point iteration method. Wei et al. in [18] used a boundary element method with a generalized Tikhonov regularization to solve a Cauchy problem. In [5], Liu et al. changed the fractional diffusion equation into an ordinary differential equation and constructed a regularizing scheme by a mollified operator for the reconstruction of boundary flux in a finite slab sideways problem. In [20], Xiong et al. proved a conditional stability estimate on the solution of a time fractional diffusion equation, then solved the Cauchy problem with a modified Tikhonov method. Recently, in [15], A. Taghavi et al. presented a convergent numerical algorithm for solving a time fractional inverse heat conduction problem, which is based on the finite difference scheme.

From a theoretical point of view, the recovery of unknown solutions is more difficult when the spatial position of the unknown solutions is far away from the location of the observation data. Especially, it is not easy to get the convergent regularized solutions on the inaccessible boundary $x = L$. The authors in [20, 25] gave convergent approximation solutions on $x = L$ under H^p norm priori bound assumption ($\|u(L,\cdot)\|_{H^p} < \infty$) and in interior domain under L^2 norm priori bound assumption ($\|u(L,\cdot)\|_{L^2} < \infty$). In this paper, we solve a Cauchy problem of a TFDE in $\{x|x \in (0, L]\}$ through formulating it into a minimization problem with a modified Tikhonov regularization method with H^p penalty functional. The standard conjugate gradient method is applied with the adjoint problem. Moreover, error estimates for any $\{x|x \in (0, L]\}$ is presented under H^p norm priori bound assumption. From the definition of H^p norm, L^2 norm is a special case of H^p norm, i.e., $p = 0$. According to Theorem 4.4, with a proper p, the regularized solution can have higher convergence rates than that under L^2 norm priori assumption.

The reminder of this paper is constructed as follows. In Section 2, we introduce some preliminaries which will be needed in the following sections. The ill-posedness of the Cauchy problem is studied in Section 3. We reformulate the Cauchy problem into a minimization problem and give the error estimates of the regularized solutions in Section 4. Section 5 is devoted to a conjugate gradient algorithm and two numerical examples are presented to show the validity of the algorithm. Finally, we give a conclusion in Section 6.

§2 Preliminary

Definition 2.1. The Fourier transform of a continuous function $h(t)$ absolutely integrable in $(-\infty, +\infty)$ is defined by

$$\widehat{h}\left(\xi\right)=F\{h\left(t\right);\xi\}=\int_{-\infty}^{\infty}e^{-i\xi t}h\left(t\right)dt.$$

The original $h(t)$ can be restored from its Fourier transform $\widehat{h}(\xi)$ with the help of inverse Fourier transform

$$h\left(t\right)=F^{-1}\{\widehat{h}\left(\xi\right);t\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{h}\left(\xi\right)e^{i\xi t}d\xi.$$

Definition 2.2. [2]. Assume $0<p<\infty$ and $u\in L^2\left(\mathbb{R}\right)$, then $u\in H^p\left(\mathbb{R}\right)$ if $\left(1+|\xi|^p\right)\widehat{u}\left(\xi\right)\in L^2\left(\mathbb{R}\right)$. For a noninteger p, we set

$$\|u\|_{H^p}:=\|\left(1+|\xi|^p\right)\widehat{u}\left(\xi\right)\|_{L^2(\mathbb{R})}.$$

Extend a function $u\in L^2(0,T)$ to the whole line $-\infty<t<+\infty$ with zero to the extension part, then $u\in H^p(0,T)$ if the extended function belongs to $H^p\left(\mathbb{R}\right)$. And we have

$$\|u\|_{H^p(0,T)}:=\|u\|_{H^p}.$$

Definition 2.3. [4]. Let $u(t)\in L(0,T)$. The Riemann-Liouville fractional left-sided integral $(I^\alpha_{0+}u)(t)$ and right-sided integral $(I^\alpha_{T-}u)(t)$ of the order $\alpha(\alpha\in(0,1])$ are defined by

$$(I^\alpha_{0+,t}u)(t):=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}u(s)ds\qquad(0<t\le T)$$

and

$$(I^\alpha_{T-,t}u)(t):=\frac{1}{\Gamma\left(\alpha\right)}\int_t^T\left(s-t\right)^{\alpha-1}u(s)ds\qquad(0\le t<T),$$

respectively. Here $\Gamma(\beta)$ is the Gamma function.

Definition 2.4. [4]. If $u(t)\in AC[0,T]$ ($AC[0,T]$ is the space of functions $u(t)$ which are absolutely continuous on $[0,T]$.), then for $\alpha(\alpha\in[0,1])$ the Caputo fractional left-sided derivative $({}^CD^\alpha_{0+}u)(t)$ and right-sided derivative $({}^CD^\alpha_{T-}u)(t)$ are defined by

$$({}^CD^\alpha_{0+,t}u)(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}u^{'}(\tau)d\tau, & 0\le\alpha<1,\\ v^{'}(t), & \alpha=1,\end{cases}$$

and

$$({}^CD^\alpha_{T-,t}u)(t)=\begin{cases}\frac{-1}{\Gamma(1-\alpha)}\int_t^T(\tau-t)^{-\alpha}u^{'}(\tau)d\tau, & 0\le\alpha<1,\\ -v^{'}(t), & \alpha=1.\end{cases}$$

Lemma 2.5. *[17]. Suppose $u(t),v(t)\in AC[0,T]$, for $0<\alpha<1$, then*

$$\int_0^T\left({}^CD^\alpha_{0+,t}u\right)(t)\ v(t)dt=-u(0)\left(I^{1-\alpha}_{T-,t}v\right)(0)+\left(I^{1-\alpha}_{0+,t}u\right)(T)\ v\left(T\right)+\int_0^Tu(t)\ \left({}^CD^\alpha_{T-,t}v\right)(t)dt.$$

§3 Mathematical formulation and ill-posedness of Cauchy problem

At the beginning of this section, we give the following note for the paper: in order to apply the Fourier transform, we extend all the functions to the whole line $-\infty<t<+\infty$ with zero for the extension part and assume that all involved functions are L^2 in $\mathbb{R}$ with respect to t. Here, and in the following sections, $\|\cdot\|$ denotes the L^2 norm, i.e.

$$\|f(t)\|=\left(\int_{\mathbb{R}}|f|^2dt\right)^{\frac{1}{2}}.$$

Consider the following Cauchy problem of the time fractional diffusion equation

$$\begin{cases} \left({}^C D^{\alpha}_{0^+,t} u\right)(x,t) = & u_{xx}(x,t), & 0<x<L,\ 0<t\le T, \\ u(x,0) = & 0, & 0<x<L,\ t=0, \\ u(0,t) = & f(t), & x=0,\ 0<t\le T, \\ u_x(0,t) = & g(t), & x=0,\ 0<t\le T, \end{cases} \tag{1}$$

where $\alpha (0<\alpha\le 1)$. The Cauchy problem is to solve the unknown solutions $u(x,t), u_x(x,t) (0<x\le L)$ from the given data $f(t), g(t)$. In reality, the measurement data $f(t)$, $g(t)$ contain noises and the solutions, therefore, have to be sought from the noisy data functions.

Using Fourier transform to (1) with respect to t, we get

$$\begin{cases} (i\xi)^{\alpha}\widehat{u}(x,\xi) - (i\xi)^{\alpha-1}u(x,0) = & \widehat{u}_{xx}(x,\xi), & 0<x<L,\ -\infty<\xi<+\infty, \\ \widehat{u}(0,\xi) = & \widehat{f}(\xi), & x=0,\ -\infty<\xi<+\infty, \\ \widehat{u}_x(0,\xi) = & \widehat{g}(\xi), & x=0,\ -\infty<\xi<+\infty, \end{cases}$$

where ξ is the frequency parameter. When ξ is fixed, the above formula can be regarded as an ordinary differential equation with respect to x, and the solution in frequency domain can be obtained from the initial value condition of (1),

$$\widehat{u}(x,\xi) = \cosh(\eta(\xi)x)\widehat{f}(\xi) + \frac{\sinh(\eta(\xi)x)}{\eta(\xi)}\widehat{g}(\xi), \tag{2}$$

$$\widehat{u_x}(x,\xi) = \eta(\xi)\sinh(\eta(\xi)x)\widehat{f}(\xi) + \cosh(\eta(\xi)x)\widehat{g}(\xi), \tag{3}$$

where $\eta(\xi) = (i\xi)^{\alpha/2} = |\xi|^{\alpha/2}\left(\cos\left(\frac{\pi}{4}\alpha\right) + i\ \mathrm{sign}(\xi)\ \sin\left(\frac{\pi}{4}\alpha\right)\right)$. When $\xi=0$, the above expression has its meaning because $\frac{\sinh(\eta(\xi)x)}{\eta(\xi)}$ approaches x as ξ tends to zero.

Now, we will explain the ill-posedness of this Cauchy problem. For any $x (0<x\le L)$, the values of $|\cosh(\eta(\xi)x)|$ and $|\frac{\sinh(\eta(\xi)x)}{\eta(\xi)}|$ are unbounded as $|\xi|\to\infty$. In order to get a solution $u(x,t)\in L^2(\mathbb{R})$, the Cauchy data $\widehat{f}$ and $\widehat{g}$ must decay rapidly to zero as $|\xi|\to\infty$. However, if $\widehat{f}$ and $\widehat{g}$ contain noise, such a decay is less likely to occur. Thus, the small perturbations of $\widehat{f}$ and $\widehat{g}$ in high frequency components will be amplified by $|\cosh(\eta(\xi)x)|$ and $|\frac{\sinh(\eta(\xi)x)}{\eta(\xi)}|$, respectively. Especially, when $|\xi|$ is big enough, the perturbation of $\widehat{f}$ in high frequency components is more sensitive to the solution than that of $\widehat{g}$, because $|\cosh(\eta(\xi)x)|$ is bigger than $|\frac{\sinh(\eta(\xi)x)}{\eta(\xi)}|$. Similarly to (2), the flux $\widehat{u_x}(x,\xi)$ in (3) does not depend continuously on the given Cauchy data. Thus the Cauchy problem is ill-posed and some kinds of regularization techniques must be required for the stable numerical reconstruction of the solutions.

§4 A modified Tikhonov regularization method and error estimates

4.1 A modified Tikhonov regularization method

Assume that there exists a constant $M>0$ such that the following priori bound exists

$$\begin{aligned} &\|u(L,t)\|_{H^p} = \|(1+|\xi|^p)\widehat{u}(L,\xi)\| \\ \le\ & \left\|(1+|\xi|^p)\cosh(\eta(\xi)L)\widehat{f}(\xi)\right\| + \left\|(1+|\xi|^p)\frac{\sinh(\eta(\xi)L)}{\eta(\xi)}\widehat{g}(\xi)\right\| < M. \end{aligned} \tag{4}$$

For a fixed $x_0 \in (0, L]$, let $h^{(x_0)}(t) = u(x_0, t)$, $h_x^{(x_0)}(t) = u_x(x_0, t)$. In the absence of confusion, we use $h(t)$ and $h_x(t)$ instead of $h^{(x_0)}(t)$ and $h_x^{(x_0)}(t)$, respectively. Then the inverse problem we'll study below is to recover $h(t)$ and $h_x(t)$ from the given $f(t)$ and $g(t)$. We consider a modified Tikhonov regularization method with H^p penalty functional and define the following quadratic functional

$$J(h(t)) = \frac{1}{2}\|Ah(t) - f(t)\|_{L^2}^2 + \frac{\epsilon}{2}\|h(t)\|_{H^p}^2. \tag{5}$$

The positive constant ϵ is the regularization parameter and $A : h(t) \to u(0,t)$ is the forward operator of the following problem

$$\begin{cases} ({}^C D_{0^+,t}^{\alpha} u)(x,t) = & u_{xx}(x,t), & 0 < x < x_0,\ 0 < t \le T, \\ u(x,0) = & 0, & 0 < x < x_0,\ t = 0, \\ u(x_0,t) = & h(t), & x = x_0,\ 0 < t \le T, \\ u_x(0,t) = & g(t), & x = 0,\ \ 0 < t \le T. \end{cases} \tag{6}$$

We now reformulate the inverse problem to the following minimization problem

$$J(h^{\epsilon}(t)) = \min_{h(t) \in H^p(0,T)} J(h(t)). \tag{7}$$

The first order necessary optimality condition of (7) takes the form

$$J'(h^{\epsilon}(t))(q(t)) = 0, \qquad \forall q(t) \in H^p(0,T), \tag{8}$$

Here $J'(h^{\epsilon}(t))(q(t))$ is the gradient of $J(h(t))$, which is defined through the Gâteaux differential of $J(h(t))$ at $h^{\epsilon}(t)$ along the direction $q(t)$. Note that the equality (8) is also the sufficient condition of (7) because the quadratic functional $J(h(t))$ is convex, which will be proved in Theorem 4.1.

Now the key point is how to efficiently compute the gradient of the objective functional $J(h(t))$. To this end, we introduce the adjoint state equation of the forward problem (6)

$$\begin{cases} ({}^C D_{T^-,t}^{\alpha} z)(x,t) = z_{xx}(x,t), & x \in (0, x_0),\ t \in (0,T) \\ z_x(0,t) = u(0,t) - f(t), & x = 0,\ t \in (0,T), \\ z(x_0,t) = 0, & x = x_0,\ t \in (0,T), \\ z(x,T) = 0, & x \in (0,x_0),\ t = T. \end{cases} \tag{9}$$

***Theorem* 4.1.** *Let $z(x,t)$ be the solution of the adjoint state equation (9), then there exists a unique solution $h^{\epsilon}(t) \in H^p(0,T)$ of the minimization problem (7). Furthermore, the gradient of $J(h(t))$ at $h(t)$ along the direction $q(t)$ can be obtained through*

$$J'(h(t))(q(t)) = \int_0^T q(t) z_x(x_0,t)\,dt + \epsilon \int_{\mathbb{R}} \Re\left(\overline{F^{-1}\{(1+|\xi|^p)\,\widehat{h}(\xi)\}} F^{-1}\{(1+|\xi|^p)\,\widehat{q}(\xi)\}\right) dt. \tag{10}$$

The symbol $\Re(\cdot)$ represents the real part of a complex function.

Proof. Let us consider the perturbation of $h(t)$, $h(t) \to \widetilde{h}(t) := h(t) + \tau q(t)$, where the real parameter τ tends to 0 and $q(t) \in H^p(\mathbb{R})$. Let $\widetilde{u}(x,t)$ be the solution of the forward problem subject to the above perturbed $\widetilde{h}(t)$. We define ω as $\omega(x,t) = \lim_{\tau \to 0} \frac{\widetilde{u}(x,t) - u(x,t)}{\tau}$. Then it is readily seen that ω is the solution of the following problem

$$\begin{cases} ({}^C D_{0^+,t}^{\alpha}\omega)(x,t) & = \omega_{xx}(x,t), & x \in (0,x_0)\,,\ t \in (0,T), \\ \omega(x,0) & = 0, & x \in (0,x_0)\,,\ t = 0, \\ \omega\,(x_0,t) & = q(t), & x = x_0,\ t \in (0,T), \\ \omega_x(0,t) & = 0, & x = 0,\ t \in (0,T). \end{cases} \tag{11}$$

By virtue of (5), we have

$$\begin{aligned} J^{'}\,(h(t))\,(q(t)) &= \lim_{\tau\to 0}\frac{J\left(\widetilde{h}(t)\right) - J\,(h(t))}{\tau} \\ &= \lim_{\tau\to 0}\frac{\int_0^T[\widetilde{u}(0,t)-f(t)]^2-[u(0,t)-f(t)]^2dt+\epsilon\int_{\mathbb{R}}(1+|\xi|^p)^2\left|\widehat{\widetilde{h}}\,(\xi)\right|^2-(1+|\xi|^p)^2\left|\widehat{h}\,(\xi)\right|^2d\xi}{2\tau} \\ &= \lim_{\tau\to 0}\int_0^T\frac{\widetilde{u}(0,t)-u(0,t)}{\tau}\,\frac{\widetilde{u}(0,t)+u(0,t)-2f(t)}{2}+\epsilon\int_{\mathbb{R}}(1+|\xi|^p)^2\,\frac{\widehat{\widetilde{h}}\,(\xi)\,\overline{\widehat{\widetilde{h}}\,(\xi)}-\widehat{h}\,(\xi)\,\overline{\widehat{h}\,(\xi)}}{2\tau}d\xi \\ &= \int_0^T\omega(0,t)\,(u(0,t)-f(t))\,dt+\lim_{\tau\to 0}\epsilon\int_{\mathbb{R}}(1+|\xi|^p)^2\,\frac{\widehat{\widetilde{h}}\,(\xi)\,\overline{\widehat{\widetilde{h}}\,(\xi)}-\widehat{h}\,(\xi)\,\overline{\widehat{h}\,(\xi)}}{2\tau}d\xi, \end{aligned}$$

where

$$\begin{aligned} &\lim_{\tau\to 0}\epsilon\int_{\mathbb{R}}(1+|\xi|^p)^2\,\frac{\widehat{\widetilde{h}}\,(\xi)\,\overline{\widehat{\widetilde{h}}\,(\xi)}-\widehat{h}\,(\xi)\,\overline{\widehat{h}\,(\xi)}}{2\tau}d\xi \\ =\ & \frac{\epsilon}{2}\int_{\mathbb{R}}\left((1+|\xi|^p)\,\widehat{h}\,(\xi)\,\overline{(1+|\xi|^p)\,\widehat{q}\,(\xi)}+(1+|\xi|^p)\,\widehat{q}\,(\xi)\,\overline{(1+|\xi|^p)\,\widehat{h}\,(\xi)}\right)d\xi \\ =\ & \frac{\epsilon}{2}\int_{\mathbb{R}}F^{-1}\{(1+|\xi|^p)\,\widehat{h}\,(\xi)\}\overline{F^{-1}\{(1+|\xi|^p)\,\widehat{q}\,(\xi)\}}+F^{-1}\{(1+|\xi|^p)\,\widehat{q}\,(\xi)\}\overline{F^{-1}\{(1+|\xi|^p)\,\widehat{h}\,(\xi)\}}dt \\ =\ & \epsilon\int_{\mathbb{R}}\Re\left(\overline{F^{-1}\{(1+|\xi|^p)\,\widehat{h}\,(\xi)\}}F^{-1}\{(1+|\xi|^p)\,\widehat{q}\,(\xi)\}\right)dt, \end{aligned}$$

then

$$J^{'}(h(t))\,(q(t))=\int_0^T\omega(0,t)\,(u(0,t)-f(t))\,dt+\epsilon\int_{\mathbb{R}}\Re\left(\overline{F^{-1}\{(1+|\xi|^p)\,\widehat{h}\,(\xi)\}}F^{-1}\{(1+|\xi|^p)\,\widehat{q}\,(\xi)\}\right)dt. \tag{12}$$

Similarly, the second order Gâteaux derivative of $J\,(h(t))$ is given by

$$\begin{aligned} J^{''}\,(h(t))\,(q(t),q(t)) &= \lim_{\tau\to 0}\frac{J^{'}\left(\widetilde{h}(t)\right)(q(t))-J^{'}\,(h(t))\,(q(t))}{\tau} \\ &= \int_0^T\omega^2(0,t)dt+\epsilon\int_{\mathbb{R}}\Re\left(\overline{F^{-1}\{(1+|\xi|^p)\,\widehat{q}\,(\xi)\}}F^{-1}\{(1+|\xi|^p)\,\widehat{q}\,(\xi)\}\right)dt \\ &= \int_0^T\omega^2(0,t)dt+\epsilon\int_{\mathbb{R}}(1+|\xi|^p)^2\,|\widehat{q}\,(\xi)|^2\,d\xi>0 \end{aligned} \tag{13}$$

which means that J is uniformly convex and therefore (7) admits a unique solution $h^{\epsilon}(t)$.

To prove (10), we multiply each side of the first equation in (11) by $z(x,t)$ the solution of the adjoint state equation (9). Then we integrate the resulted equation on the domain

$\Omega=[0,x_0]\times[0,T]$ and get

$$\int_\Omega \left({}^C D^\alpha_{0^+,t}\omega\right)(x,t)z(x,t)dxdt=\int_\Omega \omega_{xx}(x,t)z(x,t)dxdt.$$

According to Lemma 2.5 in Section 2, the above equation is converted to

$$\begin{aligned}0=&\int_0^{x_0} z(x,T)\ \left(I^{1-\alpha}_{0^+,t}\omega\right)(x,T)dx+\int_\Omega\left(\left({}^C D^\alpha_{T^-,t}z\right)(x,t)-z_{xx}(x,t)\right)\omega(x,t)dxdt\\ &-\int_0^{x_0}\omega(x,0)\ \left(I^{1-\alpha}_{T^-,t}z\right)(x,0)dx-\int_0^T[\omega_x(x,t)\ z\,(x,t)]_{x=0}^{x=x_0}dt+\int_0^T[\omega(x,t)\ z_x(x,t)]_{x=0}^{x=x_0}dt.\end{aligned}$$

Taking into account the initial and boundary conditions in (9) and (11), we get

$$\int_0^T\omega(0,t)\,(u(0,t)-f(t))\,dt=\int_0^T q(t)z_x\,(x_0,t)\,dt,$$

Substituting the above expression into (12) yields (10). □

According to (2) and Plancherel's theorem, the problem (7) can be rewritten as

$$\min\frac{1}{2}\left\|\frac{\widehat{h}(\xi)}{\cosh(\eta(\xi)x_0)}-\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)\cosh(\eta(\xi)x_0)}\widehat{g}(\xi)-\widehat{f}(\xi)\right\|^2+\frac{\epsilon}{2}\left\|(1+|\xi|^p)\widehat{h}(\xi)\right\|^2. \tag{14}$$

Let $\widehat{h^\epsilon}(\xi)$ be the minimal solution which satisfies the following Euler equation

$$\left(\frac{1}{|\cosh\left(\eta\left(\xi\right)x_0\right)|^2}+\epsilon\left(1+|\xi|^p\right)^2\right)\widehat{h^\epsilon}\left(\xi\right)=\frac{\sinh\left(\eta\left(\xi\right)x_0\right)}{\eta\left(\xi\right)|\cosh\left(\eta\left(\xi\right)x_0\right)|}\widehat{g}\left(\xi\right)+\frac{1}{\overline{\cosh\left(\eta\left(\xi\right)x_0\right)}}\widehat{f}(\xi).$$

Thus the regularized solutions $\widehat{h^\epsilon}\left(\xi\right)$ and $\widehat{h^\epsilon_x}\left(\xi\right)$ can be given as

$$\begin{aligned}\widehat{h^\epsilon}\left(\xi\right)&=\frac{\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}}{1+\epsilon\left(1+|\xi|^p\right)^2|\cosh\left(\eta\left(\xi\right)x_0\right)|^2}\widehat{g}\left(\xi\right)+\frac{\cosh\left(\eta\left(\xi\right)x_0\right)}{1+\epsilon\left(1+|\xi|^p\right)^2|\cosh\left(\eta\left(\xi\right)x_0\right)|^2}\widehat{f}\left(\xi\right),\\ \widehat{h^\epsilon_x}\left(\xi\right)&=\frac{\cosh\left(\eta\left(\xi\right)x_0\right)}{1+\epsilon\left(1+|\xi|^p\right)^2|\cosh\left(\eta\left(\xi\right)x_0\right)|^2}\widehat{g}\left(\xi\right)+\frac{\eta\left(\xi\right)\sinh\left(\eta\left(\xi\right)x_0\right)}{1+\epsilon\left(1+|\xi|^p\right)^2|\cosh\left(\eta\left(\xi\right)x_0\right)|^2}\widehat{f}\left(\xi\right)\\ &\quad-\frac{\epsilon(1+|\xi|^p)^2}{(1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2)^2}\left(\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}\widehat{g}+\cosh(\eta(\xi)x_0)\widehat{f}\right)\times\\ &\quad\left(\cosh(\overline{\eta(\xi)}x_0)\eta(\xi)\sinh(\eta(\xi)x_0)+\cosh(\eta(\xi)x_0)\overline{\eta(\xi)}\sinh(\overline{\eta(\xi)}x_0)\right).\end{aligned} \tag{15}$$

4.2 Error estimates

For a clear explanation, we now distinguish between the exact Cauchy data $f(t),g(t)$ and the measured noisy Cauchy data $f^\delta(t),g^\delta(t)$:

$$\left\|f^\delta(t)-f(t)\right\|+\left\|g^\delta(t)-g(t)\right\|<\delta, \tag{16}$$

where the level of the tolerance $\delta>0$ represents a bound on the measurement error. We denote $h^{\epsilon,\delta}(t),h^{\epsilon,\delta}_x(t)$ as the regularized solutions of the Cauchy problem at $x=x_0$ with $f^\delta(t)$ and $g^\delta(t)$, and have

$$\widehat{h^{\epsilon,\delta}}\left(\xi\right)=\frac{\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}}{1+\epsilon\left(1+|\xi|^p\right)^2|\cosh\left(\eta\left(\xi\right)x_0\right)|^2}\widehat{g^\delta}\left(\xi\right)+\frac{\cosh\left(\eta\left(\xi\right)x_0\right)}{1+\epsilon\left(1+|\xi|^p\right)^2|\cosh\left(\eta\left(\xi\right)x_0\right)|^2}\widehat{f^\delta}\left(\xi\right),$$

$$
\begin{aligned}
\widehat{h_x^{\epsilon,\delta}}(\xi) \quad = \quad & \frac{\cosh\left(\eta\left(\xi\right)x_0\right)}{1+\epsilon\left(1+|\xi|^p\right)^2\left|\cosh\left(\eta\left(\xi\right)x_0\right)\right|^2}\widehat{g^\delta}\left(\xi\right)+\frac{\eta\left(\xi\right)\sinh\left(\eta\left(\xi\right)x_0\right)}{1+\epsilon\left(1+|\xi|^p\right)^2\left|\cosh\left(\eta\left(\xi\right)x_0\right)\right|^2}\widehat{f^\delta}\left(\xi\right)\\
&-\frac{\epsilon(1+|\xi|^p)^2}{(1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2)^2}\left(\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}\widehat{g^\delta}+\cosh(\eta(\xi)x_0)\widehat{f^\delta}\right)\times\\
&\left(\cosh(\overline{\eta(\xi)}x_0)\eta(\xi)\sinh(\eta(\xi)x_0)+\cosh(\eta(\xi)x_0)\overline{\eta(\xi)}\sinh(\overline{\eta(\xi)}x_0)\right). \qquad (17)
\end{aligned}
$$

In order to obtain L^2 estimates for the differences $h^{\epsilon,\delta}(t)-h(t)$ and $h_x^{\epsilon,\delta}(t)-h_x(t)$, we can equivalently, in view of Parseval relation, estimate L^2 norm of the Fourier transform of these quantities

$$
\left\|h^{\epsilon,\delta}(t)-h(t)\right\|^2=\int_0^{+\infty}\left|h^{\epsilon,\delta}(t)-h(t)\right|^2dt=\int_{-\infty}^{+\infty}\left|h^{\epsilon,\delta}(t)-h(t)\right|^2dt=\int_{-\infty}^{+\infty}\left|\widehat{h}^{\epsilon,\delta}(\xi)-\widehat{h}(\xi)\right|^2d\xi,
$$

$$
\left\|h_x^{\epsilon,\delta}(t)-h_x(t)\right\|^2=\int_{-\infty}^{+\infty}\left|\widehat{h}_x^{\epsilon,\delta}(\xi)-\widehat{h}_x(\xi)\right|^2d\xi.
$$

We set

$$
\begin{aligned}
\overline{C}=\max\Big\{&\sup_{\xi\in\mathbb{R}}\left|\frac{1-e^{-2x_0\eta(\xi)}}{1-e^{-2L\eta(\xi)}}\right|,\sup_{\xi\in\mathbb{R}}\left|\frac{1+e^{-2x_0\eta(\xi)}}{1+e^{-2L\eta(\xi)}}\right|,\sup_{\xi\in\mathbb{R}}\left|\frac{\eta(\xi)x_0}{\sinh(\eta(\xi)x_0)}\right|,\\
&\sup_{\xi\in\mathbb{R}}\left|\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)x_0\cosh(\eta(\xi)x_0)}\right|,\sup_{\xi\in\mathbb{R}}\left|\frac{\sinh(\eta(\xi)x_0)}{\cosh(\eta(\xi)x_0)}\right|\Big\}. \qquad (18)
\end{aligned}
$$

We note here that the positive constant $\overline{C}$ is bounded. Now we give two lemmas for error estimates.

***Lemma* 4.2.** *Let $0<\alpha\le 1$, $0<x<L$, $p>0$, $\epsilon>0$. Then*

$$
\sup_{\xi\in\mathbb{R}}\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x)|^2}\left|\frac{\sinh(\eta(\xi)x)}{\sinh(\eta(\xi)L)}\right| \le 2\overline{C}\left(r\ln\frac{1}{\epsilon}\right)^{-\frac{2p}{\alpha}}\epsilon^{r(L-x)\cos(\frac{\pi}{4}\alpha)}, \qquad (19)
$$

$$
\sup_{\xi\in\mathbb{R}}\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x)|^2}\left|\frac{\cosh(\eta(\xi)x)}{\cosh(\eta(\xi)L)}\right| \le 2\overline{C}\left(r\ln\frac{1}{\epsilon}\right)^{-\frac{2p}{\alpha}}\epsilon^{r(L-x)\cos(\frac{\pi}{4}\alpha)}, \qquad (20)
$$

where $r=\left(\dfrac{4p}{e\alpha}+(x+L)\cos\left(\frac{\pi}{4}\alpha\right)\right)^{-1}$.

Proof. Set $r=\left(\frac{4p}{e\alpha}+(x+L)\cos\left(\frac{\pi}{4}\alpha\right)\right)^{-1}$. Firstly, let's proof

$$
\sup_{\xi\in\mathbb{R}}\frac{\epsilon(1+|\xi|^p)e^{(3x-L)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}}{1+\epsilon(1+|\xi|^p)^2e^{2x|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}}\le 2\left(r\ln\frac{1}{\epsilon}\right)^{-\frac{2p}{\alpha}}\epsilon^{r(L-x)\cos(\frac{\pi}{4}\alpha)} \qquad (21)
$$

by analyzing the two cases $|\xi|^{\frac{\alpha}{2}}\ge r\ln\frac{1}{\epsilon}$ and $|\xi|^{\frac{\alpha}{2}}<r\ln\frac{1}{\epsilon}$. Secondly, by (21) and the monotonicity of the left side of (19) with respect to $|\cosh(\eta(\xi)x)|^2$, we can easily obtain the results in the lemma. Let's prove them in detail as following.

a) If $|\xi|^{\frac{\alpha}{2}}\ge r\ln\frac{1}{\epsilon}$, then

$$
\frac{\epsilon(1+|\xi|^p)e^{(3x-L)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}}{1+\epsilon(1+|\xi|^p)^2e^{2x|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}}\le\frac{e^{(x-L)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}}{1+|\xi|^p}\le\left(r\ln\frac{1}{\epsilon}\right)^{-\frac{2p}{\alpha}}\epsilon^{r(L-x)\cos(\frac{\pi}{4}\alpha)}.
$$

b) If $|\xi|^{\frac{\alpha}{2}} < r\ln\frac{1}{\epsilon}$, then

$$\frac{\epsilon(1+|\xi|^p)e^{(3x-L)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}}{1+\epsilon(1+|\xi|^p)^2e^{2x|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}} \le \epsilon(1+|\xi|^p)e^{(3x-L)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}.$$

(i) For $x\in(0,\frac{L}{3}]$, we have

$$\epsilon(1+|\xi|^p)e^{(3x-L)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)} \le \epsilon(1+|\xi|^p) \le \epsilon\left(1+(r\ln\frac{1}{\epsilon})^{\frac{2p}{\alpha}}\right).$$

(ii) For $x\in(\frac{L}{3},L]$, we have

$$\epsilon(1+|\xi|^p)e^{(3x-L)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)} \le \epsilon\left(1+(r\ln\frac{1}{\epsilon})^{\frac{2p}{\alpha}}\right)e^{(3x-L)r\ln\frac{1}{\epsilon}\cos(\frac{\pi}{4}\alpha)}$$
$$=\left(1+(r\ln\frac{1}{\epsilon})^{\frac{2p}{\alpha}}\right)\epsilon^{1-r(3x-L)\cos(\frac{\pi}{4}\alpha)}.$$

It is simple to show that

$$\max\left\{\epsilon\left(1+(r\ln\frac{1}{\epsilon})^{\frac{2p}{\alpha}}\right),\left(1+(r\ln\frac{1}{\epsilon})^{\frac{2p}{\alpha}}\right)\epsilon^{1-r(3x-L)\cos(\frac{\pi}{4}\alpha)}\right\}\le 2\left(r\ln\frac{1}{\epsilon}\right)^{-\frac{2p}{\alpha}}\epsilon^{r(L-x)\cos(\frac{\pi}{4}\alpha)}.$$

So we obtain (21).

Because $\dfrac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x)|^2}$ is monotonically increasing with respect to $|\cosh(\eta(\xi)x)|^2$ and $|\cosh(\eta(\xi)x)|\le e^{x|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}$, we have

$$\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x)|^2} \le \frac{\epsilon(1+|\xi|^p)e^{2x|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}}{1+\epsilon(1+|\xi|^p)^2e^{2x|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)}}. \tag{22}$$

Furthermore, by (18), we have

$$\left|\frac{\sinh(\eta(\xi)x)}{\sinh(\eta(\xi)L)}\right| = \left|e^{-(L-x)\eta(\xi)}\right|\left|\frac{1-e^{-2x\eta(\xi)}}{1-e^{-2L\eta(\xi)}}\right| < \overline{C}e^{-(L-x)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)} \tag{23}$$

and

$$\left|\frac{\cosh(\eta(\xi)x)}{\cosh(\eta(\xi)L)}\right| = \left|e^{-(L-x)\eta(\xi)}\right|\left|\frac{1+e^{-2x\eta(\xi)}}{1+e^{-2L\eta(\xi)}}\right| < \overline{C}e^{-(L-x)|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)} \tag{24}$$

Substitute (22), (23) and (24) into the left side of (19) and (20). Then by (21), we prove the lemma.

□

Lemma 4.3. *Let $0<\alpha\le 1$, $0<x<L$, $p>\frac{\alpha}{2}$, $0<\epsilon<(\frac{1}{2})^{\frac{1}{\mu L}}$. Then*

$$\sup_{\xi\in\mathbb{R}}\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x)|^2}\left|\frac{\eta(\xi)\cosh(\eta(\xi)x)}{\sinh(\eta(\xi)L)}\right| \le \left(\frac{2\overline{C}}{L}+3\right)\left(\mu\ln\frac{1}{\epsilon}\right)^{-(\frac{2p}{\alpha}-1)}\epsilon^{(L-x)\mu\cos(\frac{\pi}{4}\alpha)}, \tag{25}$$

$$\sup_{\xi\in\mathbb{R}}\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x)|^2}\left|\frac{\eta(\xi)\sinh(\eta(\xi)x)}{\cosh(\eta(\xi)L)}\right| \le \left(\frac{2\overline{C}}{L}+3\right)\left(\mu\ln\frac{1}{\epsilon}\right)^{-(\frac{2p}{\alpha}-1)}\epsilon^{(L-x)\mu\cos(\frac{\pi}{4}\alpha)}, \tag{26}$$

$$\sup_{\xi\in\mathbb{R}}\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x)|^2}\left|\frac{\eta(\xi)\sinh(\eta(\xi)x)}{\sinh(\eta(\xi)L)}\right| \le \left(\frac{2\overline{C}}{L}+3\right)\left(\mu\ln\frac{1}{\epsilon}\right)^{-(\frac{2p}{\alpha}-1)}\epsilon^{(L-x)\mu\cos(\frac{\pi}{4}\alpha)}, \tag{27}$$

where $\mu=\left(\dfrac{\frac{4p}{\alpha}-1}{e}+(2x+L)\cos\left(\frac{\pi}{4}\alpha\right)\right)^{-1}$.

Proof. Set $\mu = \left(\dfrac{\frac{4p}{\alpha}-1}{e} + (2x+L)\cos\left(\frac{\pi}{4}\alpha\right)\right)^{-1}$ and $p > \frac{\alpha}{2}$. We only prove the first inequality (25) and similarly we can prove (25) and (27). We illustrate (1) by discussing the cases of $|\xi|^{\alpha/2} \geq \mu \ln \frac{1}{\epsilon}$ and $|\xi|^{\alpha/2} < \mu \ln \frac{1}{\epsilon}$.

a) If $|\xi|^{\alpha/2} \geq \mu \ln \frac{1}{\epsilon}$ and ϵ is sufficiently small, i.e., $\epsilon < (\frac{1}{2})^{\frac{1}{\mu L}}$, we have $\dfrac{1+\epsilon^{2(\mu \ln \frac{1}{\epsilon})x\cos(\frac{\pi}{4}\alpha)}}{1-\epsilon^{2(\mu \ln \frac{1}{\epsilon})L\cos(\frac{\pi}{4}\alpha)}} < 3$. Then we have

$$
\begin{aligned}
&\sup_{|\xi|^{\alpha/2}\geq \mu \ln \frac{1}{\epsilon}} \frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\left|\frac{\eta(\xi)\cosh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)}\right| \\
\leq\ & \frac{|\xi|^{\alpha/2}}{1+|\xi|^p}\left|\frac{\cosh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)}\right| \leq \frac{|\xi|^{\alpha/2}}{1+|\xi|^p}\frac{e^{|\xi|^{\alpha/2}\cos(\frac{\pi}{4}\alpha)x}+e^{-|\xi|^{\alpha/2}\cos(\frac{\pi}{4}\alpha)x}}{e^{|\xi|^{\alpha/2}\cos(\frac{\pi}{4}\alpha)L}-e^{-|\xi|^{\alpha/2}\cos(\frac{\pi}{4}\alpha)L}} \\
\leq & |\xi|^{-(p-\frac{\alpha}{2})}e^{-(L-x)|\xi|^{\alpha/2}\cos(\frac{\pi}{4}\alpha)}\frac{1+e^{-2|\xi|^{\alpha/2}\cos(\frac{\pi}{4}\alpha)x}}{1-e^{-2|\xi|^{\alpha/2}\cos(\frac{\pi}{4}\alpha)L}} \\
\leq & \left(\mu \ln\frac{1}{\epsilon}\right)^{-(\frac{2p}{\alpha}-1)}\epsilon^{(L-x)\mu\cos(\frac{\pi}{4}\alpha)}\frac{1+e^{-2(\mu \ln \frac{1}{\epsilon})\cos(\frac{\pi}{4}\alpha)x}}{1-e^{-2(\mu \ln \frac{1}{\epsilon})\cos(\frac{\pi}{4}\alpha)L}} \leq 3\left(\mu \ln\frac{1}{\epsilon}\right)^{-(\frac{2p}{\alpha}-1)}\epsilon^{(L-x)\mu\cos(\frac{\pi}{4}\alpha)}.
\end{aligned}
$$

b) If $|\xi|^{\alpha/2} < \mu \ln \frac{1}{\epsilon}$, from (18), we know $\left|\dfrac{\eta(\xi)L}{\sinh(\eta(\xi)L)}\right| \leq \overline{C}$, then we have

$$
\begin{aligned}
&\sup_{|\xi|^{\alpha/2}< \mu \ln \frac{1}{\epsilon}} \frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\left|\frac{\eta(\xi)\cosh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)}\right| \\
\leq & \epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x)|^2\frac{|\eta(\xi)|}{|\sinh(\eta(\xi)L)|}|\cosh(\eta(\xi)x)| \leq \frac{\overline{C}}{L}\epsilon(1+|\xi|^p)e^{3x|\xi|^{\frac{\alpha}{2}}\cos(\frac{\pi}{4}\alpha)} \\
\leq & \frac{\overline{C}}{L}\epsilon\left(1+\left(\mu \ln\frac{1}{\epsilon}\right)^{\frac{2p}{\alpha}}\right)e^{3x(\mu \ln \frac{1}{\epsilon})\cos(\frac{\pi}{4}\alpha)} \leq \frac{\overline{C}}{L}\left(1+\left(\mu \ln\frac{1}{\epsilon}\right)^{\frac{2p}{\alpha}}\right)\epsilon^{1-3x\mu\cos(\frac{\pi}{4}\alpha)}.
\end{aligned}
$$

It is easy to verify the following inequality

$$
\left(1+(\mu \ln\frac{1}{\epsilon})^{\frac{2p}{\alpha}}\right)\epsilon^{1-3x\mu\cos(\frac{\pi}{4}\alpha)} \leq 2(\mu \ln\frac{1}{\epsilon})^{-(\frac{2p}{\alpha}-1)}\epsilon^{(L-x)\mu\cos(\frac{\pi}{4}\alpha)}.
$$

Together with a) and b), we obtain (25). □

Theorem 4.4. *Suppose $0 < x_0 \leq L$. Let $h(t)$ and $h_x(t)$ be the exact solutions of $u(x,t)$ and $u_x(x,t)$ of (1) at $x = x_0$. And let $h^{\epsilon,\delta}(t)$, $h_x^{\epsilon,\delta}(t)$ be the regularized approximations of $h(t)$, $h_x(t)$, as in (17) and (17). Assume that the measured data $f^\delta(t)$ and $g^\delta(t)$ satisfy (16) and $u(L,t)$ satisfies the prior condition (4). If $\epsilon = \frac{\delta}{M}$, then for sufficiently small δ, we have the following error estimates*

$$
\left\|h^{\epsilon,\delta}(t)-h(t)\right\| \leq \frac{x_0\overline{C}+1}{2}\sqrt{M}\sqrt{\delta}+4\overline{C}M(r\ln\frac{M}{\delta})^{-2m}(\frac{\delta}{M})^{r(L-x_0)\cos(\frac{\pi}{4}\alpha)},\ m>0, \tag{28}
$$

and

$$
\begin{aligned}
&\|h_x^{\epsilon,\delta}(t)-h_x(t)\| \\
\leq & \left(\frac{1}{2}+\frac{3\overline{C}}{2}+\overline{C}^2\right)\sqrt{M}\sqrt{\delta}+(2\overline{C}+4)M(\frac{2\overline{C}}{L}+3)(\mu\ln\frac{M}{\delta})^{-(2m-1)}(\frac{\delta}{M})^{(L-x_0)\mu\cos(\frac{\pi}{4}\alpha)}, m>\frac{1}{2},
\end{aligned}
\tag{29}
$$

where $r=\left(\frac{4m}{e}+(x_0+L)\cos\left(\frac{\pi}{4}\alpha\right)\right)^{-1}$, $\mu=\left(\frac{4m-1}{e}+(2x_0+L)\cos\left(\frac{\pi}{4}\alpha\right)\right)^{-1}$.

Proof. Because the proof is quite involved, using (15), (17) and triangle inequality, we divide the estimates of $h^{\epsilon,\delta}(t)-h(t)$ and $h_x^{\epsilon,\delta}(t)-h_x(t)$ into several steps in terms of A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2.

$$\|h^{\epsilon,\delta}(t)-h(t)\|\le A_1+B_1+C_1+D_1, \tag{30}$$

where

$$A_1=\left\|\frac{\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\left(\widehat{g^\delta}(\xi)-\widehat{g}(\xi)\right)\right\|,$$
$$B_1=\left\|\frac{\cosh(\eta(\xi)x_0)}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\left(\widehat{f^\delta}(\xi)-\widehat{f}(\xi)\right)\right\|,$$
$$C_1=\left\|\left(\frac{\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}-\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}\right)\widehat{g}(\xi)\right\|,$$
$$D_1=\left\|\left(\frac{\cosh(\eta(\xi)x_0)}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}-\cosh(\eta(\xi)x_0)\right)\widehat{f}(\xi)\right\|.$$

For A_1 and B_1, by (18), we know $\left|\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)x_0\cosh(\eta(\xi)x_0)}\right|<\overline{C}$, then we can deduce

$$\begin{aligned}A_1&\le\sup_{\xi\in\mathbb{R}}\left|\frac{\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\right|\|\widehat{g^\delta}(\xi)-\widehat{g}(\xi)\|\\&\le\sup_{\xi\in\mathbb{R}}\frac{x_0\delta}{2\sqrt{\epsilon}}\left|\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)x_0\cosh(\eta(\xi)x_0)}\right|\le\frac{x_0\overline{C}\delta}{2\sqrt{\epsilon}}\end{aligned}$$

and

$$B_1\le\sup_{\xi\in\mathbb{R}}\left|\frac{\cosh(\eta(\xi)x_0)}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\right|\|\widehat{f^\delta}(\xi)-\widehat{f}(\xi)\|\le\sup_{\xi\in\mathbb{R}}\left|\frac{\cosh(\eta(\xi)x_0)}{2\sqrt{\epsilon}(1+|\xi|^p)\cosh(\eta(\xi)x_0)}\right|\delta\le\frac{\delta}{2\sqrt{\epsilon}}.$$

For C_1 and D_1, we have

$$\begin{aligned}C_1&=\left\|\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\frac{\sinh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)}(1+|\xi|^p)\frac{\sinh(\eta(\xi)L)}{\eta(\xi)}\widehat{g}(\xi)\right\|\\&\le\sup_{\xi\in\mathbb{R}}\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\left|\frac{\sinh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)}\right|M,\end{aligned}$$

and

$$\begin{aligned}D_1&\le\left\|\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\frac{\cosh(\eta(\xi)x_0)}{\cosh(\eta(\xi)L)}(1+|\xi|^p)\cosh(\eta(\xi)L)\widehat{f}(\xi)\right\|\\&\le\sup_{\xi\in\mathbb{R}}\frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\frac{\cosh(\eta(\xi)x_0)}{\cosh(\eta(\xi)L)}M.\end{aligned}$$

Let $p=m\alpha$, $m>0$ and $r=\left(\frac{4m}{e}+(x_0+L)\cos\left(\frac{\pi}{4}\alpha\right)\right)^{-1}$. We can apply Lemma 4.2 and obtain

$$C_1\le2\overline{C}M\left(r\ln\frac{1}{\epsilon}\right)^{-2m}\epsilon^{r(L-x_0)\cos(\frac{\pi}{4}\alpha)},\ D_1\le2\overline{C}M(r\ln\frac{1}{\epsilon})^{-2m}\epsilon^{r(L-x_0)\cos(\frac{\pi}{4}\alpha)}.$$

The sum of A_1,B_1,C_1 and D_1 is

$$\|h^{\epsilon,\delta}(t)-h(t)\|\le\frac{x_0\overline{C}+1}{2}\epsilon^{-\frac{1}{2}}\delta+4\overline{C}M\left(r\ln\frac{1}{\epsilon}\right)^{-2m}\epsilon^{r(L-x_0)\cos(\frac{\pi}{4}\alpha)}.$$

Choosing $\epsilon=\frac{\delta}{M}$, we have (28).

We are now on a position to prove (29).

$$\|h_x^{\epsilon,\delta}(t)-h_x(t)\|\le A_2+B_2+C_2+D_2+E_2+F_2+G_2+H_2,$$

where

$$A_2=\left\|\frac{\cosh(\eta(\xi)x_0)}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}(\widehat{g^\delta}(\xi)-\widehat{g}(\xi))\right\|,$$
$$B_2=\left\|\frac{\eta(\xi)\sinh(\eta(\xi)x_0)}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}(\widehat{f^\delta}(\xi)-\widehat{f}(\xi))\right\|,$$
$$C_2=\left\|\frac{2\epsilon(1+|\xi|^p)^2|\eta(\xi)\sinh(\eta(\xi)x_0)\cosh(\eta(\xi)x_0)|\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}(\widehat{g^\delta}(\xi)-\widehat{g}(\xi))}{(1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2)^2}\right\|,$$
$$D_2=\left\|\frac{2\epsilon(1+|\xi|^p)^2|\eta(\xi)\sinh(\eta(\xi)x_0)\cosh(\eta(\xi)x_0)|\cosh(\eta(\xi)x_0)(\widehat{f^\delta}(\xi)-\widehat{f}(\xi))}{(1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2)^2}\right\|,$$
$$E_2=\left\|\left(\frac{\cosh(\eta(\xi)x_0)}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}-\cosh(\eta(\xi)x_0)\right)\widehat{g}(\xi)\right\|,$$
$$F_2=\left\|\left(\frac{\eta(\xi)\sinh(\eta(\xi)x_0)}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}-\eta(\xi)\sinh(\eta(\xi)x_0)\right)\widehat{f}(\xi)\right\|,$$
$$G_2=\left\|\frac{2\epsilon(1+|\xi|^p)^2|\eta(\xi)\sinh(\eta(\xi)x_0)\cosh(\eta(\xi)x_0)|\frac{\sinh(\eta(\xi)x_0)}{\eta(\xi)}\widehat{g}(\xi)}{(1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2)^2}\right\|,$$
$$H_2=\left\|\frac{2\epsilon(1+|\xi|^p)^2|\eta(\xi)\sinh(\eta(\xi)x_0)\cosh(\eta(\xi)x_0)|\cosh(\eta(\xi)x_0)\widehat{f}(\xi)}{(1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2)^2}\right\|.$$

$\mathbf{A_2}$:

$$A_2=\left\|\frac{\cosh(\eta(\xi)x_0)}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}(\widehat{g^\delta}(\xi)-\widehat{g}(\xi))\right\|\le\frac{\delta}{2\sqrt{\epsilon}}.$$

$\mathbf{B_2}$: Choosing $p\ge\frac{\alpha}{2}$, by (18), we have $\left|\frac{\sinh(\eta(\xi)x_0)}{\cosh(\eta(\xi)x_0)}\right|<\overline{C}$, we have

$$B_2\le\frac{1}{2\sqrt{\epsilon}}\frac{|\xi|^{\alpha/2}}{1+|\xi|^p}\left|\frac{\sinh(\eta(\xi)x_0)}{\cosh(\eta(\xi)x_0)}\right|\delta\le\frac{\overline{C}\delta}{2\sqrt{\epsilon}}.$$

$\mathbf{C_2}$:

$$\begin{aligned}C_2&\le\sup_{\xi\in\mathbb{R}}\frac{2\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\frac{\frac{|\sinh(\eta(\xi)x_0)|^2}{|\cosh(\eta(\xi)x_0)|}}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\|\widehat{g^\delta}(\xi)-\widehat{g}(\xi)\|\\&\le\sup_{\xi\in\mathbb{R}}\frac{|\sinh(\eta(\xi)x_0)|^2}{\sqrt{\epsilon}|\cosh(\eta(\xi)x_0)|^2}\delta\le\frac{\overline{C}^2\delta}{\sqrt{\epsilon}}\quad\left(by\ \frac{|\sinh(\eta(\xi)x_0)|}{|\cosh(\eta(\xi)x_0)|}<\overline{C}\right).\end{aligned}$$

$\mathbf{D_2}$:

$$\begin{aligned}D_2&\le\sup_{\xi\in\mathbb{R}}\frac{2\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\frac{|\eta(\xi)\sinh(\eta(\xi)x_0)|}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2}\|\widehat{f^\delta}(\xi)-\widehat{f}(\xi)\|\\&\le\sup_{\xi\in\mathbb{R}}\frac{|\xi|^{\frac{\alpha}{2}}|\sinh(\eta(\xi)x_0)|}{\sqrt{\epsilon}(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|}\delta\le\frac{\overline{C}\delta}{\sqrt{\epsilon}},\ m>\frac{1}{2}\quad\left(by\ \frac{|\sinh(\eta(\xi)x_0)|}{|\cosh(\eta(\xi)x_0)|}<\overline{C}\right).\end{aligned}$$

Let $p=m\alpha$, $m>\frac{1}{2}$, $\mu=(\frac{4m-1}{e}+(2x_0+L)\cos(\frac{\pi}{4}\alpha))^{-1}$ and $\epsilon<(\frac{1}{2})^{\frac{1}{\mu L}}$. From Lemma 4.3, we obtain

$\mathbf{E_2}$:

$$\begin{aligned} E_2 &= \left\| \frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2} \frac{\eta(\xi)\cosh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)}(1+|\xi|^p)\frac{\sinh(\eta(\xi)L)}{\eta(\xi)}\widehat{g}(\xi) \right\| \\ &\leq \sup_{\xi\in\mathbb{R}} \frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2} \left| \frac{\eta(\xi)\cosh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)} \right| M \\ &\leq M\left(\frac{2\overline{C}}{L}+3\right)\left(\mu\ln\frac{1}{\epsilon}\right)^{-(2m-1)} \epsilon^{(L-x_0)\mu\cos(\frac{\pi}{4}\alpha)}, \quad m>\frac{1}{2}. \end{aligned}$$

$\mathbf{F_2}$:

$$\begin{aligned} F_2 = &\left\| \frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2} \frac{\eta(\xi)\sinh(\eta(\xi)x_0)}{\cosh(\eta(\xi)L)}(1+|\xi|^p)\cosh(\eta(\xi)L)\widehat{f}(\xi) \right\| \\ &\leq \sup_{\xi\in\mathbb{R}} \frac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2} \frac{\eta(\xi)\sinh(\eta(\xi)x_0)}{\cosh(\eta(\xi)L)} M \\ &\leq M\left(\frac{2\overline{C}}{L}+3\right)\left(\mu\ln\frac{1}{\epsilon}\right)^{-(2m-1)} \epsilon^{(L-x_0)\mu\cos(\frac{\pi}{4}\alpha)}, \quad m>\frac{1}{2}. \end{aligned}$$

$\mathbf{G_2}$:

$$\begin{aligned} G_2 &= \left\| \tfrac{2\epsilon(1+|\xi|^p)\eta(\xi)\sinh(\eta(\xi)x_0)\cosh(\eta(\xi)x_0)}{(1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2)^2} \tfrac{\sinh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)} \left((1+|\xi|^p)\tfrac{\sinh(\eta(\xi)L)}{\eta(\xi)}\widehat{g}(\xi)\right) \right\| \\ &\leq \sup_{\xi\in\mathbb{R}} \left| \tfrac{2\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2} \tfrac{\eta(\xi)\sinh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)} \tfrac{\sinh(\eta(\xi)x_0)}{\cosh(\eta(\xi)x_0)} \right| M \\ &\leq 2\overline{C}M \ \sup_{\xi\in\mathbb{R}} \tfrac{\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2} \left| \tfrac{\eta(\xi)\sinh(\eta(\xi)x_0)}{\sinh(\eta(\xi)L)} \right| \\ &\leq 2\overline{C}M(\tfrac{2\overline{C}}{L}+3)(\mu\ln\tfrac{1}{\epsilon})^{-(2m-1)}\epsilon^{(L-x_0)\mu\cos(\frac{\pi}{4}\alpha)}, \qquad p=m\alpha, \ m>\tfrac{1}{2}. \end{aligned}$$

Here the penultimate inequality is proved by $\dfrac{|\sinh(\eta(\xi)x_0)|}{|\cosh(\eta(\xi)x_0)|} < \overline{C}$.

$\mathbf{H_2}$:

$$\begin{aligned} H_2 &= \left\| \tfrac{2\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{(1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2)^2} \tfrac{\eta(\xi)\sinh(\eta(\xi)x_0)}{\cosh(\eta(\xi)L)} \left((1+|\xi|^p)\cosh(\eta(\xi)L)\widehat{f}(\xi)\right) \right\| \\ &\leq \sup_{\xi\in\mathbb{R}} \left| \tfrac{2\epsilon(1+|\xi|^p)|\cosh(\eta(\xi)x_0)|^2}{1+\epsilon(1+|\xi|^p)^2|\cosh(\eta(\xi)x_0)|^2} \tfrac{\eta(\xi)\sinh(\eta(\xi)x_0)}{\cosh(\eta(\xi)L)} \right| M \\ &\leq 2M\left(\tfrac{2\overline{C}}{L}+3\right)\left(\mu\ln\tfrac{1}{\epsilon}\right)^{-(2m-1)}\epsilon^{(L-x_0)\mu\cos(\frac{\pi}{4}\alpha)}, \qquad p=m\alpha, \ m>\tfrac{1}{2}. \end{aligned}$$

The sum of $A_2, B_2, C_2, D_2, E_2, F_2, G_2$ and H_2 is

$$\|h_x^{\epsilon,\delta}(t)-h_x(t)\| \leq \left(\frac{1}{2}+\frac{3\overline{C}}{2}+\overline{C}^2\right)\frac{\delta}{\sqrt{\epsilon}} + (2\overline{C}+4)M\left(\frac{1}{L}+1\right)\left(\mu\ln\frac{1}{\epsilon}\right)^{-(2m-1)}\epsilon^{(L-x_0)\mu\cos(\frac{\pi}{4}\alpha)},$$
$$p=m\alpha, \ m>\frac{1}{2}.$$

Choosing $\epsilon=\frac{\delta}{M}$, we can get (29). □

Note 1. It can be observed from Theorem 4.4 that the rate of convergence degenerates to being logarithmic when $x_0=L$, since both $(\frac{\delta}{M})^{r(L-x_0)\cos(\frac{\pi}{4}\alpha)}$ in (28) and $(\frac{\delta}{M})^{(L-x_0)\mu\cos(\frac{\pi}{4}\alpha)}$ in (29) become 1.

§5 Numerical implementation

5.1 Numerical expression for the gradient of the regularization functional

We will deduce the gradient of the quadratic functional (5) with respect to $h(t)$. Denote

$$J(h(t)) = \frac{1}{2}\left\|Ah(t) - f^\delta(t)\right\|_{L^2}^2 + \frac{\epsilon}{2}\left\|h(t)\right\|_{H^p}^2 \triangleq J_1(h(t)) + \epsilon J_2(h(t)),$$

where

$$\begin{aligned} J_2(h(t)) &= \frac{1}{2}\|(1+|\xi|^p)\widehat{h}(\xi)\|^2 = \frac{1}{2}\|\widehat{h}(\xi)\|^2 + \||\xi|^{\frac{p}{2}}\widehat{h}(\xi)\|^2 + \frac{1}{2}\||\xi|^p\widehat{h}(\xi)\|^2 \\ &= \frac{1}{2}\|h(t)\|^2 + \|({}^C D_{0+}^{\frac{p}{2}} h)(t)\|^2 + \frac{1}{2}\|({}^C D_{0+}^{p} h)(t)\|^2. \end{aligned}$$

For $0 < \beta < \infty$, the bigger the fractional derivative order β is, the more complex numerical expression of $\|({}^C D_{0+}^{\beta} h)(t)\|^2$ is. In fact, $\beta \in (0,1]$ is enough from numerical investigation in the subsection of numerical examples. So we only consider this case in the following sections. Denote temporal step size by Δt. Let $t_k = k\Delta t$, $h_k = h(t_k)$ and $h = [h_0, h_1, \cdots, h_n]^T$, $k = 0, 1, \cdots, n$.

$$\begin{aligned} \|({}^C D_{0+}^{\beta} h)(t)\|^2 &= \int_0^T \left(({}^C D_{0+}^{\beta} h)(t)\right)^2 dt = \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left(({}^C D_{0+}^{\beta} h)(t)\right)^2 dt \\ &\approx \sum_{k=0}^{n-1}\left(({}^C D_{0+}^{\beta} h)(t_{k+1})\right)^2 \Delta t \approx \sum_{k=0}^{n-1}\left(\frac{1}{\Gamma(2-\beta)(\Delta t)^\beta}\sum_{j=0}^{k} b_j(h_{k+1-j} - h_{k-j})\right)^2 \Delta t \\ &= \sum_{k=0}^{n-1}(A_k h)^2\Delta t = \sum_{k=0}^{n-1} h^T A_k^T A_k h \Delta t = h^T A^T A h \Delta t, \end{aligned}$$

where $b_0 = 1, b_k = (k+1)^{1-\beta} - k^{1-\beta}, k = 1, \cdots, n-1$. The n-dimensional matrix A has the form $A^T = [A_0^T \ A_1^T \ \cdots \ A_{n-1}^T]$, where the size of A_k is $1 \times (n+1)$ and

$$A_0 = \frac{1}{\Gamma(2-\beta)(\Delta t)^\beta}[-b_0, b_0, 0, \cdots, 0],$$

$$A_k = \frac{1}{\Gamma(2-\beta)(\Delta t)^\beta}[-b_k, b_k - b_{k-1}, \cdots, b_1 - b_0, b_0, 0, \cdots, 0], \qquad k = 1, 2, \cdots, n-1.$$

Due to

$$\|({}^C D_{0+}^{\frac{p}{2}} h)(t)\|^2 \approx h^T A^T A h \Delta t, \qquad \|({}^C D_{0+}^{p} h)(t)\|^2 \approx h^T B^T B h \Delta t,$$

the discrete form of the gradient of $J_2(h(t))$ is

$$J_2^{'}(h) \approx h + 2A^T A h + B^T B h.$$

From Theorem 4.1, we have

$$J_1^{'}(h(t))(q(t)) = \int_0^T q(t)(z_x(x_0, t))dt.$$

Denote $z_x(x_0) = [z_x(x_0, t_0), z_x(x_0, t_1), \cdots, z_x(x_0, t_n)]^T$, and have

$$J^{'}(h) = z_x(x_0) + \epsilon h + 2\epsilon A^T A h + \epsilon B^T B h. \tag{31}$$

5.2 Reformulation of the adjoint problem

Let $z(x,t)$ be the solution of the adjoint problem (9). We introduce $\widetilde{z}(x,t) = z(x,T-t)$ and get

$$\begin{aligned}
&({}^C D^{\alpha}_{T-,t} z)(x,t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T \frac{z_s(x,s)}{(s-t)^{\alpha}} ds = \frac{-1}{\Gamma(1-\alpha)} \int_{T-t}^0 \frac{z_{\zeta}(x,T-\zeta)}{(T-\zeta-t)^{\alpha}} d\zeta \\
= &\frac{1}{\Gamma(1-\alpha)} \int_0^{T-t} \frac{z_{\zeta}(x,T-\zeta)}{(T-t-\zeta)^{\alpha}} d\zeta = \frac{1}{\Gamma(1-\alpha)} \int_0^{T-t} \frac{\widetilde{z}_{\zeta}(x,\zeta)}{(T-t-\zeta)^{\alpha}} d\zeta \qquad (\text{let } \tau = T-t) \\
= &\frac{1}{\Gamma(1-\alpha)} \int_0^{\tau} \frac{\widetilde{z}_{\zeta}(x,\zeta)}{(\tau-\zeta)^{\alpha}} d\zeta = ({}^C D^{\alpha}_{0+,\tau} \widetilde{z})(x,\tau).
\end{aligned}$$

The adjoint problem (9) can be transformed into a forward initial boundary value problem

$$\begin{cases}
({}^C D^{\alpha}_{0+,\tau} \widetilde{z})(x,t) = \widetilde{z}_{xx}(x,t), & x \in (0,x_0)\,,\ \tau \in (0,T) \\
\widetilde{z}_x(0,t) = u(0,T-\tau) - f(T-\tau), & x = 0,\ \tau \in (0,T), \\
\widetilde{z}\,(x_0,\tau) = 0, & x = x_0,\ \tau \in (0,T), \\
\widetilde{z}\,(x,0) = 0, & x \in (0,x_0),\ \tau = 0,
\end{cases} \tag{32}$$

5.3 The conjugate gradient algorithm for solving the minimization problem

We present the conjugate gradient procedure for the Cauchy problem of time fractional diffusion equation.

1. Give the initial guess h_0;

2. Solve the forward problem (6) with $h = h_0$ by the unconditionally stable implicit finite difference scheme (IFDS) of [11];

3. Solve the adjoint problem (32) and determine $J^{'}(h_0)$ in (31). Since $h(0)$ can be determined from the initial condition, we take $J^{'}(0) = 0$;

4. Let $q_0 = -J^{'}(h_0)$; $\rho_0 = \|J^{'}(h_0)\|^2$; $\nu := 0$;
   Begin CG iterations:

5. The step size $\tau_{\nu} = \arg\min_{\tau>0} J(h_{\nu}(t)+\tau q_{\nu}(t))$ can be obtained in the following deduction. From (5), we have

$$\begin{aligned}
J(h_{\nu}(t) + \tau q_{\nu}(t)) =& \frac{1}{2} \int_0^T (u_{\nu}(0,t) + \tau\omega_{\nu}(0,t) - f^{\delta}(t))^2 dt + \frac{\epsilon}{2} \|h_{\nu}(t) + \tau q_{\nu}(t)\|^2_{H^p} \\
=& \frac{1}{2} \int_0^T (u_{\nu}(0,t) + \tau\omega_{\nu}(0,t) - f^{\delta}(t))^2 dt + \frac{\epsilon}{2} \int_0^T (h_{\nu}(t) + \tau q_{\nu}(t))^2 dt \\
&+ \epsilon \|{}^C D^{\frac{p}{2}}_{0+}(h_{\nu}(t) + \tau q_{\nu}(t))\|^2 + \frac{\epsilon}{2} \|{}^C D^{p}_{0+}(h_{\nu}(t) + \tau q_{\nu}(t))\|^2,
\end{aligned}$$

where h_{ν} and q_{ν} are the vector forms of $h_{\nu}(t)$ and $q_{\nu}(t)$, and $\omega_{\nu}(x,t)$ is the solution of

Table 1: Error1 for Example 1.

	$\alpha = 0.8$			$\alpha = 0.5$			$\alpha = 0.2$		
error level	$x_0 = 1.0$	$x_0 = 0.7$	$x_0 = 0.4$	$x_0 = 1.0$	$x_0 = 0.7$	$x_0 = 0.4$	$x_0 = 1.0$	$x_0 = 0.7$	$x_0 = 0.4$
$\delta_0 = \delta_1 = 0.1$	0.1233	0.0496	0.0238	0.0976	0.0541	0.0289	0.0815	0.0525	0.0308
$\delta_0 = \delta_1 = 0.05$	0.1115	0.0352	0.0140	0.0602	0.0367	0.0175	0.0556	0.0332	0.0216
$\delta_0 = \delta_1 = 0.01$	0.0872	0.0220	0.0049	0.0200	0.0088	0.0047	0.0274	0.0232	0.0049

the sensitivity problem (11) with $q(t) = q_\nu(t)$. From

$$\frac{dJ(h_\nu(t) + \tau_\nu q_\nu(t))}{d\tau_\nu} = \int_0^T (u_\nu(0,t) + \tau_\nu \omega_\nu(0,t) - f^\delta(t))\omega_\nu(0,t)dt$$

$$+\epsilon \int_0^T (h_\nu(t) + \tau_\nu q_\nu(t))q_\nu(t)dt + 2\epsilon q_\nu^T A^T A(h_\nu + \tau_\nu q_\nu)\Delta t + \epsilon q_\nu^T B^T B(h_\nu + \tau_\nu q_\nu)\Delta t = 0,$$

we can get a step size

$$\tau_\nu = \frac{-[\int_0^T (u_\nu(0,t) - f^\delta(t))\omega_\nu(0,t)dt + \epsilon \int_0^T h_\nu(t)q_\nu(t)dt + \epsilon q_\nu^T(2A^T A + B^T B)h_\nu \Delta t]}{\int_0^T \omega_\nu^2(0,t)dt + \epsilon \int_0^T q_\nu^2(t)dt + \epsilon q_\nu^T(2A^T A + B^T B)q_\nu \Delta t};$$

6. Update $h_{\nu+1} := h_\nu + \tau_\nu q_\nu$;
7. Solve the forward problem (1) with $h = h_{\nu+1}$; And compute the residual $E_\nu = \|u_\nu(0,t) - f^\delta(t)\| + \|(u_\nu)_x(0,t) - g^\delta(t)\|$.
8. Compute $J'(h_{\nu+1})$;
9. $\rho_{\nu+1} = \|J'(h_{\nu+1})\|^2$;
10. $\lambda_\nu := \rho_{\nu+1}/\rho_\nu$;
11. Update $q_{\nu+1} := -J'(h_{\nu+1}) + \lambda_\nu q_\nu$;
12. $\nu := \nu + 1$;
    End CG iterations when a stopping criterion is satisfied.

We use the well-known Morozov discrepancy principle [9] to find a suitable stopping rule. And ν is obtained as follows

$$E_\nu \le \sigma\delta < E_{\nu-1},$$

where $\sigma > 1$ is a constant. In the following two examples, we take $\sigma = 1.01$ [3] and $h_0(t) = 0$.

5.4 Numerical examples

In this section, we present two examples to demonstrate the efficiency of conjugate gradient algorithm in Section 5.3. The exact solution $u(x,t)$ is obtained by solving the following problem

$$\begin{cases} ({}^C D_{0^+,t}^\alpha u)(x,t) = & u_{xx}(x,t), & 0 < x < L,\ 0 < t \le T, \\ u(0,t) = & f(t), & x = 0,\ 0 \le t \le T, \\ u(L,t) = & \phi(t), & x = L,\ 0 \le t \le T, \\ u(x,0) = & 0, & 0 < x < L,\ t = 0, \end{cases} \tag{33}$$

Table 2: Error2 for Example 1.

	$\alpha=0.8$			$\alpha=0.5$			$\alpha=0.2$		
error level	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$
$\delta_0=\delta_1=0.1$	0.5274	0.2104	0.0973	0.2398	0.1323	0.0755	0.1251	0.0908	0.0747
$\delta_0=\delta_1=0.05$	0.4814	0.1624	0.0645	0.1566	0.0917	0.0448	0.0798	0.0529	0.0457
$\delta_0=\delta_1=0.01$	0.4002	0.1108	0.0261	0.0590	0.0262	0.0142	0.0395	0.0267	0.0088

Table 3: Error1 for Example 2.

	$\alpha=0.8$			$\alpha=0.5$			$\alpha=0.2$		
error level	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$
$\delta_0=\delta_1=0.05$	0.1508	0.0473	0.0230	0.1258	0.0577	0.0.0302	0.0770	0.0500	0.0300
$\delta_0=\delta_1=0.03$	0.1346	0.0365	0.0171	0.1041	0.0442	0.0222	0.0495	0.0321	0.0218
$\delta_0=\delta_1=0.01$	0.1119	0.0235	0.0091	0.0617	0.0238	0.0097	0.0176	0.0113	0.0077

where $L=1$, $T=1$. The grid sizes for space and time domain are taken to be $h_x=\frac{1}{100}$, $h_t=\frac{1}{60}$. The exact flux $u_x(x,t)$ is approximated by the backward difference scheme. We choose $g(t)=u_x(0,t)$.

In the following examples, we consider the cases of $\alpha=0.2,0.5,0.8,1.0$ and $x_0=0.4,0.7,1$, and compute the L^2 errors

$$Error1=\|h(t)-u(x_0,t)\|_{L^2(0,T)},\ Error2=\|h_x(t)-u_x(x_0,t)\|_{L^2(0,T)}.$$

Example 1: Consider the direct problem with $f(t)=1-e^{-t}$ and $\phi(t)=2\sin(4\pi t)$. The noisy data are

$$f^\delta=f(1+\delta_1(2\text{rand}(\text{size}(f))-1)),g^\delta=g(1+\delta_0(2\text{rand}(\text{size}(g))-1))$$

where $\delta_0=0.1,0.05,0.01$ and $\delta_1=\delta_0$.

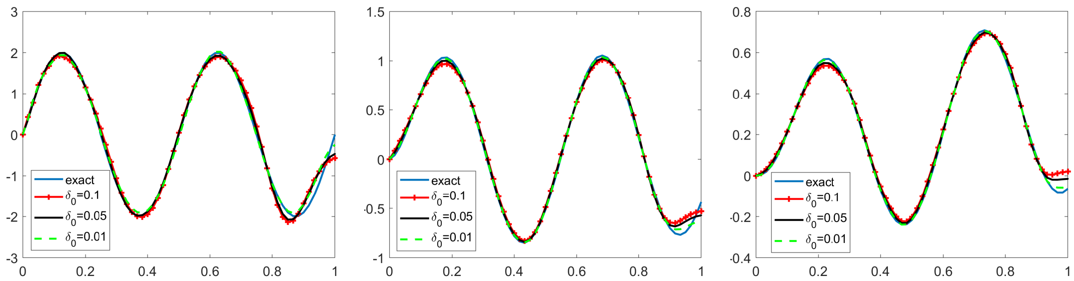
Because $\phi(t)\in H^1(0,1)$ we choose $p=1$. In Table 1, we tabulate L^2 errors of u, Error1, for various δ_0, α and x_0. In Table 2, we tabulate L^2 errors of u_x, Error2. Both Error1 and Error2 dependent on three parameters δ_0, α and x_0. With the fixed α and x_0, the error increases with δ_0. With the fixed α and δ_0, the error increases with x_0.

We plot the numerical/exact solutions in Figures 1-8. The values of u or u_x appear in the vertical direction and the time in the horizontal. From Figures 1-8, it can be observed that the curve of numerical solutions match well that of the exact solutions.

From Figure 2, it can be observed that the curve of numerical solution match well that of the exact solution except several time intervals, such as the location around $t=0.9$. Indeed, if we compare three sub-figures in Figure 2, we should agree that the sub-figure (c) has the

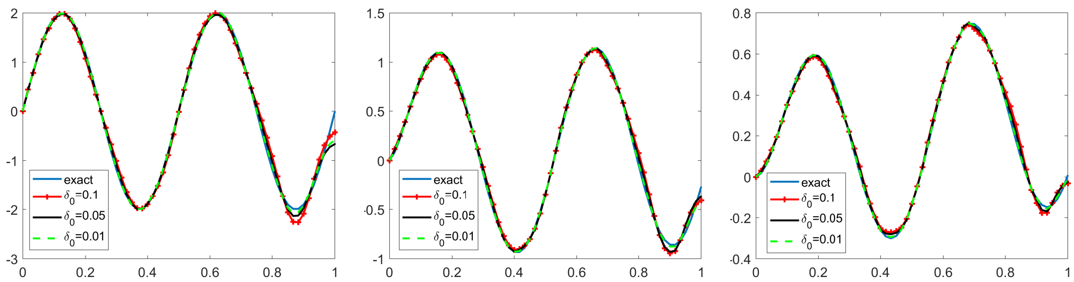
Table 4: Error2 for Example 2.

	$\alpha=0.8$			$\alpha=0.5$			$\alpha=0.2$		
error level	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$	$x_0=1.0$	$x_0=0.7$	$x_0=0.4$
$\delta_0=\delta_1=0.05$	0.7631	0.1991	0.0870	0.3748	0.1644	0.0694	0.1165	0.0728	0.0408
$\delta_0=\delta_1=0.03$	0.7236	0.1677	0.0684	0.3230	0.1316	0.0580	0.0756	0.0468	0.0280
$\delta_0=\delta_1=0.01$	0.6502	0.1259	0.0402	0.1991	0.0752	0.0267	0.0270	0.0164	0.0098



(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 1: Reconstruction of $u(x_0, t)$ for Example 1 with $\alpha = 1.0$.



(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 2: Reconstruction of $u(x_0, t)$ for Example 1 with $\alpha = 0.8$.

smallest L^2 error. In fact, for the case of u_x it is also evident from Figures 6, 7, 8 that for the fixed δ_0 and α, the error of $x_0 = 0.4$ is smaller than those of $x_0 = 0.7$ and $x_0 = 1$.

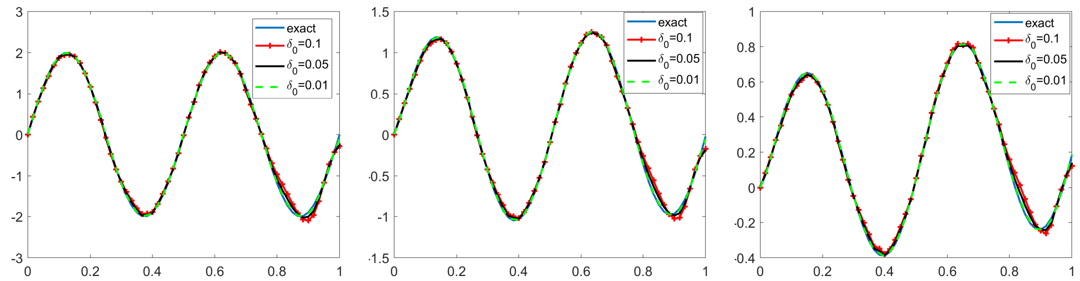
Example 2: Now consider the forward problem with $f(t) = 0$ and discontinuous right boundary condition $\phi(t) = H(t - 0.4) - H(t - 0.8)$. The noisy Cauchy data are

$$f^\delta = f + \delta_1(2\text{rand}(\text{size}(f)) - 1), \qquad g^\delta = g + \delta_0(2\text{rand}(\text{size}(g)) - 1)$$

where $\delta_0 = \delta_1 = 0.05, 0.03, 0.01$.

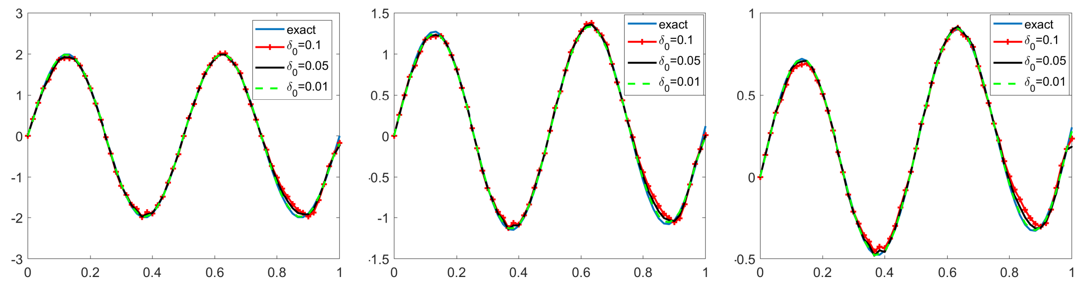
We choose $p = 0.45$, because $u(L, t) \in H^p(0, 1)$ in the case of $p \in [0, \frac{1}{2})$. The values of Error1 and Error2 are presented in Table 3 and Table 4 respectively. We observe that the smallest δ_0, α and x_0 give the smallest errors in all cases. As any one of three parameters δ_0, α and x_0 increases, the L^2 errors Error1 and Error2 increase. Figures 9-16 illustrate the numerical/exact solutions. It can be observed that the numerical solutions are in good agreement with the exact solutions.

The two numerical examples illustrate that the proposed algorithm is robust to noise for both smooth and nonsmooth examples. It also shows that this fractional inverse problem is better behaved than the standard parabolic counterpart ($\alpha = 1.0$).



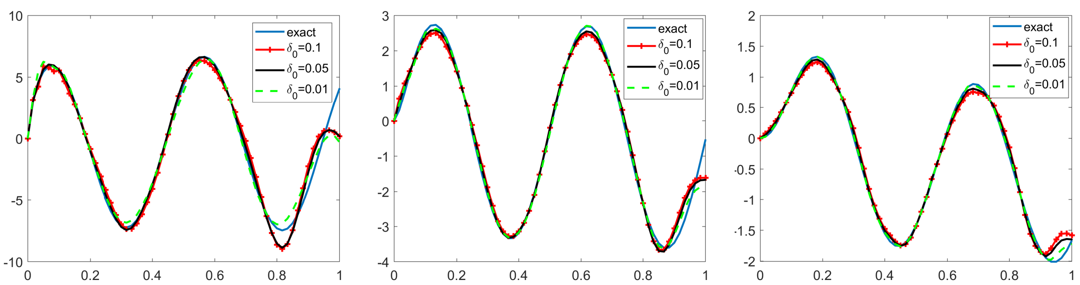
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 3: Reconstruction of $u(x_0, t)$ for Example 1 with $\alpha = 0.5$.



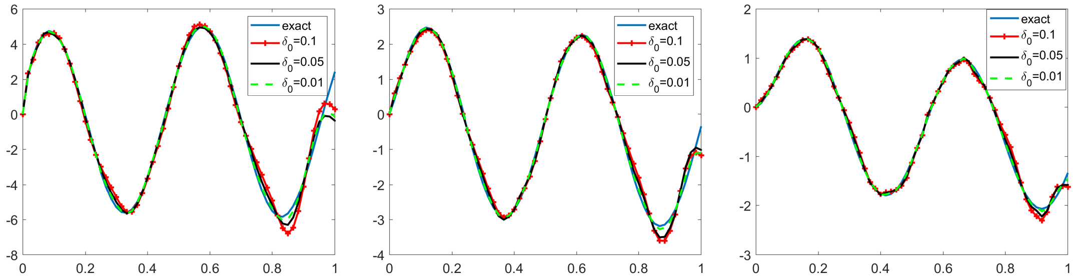
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 4: Reconstruction of $u(x_0, t)$ for Example 1 with $\alpha = 0.2$.



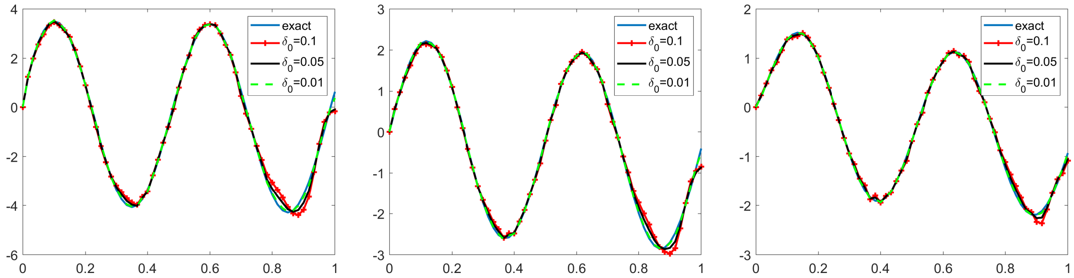
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 5: Reconstruction of $u_x(x_0, t)$ for Example 1 with $\alpha = 1.0$.



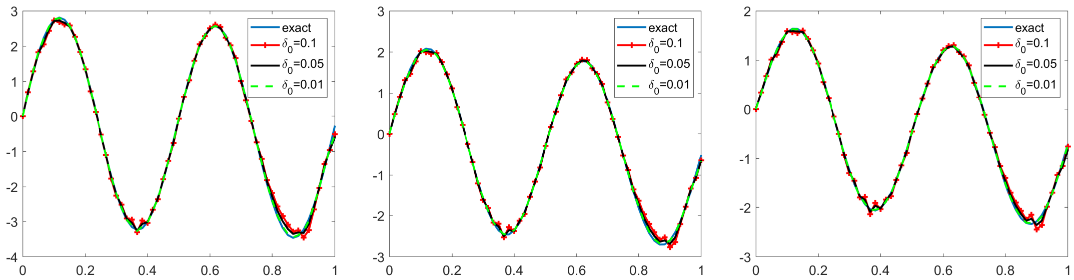
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 6: Reconstruction of $u_x(x_0, t)$ for Example 1 with $\alpha = 0.8$.



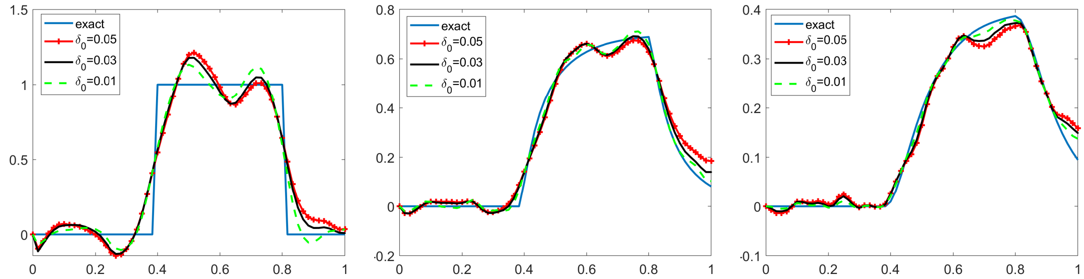
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 7: Reconstruction of $u_x(x_0, t)$ for Example 1 with $\alpha = 0.5$.



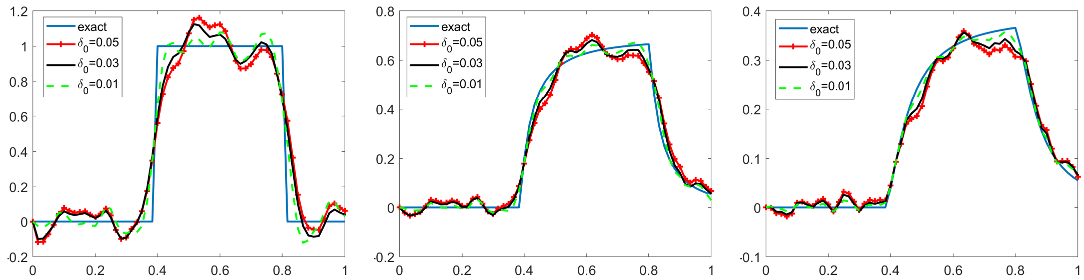
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 8: Reconstruction of $u_x(x_0, t)$ for Example 1 with $\alpha = 0.2$.



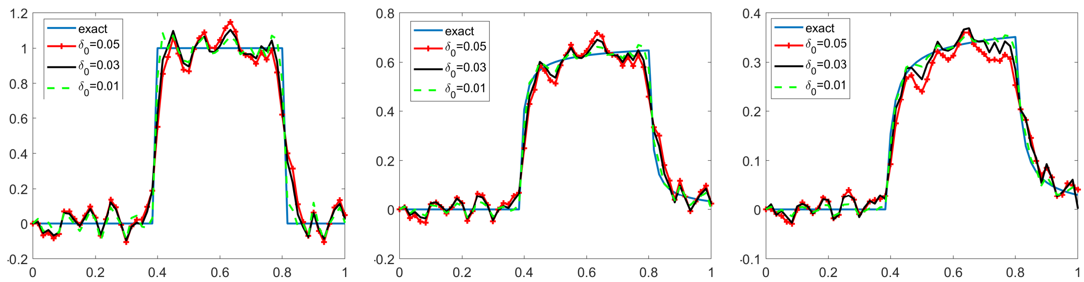
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 9: Reconstruction of $u(x_0, t)$ for Example 2 with $\alpha = 1.0$.



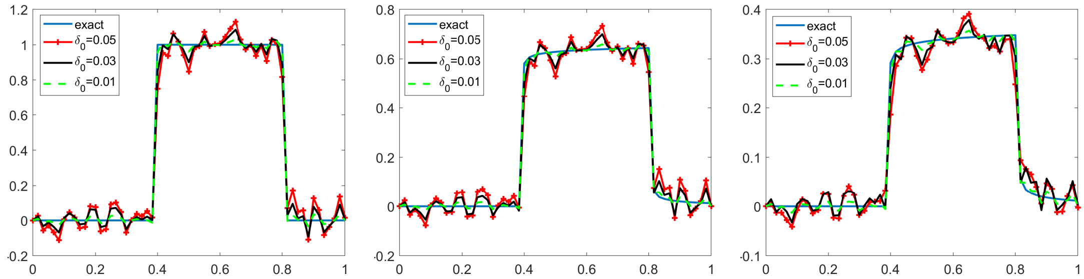
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 10: Reconstruction of $u(x_0, t)$ for Example 2 with $\alpha = 0.8$.



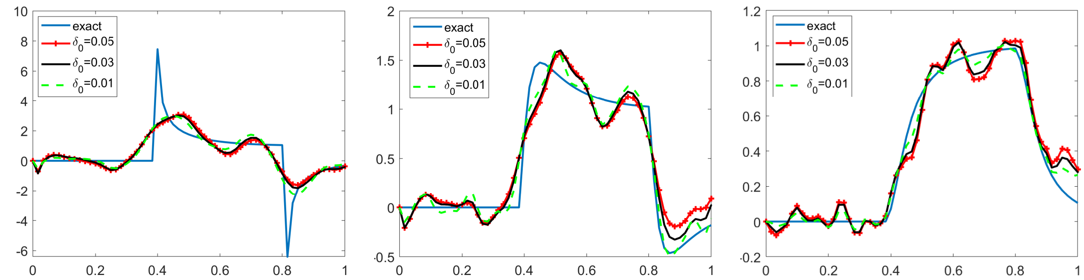
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 11: Reconstruction of $u(x_0, t)$ for Example 2 with $\alpha = 0.5$.



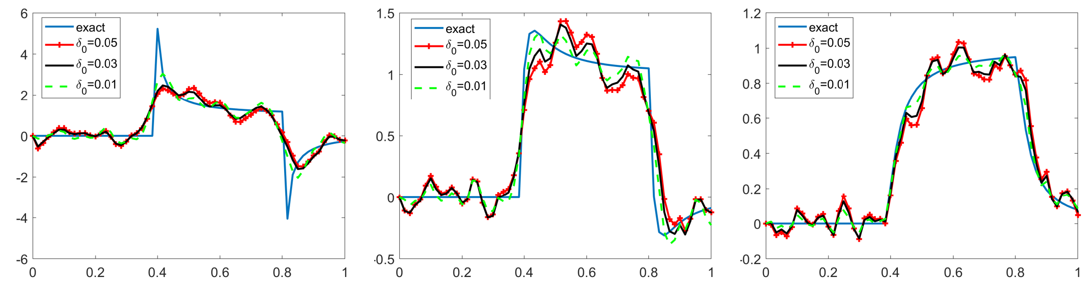
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 12: Reconstruction of $u(x_0, t)$ for Example 2 with $\alpha = 0.2$.



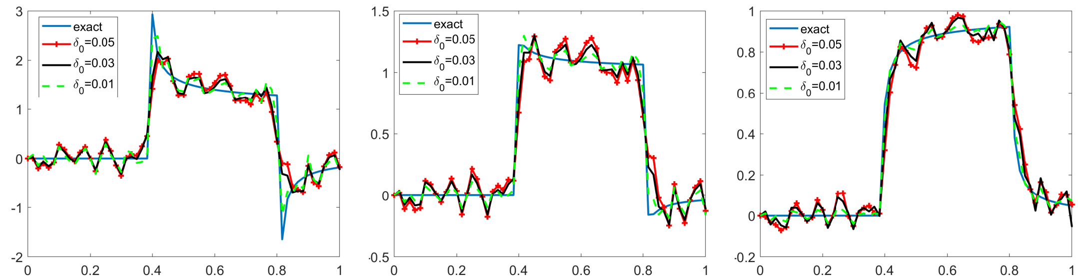
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 13: Reconstruction of $u_x(x_0, t)$ for Example 2 with $\alpha = 1.0$.



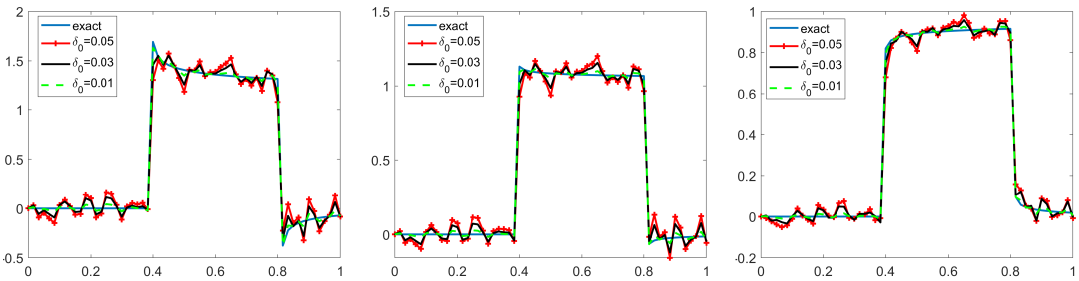
(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 14: Reconstruction of $u_x(x_0, t)$ for Example 2 with $\alpha = 0.8$.



(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 15: Reconstruction of $u_x(x_0, t)$ for Example 2 with $\alpha = 0.5$.



(a) x_0=1. (b) x_0=0.7. (c) x_0=0.4.

Figure 16: Reconstruction of $u_x(x_0, t)$ for Example 2 with $\alpha = 0.2$.

§6 Conclusions.

This paper is devoted to solving a Cauchy problem of the time fractional diffusion equation in $x \in [0, L]$. The Cauchy problem becomes much more difficult with x_0 is farther from 0. We formulate the problem into a minimization problem with a modified Tikhonov regularization method. A conjugate gradient method is employed to solve the corresponding minimization problem. Numerical examples illustrate that the proposed algorithm is robust to noise, and can more effectively recover either the smooth or the nonsmooth solutions. The error estimates of numerical solutions are provided.

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